Polymorphic Types

CS242

Lecture 6

Review: Type Rules for Simply Typed LC

$$\frac{}{A, x: t \vdash x : t} \quad [\text{Var}]$$

$$\frac{A, x: t \vdash e : t'}{A \vdash \lambda x{:}t.e : t \to t'} \quad [\text{Abs}]$$

$$\frac{}{A \vdash i : \text{int}} \quad [\text{Int}]$$

$$\frac{A \vdash e_1 : t \to t' \qquad A \vdash e_2 : t}{A \vdash e_1\, e_2 : t'} \quad [\text{App}]$$

Review: Type Inference Rules

$$\frac{}{A, x: t \vdash x : t} \quad [\text{Var}]$$

$$\frac{A, x: \alpha_x \vdash e : t}{A \vdash \lambda x: \alpha_x . e : \alpha_x \rightarrow t} \quad [\text{Abs}]$$

$$\frac{}{A \vdash i : int} \quad [\text{Int}]$$

$$\frac{t = t' \rightarrow \beta \quad A \vdash e_1 : t \quad A \vdash e_2 : t'}{A \vdash e_1\, e_2 : \beta} \quad [\text{App}]$$

Review: Algorithms

- Type checking
  - Collect assumptions from the root down to the leaves
  - Calculate types from the leaves up to the root

- Type inference
  - Generate type constraints from applications
  - Solve the constraints
  - If no solutions or only infinite solutions, type error
  - If the constraints have finite solutions, construct a type checking proof

Let Expressions

Extend the lambda calculus with one new expression

$$e \rightarrow x \mid \lambda x.e \mid e\ e \mid \text{let } f = \lambda x.e \text{ in } e \mid i$$

$$t \rightarrow \alpha \mid t \rightarrow t \mid \text{int}$$

Let Expressions

Nothing new here, really:

let f = λx.e in e'   is equivalent to   (λf.e') λx.e

And note we are getting closer to standard syntax:

let f x = e in e'   is syntactic sugar for let f = λx.e in e'

Type Rules

$$\frac{}{A, x: t \vdash x : t} \text{[Var]}$$

$$\frac{A, x: t \vdash e : t'}{A \vdash \lambda x{:}t.e : t \rightarrow t'} \text{[Abs]}$$

$$\frac{}{A \vdash i : int} \text{[Int]}$$

$$\frac{A \vdash \lambda x.e : t \qquad A, f: t \vdash e' : t'}{A \vdash \text{let } f = \lambda x.e \text{ in } e' : t'} \text{[Let]}$$

$$\frac{A \vdash e_1 : t \rightarrow t' \qquad A \vdash e_2 : t}{A \vdash e_1\, e_2 : t'} \text{[App]}$$

Recall …

The program

let f = λx.x in f f

is untypable, but

(λx.x) (λy.y)

is typable (in simply typed lambda calculus)

Polymorphic Types

$e \rightarrow x \mid \lambda x.e \mid e\ e \mid \text{let } f = \lambda x.e \text{ in } e \mid i$

$t \rightarrow \alpha \mid t \rightarrow t \mid \text{int}$

$o \rightarrow \forall \alpha.o \mid t$

Polymorphic Let Type Rule

$$\frac{A \vdash \lambda x.e : t \qquad A, f : \forall\alpha.t \vdash e' : t' \quad \text{if } \alpha \notin FV(A)}{A \vdash \text{let } f = \lambda x.e \text{ in } e' : t'} \quad [\text{Let}]$$

$FV(A, x{:}t) = FV(A) \cup FV(t)$
$FV(\emptyset) = \emptyset$
$FV(int) = \emptyset$
$F(t \rightarrow t') = FV(t) \cup FV(t')$
$FV(\forall\alpha.t) = FV(t) - \{\alpha\}$
$FV(\alpha) = \{\alpha\}$

The Idea

If we prove $e : t$ and the proof does not rely on any assumptions about α, then we have also proven $e: \forall\alpha.t$

Instantiation Rule

$$\frac{}{A, f: \forall\alpha.t \vdash f: t[\alpha := t']} \quad [\text{Inst}]$$

Example

$$\dfrac{\dfrac{x: \beta \vdash x: \beta}{\vdash \lambda x.x : \beta \to \beta} \qquad \dfrac{I: \forall \beta.\ \beta \to \beta \ \vdash I : (\rho \to \rho) \to (\rho \to \rho) \qquad I: \forall \beta.\beta \to \beta \vdash I: \rho \to \rho}{I: \forall\ \beta.\ \beta \to \beta \vdash I\ I : \rho \to \rho}}{\vdash \text{let } I = \lambda x.x \text{ in } I\ I : \rho \to \rho}$$

