Program Verification via Type Theory

CS242

Lecture 14

Program Verification

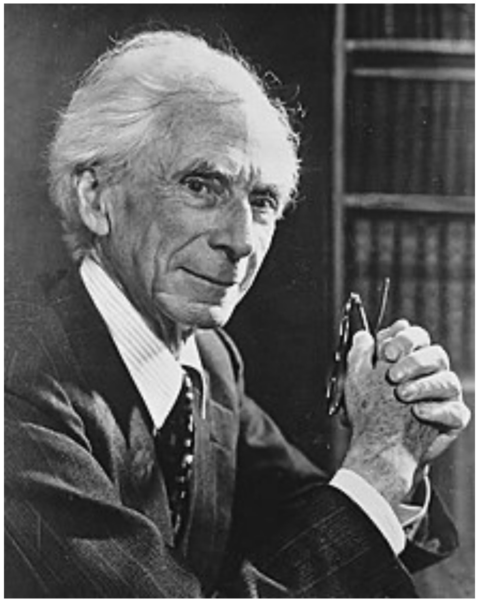
- Proving properties of programs

- But not just that programs are well-typed
  - Much deeper, almost arbitrary properties
  - And often verifying full functional correctness

- Components
  - A specification: What property the program is supposed to have
  - A proof: Written mostly manually
  - A proof assistant: Supports defining the concepts, managing the proof, checking the proof, some automation of easy parts of the proof

- Proof assistants are based on *type theory*

Type Theory



- Pioneered by Bertrand Russell in the early 20th century
  - And greatly extended in computer science
- Original goal: A basis for all mathematics
  - An alternative to set theory
- Allows the formalization of
  - Programs
  - Propositions (types)
  - Proofs that programs satisfy the propositions
  - Uniformly in one system

Caveats

- There are multiple versions of type theory
- We will look at one, and mostly by example
  - At the level we consider, there aren't significant differences with other approaches
- Type theory is a big topic
  - Whole courses are devoted to it
  - (But the same is true of other topics in this class!)

Lambda Application and Abstraction Rules

$$\frac{A \vdash e_1 : t \rightarrow t' \qquad A \vdash e_2 : t}{A \vdash e_1\, e_2 : t'} \quad \text{[App]}$$

$$\frac{A, x : t \vdash e : t'}{A \vdash \lambda x.e : t \rightarrow t'} \quad \text{[Abs]}$$

If $e_1 : t \rightarrow t'$ and $e_2 : t$,
then $e_1\, e_2$ has type t'.

If assuming $x : t$ implies $e : t'$,
then $\lambda x.e : t \rightarrow t'$.

Function Type Elimination

Function Type Introduction

Ignore the Programs for a Moment ...

$$\frac{A \vdash e_1 : t \rightarrow t' \qquad A \vdash e_2 : t}{A \vdash e_1\, e_2 : t'} \text{ [App]}$$

From a proof of $t \rightarrow t'$ and and a proof of t, we can prove t'.

Implication Elimination (modus ponens)

$$\frac{A, x : t \vdash e : t'}{A \vdash \lambda x.e : t \rightarrow t'} \text{ [Abs]}$$

If assuming t we can prove t', then we can prove $t \rightarrow t'$.

Implication Introduction

Types As Propositions

$$\frac{A \vdash e_1 : t \rightarrow t' \qquad A \vdash e_2 : t}{A \vdash e_1\, e_2 : t'} \text{ [App]}$$

$$\frac{A, x : t \vdash e : t'}{A \vdash \lambda x.e : t \rightarrow t'} \text{ [Abs]}$$

From a proof of $t \rightarrow t'$ and and a proof of t, we can prove t'.

If assuming t we can prove t', then we can prove $t \rightarrow t'$.

Here we regard the types as propositions: If we can prove certain propositions are true, then we can prove that other propositions are true.

But what are the proofs?

Programs as Proofs

$$\frac{A \vdash e_1 : t \rightarrow t' \qquad A \vdash e_2 : t}{A \vdash e_1\, e_2 : t'} \text{ [App]}$$

$$\frac{A, x : t \vdash e : t'}{A \vdash \lambda x.e : t \rightarrow t'} \text{ [Abs]}$$

From a proof of $t \rightarrow t'$ and and a proof of t, we can prove t'.

If assuming t we can prove t', then we can prove $t \rightarrow t'$.

Answer: The programs! $e: t$ is a proof that there is a program of type t.

The Curry-Howard Isomorphism

- There is a isomorphism between programs/types and proofs/propositions.

- Two interpretations of $\vdash e : t$

- We have a proof that the program e has type t
  - $\rightarrow$ is a constructor for function types

- e is a proof of t
  - $\rightarrow$ is logical implication

Discussion

- This seems interesting ... but is it useful?
- Not so far
- If we use more expressive types, we can express more propositions.
- We need more than implication!