Multiple Type Variables

$$\frac{A \vdash \lambda x.e : t \qquad A, f: \forall \alpha_1,\ldots,\alpha_n.t \vdash e' : t' \quad \text{if } \alpha_1,\ldots,\alpha_n \notin FV(A)}{A \vdash \text{let } f = \lambda x.e \text{ in } e' : t'} \text{[Let]}$$

$FV(A, x{:}t) = FV(A) \cup FV(t)$

$FV(\emptyset) = \emptyset$

$FV(int) = \emptyset$

$F(t \rightarrow t') = FV(t) \cup FV(t')$

$FV(\forall \alpha_1,\ldots,\alpha_n.t) = FV(t) - \{\alpha_1,\ldots,\alpha_n\}$

$FV(\alpha) = \{\alpha\}$

Type Inference for Polymorphic Let

- To do type inference with polymorphic let, we need to know the type derivation for $\lambda x.e$ to do the generalization step
  - Because we need to compute the set of free variables in the environment
  - And we need to know the variables in the type of the function to generalize
- Thus, we need to solve the constraints and produce a valid typing of $\lambda x.e$ to proceed
  - So we solve the constraints and substitute the solution back into the proof at each let.
  - Compute FV(A)
  - Generalize

$$\frac{A \vdash \lambda x.e : t \qquad A, f: \forall \alpha_1,\ldots,\alpha_n.t \vdash e' : t' \quad \text{if } \alpha_1,\ldots,\alpha_n \notin FV(A)}{A \vdash \text{let } f = \lambda x.e \text{ in } e' : t'} \; [\text{Let}]$$

Example – Full Derivation

$$\dfrac{x : \beta \to \beta \vdash x : \beta \to \beta}{\vdash \lambda x.x : (\beta \to \beta) \to (\beta \to \beta)}$$

$$\dfrac{y : \beta \vdash y : \beta}{\vdash \lambda y.y : \beta \to \beta}$$

$$\dfrac{\vdash \lambda x.x : (\beta \to \beta) \to (\beta \to \beta) \qquad \vdash \lambda y.y : \beta \to \beta}{\vdash (\lambda x.x)\ (\lambda y.y) : \beta \to \beta}$$

$$\dfrac{I : \forall\ \beta.\ \beta \to \beta\ \vdash I : (\rho \to \rho) \to (\rho \to \rho) \qquad I : \forall \beta.\ \beta \to \beta \vdash I : \rho \to \rho}{I : \forall\ \beta.\ \beta \to \beta \vdash I\ I : \rho \to \rho}$$

$$\dfrac{\vdash (\lambda x.x)\ (\lambda y.y) : \beta \to \beta \qquad \beta \notin FV(\emptyset) \qquad I : \forall\ \beta.\ \beta \to \beta \vdash I\ I : \rho \to \rho}{\vdash \text{let } I = (\lambda x.x)\ (\lambda y.y) \text{ in } I\ I : \rho \to \rho}$$

Outside the allowed syntax, but this example still works.

Example – Type Derivation Skeleton

x: ⊢ x:

⊢ λx.x :

y: ⊢ y:

⊢ λy.y:

I: ⊢ I : I: ⊢ I:

⊢ (λx.x) (λy.y) : ?? ∉ FV(Ø)

I: ⊢ I I :

⊢ let I = (λx.x) (λy.y) in I I :

Example – Type Inference

First we run type inference (from last lecture) on the innermost let binding.

x: ⊢ x:

⊢ λx.x :

y: ⊢ y:

⊢ λy.y:

⊢ (λx.x) (λy.y) : ?? ∉ FV(∅)

I: ⊢ I : I: ⊢ I:

I: ⊢ I I :

⊢ let I = (λx.x) (λy.y) in I I :

Example – Type Inference

$x: \alpha_x \vdash x:$ $\quad$ $y: \alpha_y \vdash y:$

$\vdash \lambda x.x :$ $\quad$ $\vdash \lambda y.y:$

$I: \quad \vdash I :$ $\quad$ $I: \quad \vdash I:$

$\vdash (\lambda x.x)\ (\lambda y.y) : \quad ?? \notin FV(\emptyset)$ $\quad$ $I: \quad \vdash I\ I :$

$\vdash \text{let } I = (\lambda x.x)\ (\lambda y.y) \text{ in } I\ I :$

Example – Type Inference

$x: \alpha_x \ \vdash x: \alpha_x$ $\qquad$ $y: \alpha_y \ \vdash y: \alpha_y$

$\vdash \lambda x.x : \alpha_x \rightarrow \alpha_x$ $\qquad$ $\vdash \lambda y.y: \alpha_y \rightarrow \alpha_y$

$I: \qquad \vdash I : \qquad I: \quad \vdash I:$

$\vdash (\lambda x.x)\ (\lambda y.y) : \beta$ $\qquad$ $?? \notin FV(\emptyset)$ $\qquad$ $I: \quad \vdash I\ I :$

$\alpha_x \rightarrow \alpha_x = (\alpha_y \rightarrow \alpha_y) \rightarrow \beta$

$\vdash \text{let } I = (\lambda x.x)\ (\lambda y.y) \text{ in } I\ I :$

Solving the Equations

$\alpha_x \rightarrow \alpha_x = (\alpha_y \rightarrow \alpha_y) \rightarrow \beta$

$\alpha_x = \alpha_y \rightarrow \alpha_y$ [Structure]

$\alpha_x = \beta$

$\beta = \alpha_x$ [Symmetry]

$\beta = \alpha_y \rightarrow \alpha_y$ [Transitivity]

Canonicalization:

$\alpha_x = \alpha_y \rightarrow \alpha_y$

$\beta = \alpha_y \rightarrow \alpha_y$

Example – Type Inference

$x: \alpha_y \to \alpha_y \vdash x: \alpha_y \to \alpha_y$ $\quad\quad$ $y: \alpha_y \vdash y: \alpha_y$

$\vdash \lambda x.x : (\alpha_y \to \alpha_y) \to (\alpha_y \to \alpha_y)$ $\quad\quad$ $\vdash \lambda y.y: \alpha_y \to \alpha_y$

$I: \quad \vdash I: \quad I: \quad \vdash I:$

$\vdash (\lambda x.x)\ (\lambda y.y) : \alpha_y \to \alpha_y$ $\quad\quad$ $?? \notin FV(\emptyset)$ $\quad\quad$ $I: \ \vdash I\ I :$

$\vdash \text{let } I = (\lambda x.x)\ (\lambda y.y) \text{ in } I\ I :$

Example – Generalization

$$\dfrac{x: \alpha_y \to \alpha_y \;\; \vdash x: \alpha_y \to \alpha_y}{\vdash \lambda x.x : (\alpha_y \to \alpha_y) \to (\alpha_y \to \alpha_y)} \qquad \dfrac{y: \alpha_y \;\; \vdash y: \alpha_y}{\vdash \lambda y.y: \alpha_y \to \alpha_y}$$

$$\dfrac{\vdash \lambda x.x : (\alpha_y \to \alpha_y) \to (\alpha_y \to \alpha_y) \qquad \vdash \lambda y.y: \alpha_y \to \alpha_y}{\vdash (\lambda x.x)\ (\lambda y.y) : \alpha_y \to \alpha_y \qquad \alpha_y \notin FV(\emptyset)}$$

$$\dfrac{I: \forall\, \alpha_y.\ \alpha_y \to \alpha_y \vdash I : \qquad I: \forall \alpha_y.\ \alpha_y \to \alpha_y \vdash I:}{I: \forall\, \alpha_y.\ \alpha_y \to \alpha_y \;\; \vdash I\ I :}$$

$$\dfrac{\vdash (\lambda x.x)\ (\lambda y.y) : \alpha_y \to \alpha_y \qquad \alpha_y \notin FV(\emptyset) \qquad I: \forall\, \alpha_y.\ \alpha_y \to \alpha_y \;\; \vdash I\ I :}{\vdash \text{let } I = (\lambda x.x)\ (\lambda y.y) \text{ in } I\ I :}$$

Example – Type Inference

Next we run type inference on the body of the let.

$$\frac{x: \alpha_y \to \alpha_y \;\vdash x: \alpha_y \to \alpha_y}{\vdash \lambda x.x : (\alpha_y \to \alpha_y) \to (\alpha_y \to \alpha_y)} \qquad \frac{y: \alpha_y \;\vdash y: \alpha_y}{\vdash \lambda y.y: \alpha_y \to \alpha_y}$$

$$\frac{}{\vdash (\lambda x.x)\ (\lambda y.y) : \alpha_y \to \alpha_y} \qquad \alpha_y \notin FV(\emptyset)$$

$$\frac{l: \forall\, \alpha_y.\ \alpha_y \to \alpha_y \;\vdash l : \qquad l: \forall \alpha_y.\ \alpha_y \to \alpha_y \vdash l:}{l: \forall\, \alpha_y.\ \alpha_y \to \alpha_y \;\vdash l\ l :}$$

$$\vdash \text{let } l = (\lambda x.x)\ (\lambda y.y) \text{ in } l\ l :$$

Example – Type Inference

$x: \alpha_y \to \alpha_y \;\; \vdash x: \alpha_y \to \alpha_y \qquad\qquad y: \alpha_y \;\; \vdash y: \alpha_y$

$\vdash \lambda x.x : (\alpha_y \to \alpha_y) \to (\alpha_y \to \alpha_y) \qquad \vdash \lambda y.y: \alpha_y \to \alpha_y$

$I: \forall\, \alpha_y.\, \alpha_y \to \alpha_y \;\vdash I : \; \gamma \to \gamma \qquad\qquad I: \forall \alpha_y.\, \alpha_y \to \alpha_y \vdash I: \rho \to \rho$

$\vdash (\lambda x.x)\ (\lambda y.y) : \alpha_y \to \alpha_y \qquad \alpha_y \notin FV(\emptyset)$

$I: \forall\, \alpha_y.\, \alpha_y \to \alpha_y \;\; \vdash I\ I : \mu$

$\gamma \to \gamma = (\rho \to \rho) \to \mu$

$\vdash \text{let } I = (\lambda x.x)\ (\lambda y.y) \text{ in } I\ I : \mu$

Solving the Equations

$\gamma \to \gamma = (\rho \to \rho) \to \mu$

$\gamma = \rho \to \rho$ [Structure]

$\gamma = \mu$

$\mu = \gamma$ [Symmetry]

$\mu = \rho \to \rho$ [Transitivity]

Canonicalization:

$\gamma = \rho \to \rho$

$\mu = \rho \to \rho$

Example – Full Derivation

$$\dfrac{x : \alpha_y \rightarrow \alpha_y \ \vdash x : \alpha_y \rightarrow \alpha_y}{\vdash \lambda x.x : (\alpha_y \rightarrow \alpha_y) \rightarrow (\alpha_y \rightarrow \alpha_y)} \qquad \dfrac{y : \alpha_y \ \vdash y : \alpha_y}{\vdash \lambda y.y : \alpha_y \rightarrow \alpha_y}$$

$$\dfrac{\vdash \lambda x.x : (\alpha_y \rightarrow \alpha_y) \rightarrow (\alpha_y \rightarrow \alpha_y) \qquad \vdash \lambda y.y : \alpha_y \rightarrow \alpha_y}{\vdash (\lambda x.x)\ (\lambda y.y) : \alpha_y \rightarrow \alpha_y}$$

$$\dfrac{I : \forall \alpha_y.\ \alpha_y \rightarrow \alpha_y \ \vdash I : (\rho \rightarrow \rho) \rightarrow (\rho \rightarrow \rho) \qquad I : \forall \alpha_y.\ \alpha_y \rightarrow \alpha_y \vdash I : \rho \rightarrow \rho}{I : \forall \alpha_y.\ \alpha_y \rightarrow \alpha_y \ \vdash I\ I : \rho \rightarrow \rho}$$

$$\dfrac{\vdash (\lambda x.x)\ (\lambda y.y) : \alpha_y \rightarrow \alpha_y \qquad \alpha_y \notin FV(\emptyset) \qquad I : \forall \alpha_y.\ \alpha_y \rightarrow \alpha_y \ \vdash I\ I : \rho \rightarrow \rho}{\vdash \text{let } I = (\lambda x.x)\ (\lambda y.y) \text{ in } I\ I : \rho \rightarrow \rho}$$

Summary

Polymorphism allows one to write and use generic functions.

Data types:

Cons: $\forall\alpha.\ \alpha \rightarrow \text{List}(\alpha) \rightarrow \text{List}(\alpha)$

Nil: $\forall\alpha.\ \text{List}(\alpha)$

Higher order functions:

Map: $\forall\alpha, \beta.\ (\alpha \rightarrow \beta) \rightarrow \text{List}(\alpha) \rightarrow \text{List}(\beta)$

Function composition: $\forall\alpha, \beta, \rho.\ (\alpha \rightarrow \rho) \rightarrow (\rho \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$

Discussion

- *Parametric polymorphism* allows functions to be defined once and used at many different types
  - Does not eliminate all cases where code must be duplicated to satisfy the type checker, but it goes a very long way

- The type inference algorithm produces the most general possible type
  - No better type is possible within the type system

- Considered a major breakthrough when it was discovered in the late 1970's
  - Robin Milner received the Turing Award for this work

Impact

- All typed functional languages use parametric polymorphism
  - ML, Haskell
  - The functional languages also use type inference

- Also the basis of templates/generics in C++ and Java

Typed vs. Untyped

- Typed languages always rule out some desirable programs
  - Response: Various kinds of polymorphism

- Typed languages require a lot more work (writing types)
  - Response: Type inference

- Typed languages provide a powerful form of program verification, guaranteeing certain behavior for all inputs
  - Response: Maybe we only care about certain inputs, not all inputs

- Bottom line: Modern typed languages cover 95%+ of what you want to write and require only a small amount of extra work
  - But programmers still need to understand the type system to use it – this is a real cost.
  - Some programs are harder to write in typed languages.
  - And sometimes we really don't care if the program is completely type correct

Utility

- Polymorphic type inference can make you a better programmer
- Especially when you program in untyped languages!
- If you learn this type discipline, you will find yourself mentally applying it to your own code
  - And making many fewer type errors, even without a type checker
  - Covers > 95% of code people write (excluding objects ...)