Propositional Logic

- As an example, we show how to define the rest of propositional logic

- This is just one of many theories we could define
  - But a particularly useful one

- We will define:
  - And
  - Or
  - Not

And

$$\frac{A \vdash e : t_1 \wedge t_2}{A \vdash \; ? \; : t_1} \text{ [And-Elim-Left]}$$

$$\frac{A \vdash e_1 : t_1 \qquad A \vdash e_2 : t_2}{A \vdash \; ? : t_1 \wedge t_2} \text{ [And-Intro]}$$

$$\frac{A \vdash e : t_1 \wedge t_2}{A \vdash \; ? \; : t_2} \text{ [And-Elim-Right]}$$

What program is a proof of $t_1 \wedge t_2$?

Pairs

$$\frac{A \vdash e_1 : t_1 \qquad A \vdash e_2 : t_2}{A \vdash (e_1, e_2) : t_1 \wedge t_2} \text{ [And-Intro]}$$

$$\frac{A \vdash e : t_1 \wedge t_2}{A \vdash e.\text{left} : t_1} \text{ [And-Elim-Left]}$$

$$\frac{A \vdash e : t_1 \wedge t_2}{A \vdash e.\text{right} : t_2} \text{ [And-Elim-Right]}$$

Or

$$\frac{A \vdash e : t_1}{A \vdash e : t_1 \vee t_2} \quad \text{[Or-Intro-Left]}$$

$$\frac{A \vdash e : t_2}{A \vdash e : t_1 \vee t_2} \quad \text{[Or-Intro-Right]}$$

$$\frac{A \vdash e : t_1 \vee t_2}{A \vdash\ ?\ :\ ?} \quad \text{[Or-Elim]}$$

Hmmmm …

- The Or-Elim rule isn't obvious

- We need to exhibit a program that works regardless of whether e is an element of t_1 or t_2.

- Solution
  - The elimination is done by another program that does a case analysis

Or Elimination

$$\frac{A \vdash e_0 : t_1 \vee t_2 \qquad A, x{:}\ t_1 \vdash e_1 : t_0 \quad A, x{:}\ t_2 \vdash e_2 : t_0}{A \vdash (\lambda x.\ \text{case } x \text{ of } t_1 \text{ -> } e_1;\ t_2 \text{ -> } e_2)\ e_0 : t_0}\ [\text{Or-Elim}]$$

Discussion

- Using a case analysis makes sense to computer scientists
  - Do one thing if the list is Nil / n = 0
  - Do something else if the list has at least one element/ n > 0
- But this is not the "or" of classical logic
  - In *constructive* logic, we must construct evidence for everything we prove
  - To use a disjunction, we must know which case we are in
- A dual explanation
  - To create a disjunction, we must compute a value of one of the types
- Thus $t \vee \neg t$ is not an axiom of this system!
  - And this is the only classical axiom that must be excluded

Negation

- $\neg p$ is defined as $p \rightarrow false$
  - Proposition p implies a contradiction

- False is the empty type – there is no evidence for false

- Thus $\neg p$ either does not have any elements, or only non-terminating functions
  - Depending on what else is included in the theory we are using

What is Negation Good For?

- There can be uses for negation
- If we are just interested in proving things, proof by contradiction is an important technique
  - Recall one goal is to formalize mathematics
- But there are also computational interpretations

Type Theory for Continuations (Sketch)

Recall $\neg p = p \rightarrow \text{false}$

In pure lambda calculus, a function of type $\neg p$ can't be called

- Because false has no elements in its type

- But in a language with continuations:
  - Recall that a continuation has the form $\lambda v.e$ *and does not return when called*
  - So it is sensible to give continuations a type $p \rightarrow \text{false} = \neg p$

Constructive vs. Classical Logic

- Constructive logic gives us programs we can run
- Type theory can also have classical axioms
  - What axioms are used is not the distinguishing feature of type theory
  - But if we use classical logic, we also lose the ability to use the proofs as programs, as they are no longer constructive
- In applications to software, we are generally interested in constructive proofs

Summary

- We have shown how to define propositional logic in type theory
  - Give sensible type rules for and, or and not
  - Show how to construct programs that have the postulated types

- Example: We can prove $(a \to b) \to (a \to c) \to (a \to b \wedge c)$

Taking It to the Next Level

- We want to be able to define new kinds of theories within the system
- and, or, & not should definable within the system
- The type checking rules should also be definable

Boolean Connectives Revisited

- What are and, or and not?
- They are functions that take types and construct new types
- Introduce a new type Type that contains all types
  - Type = { Int, Bool, Int → Int, ... }
- and: Type → Type → Type
- or: Type → Type → Type
- not: Type → Type

Inference Rules Revisited

- An inference rule is a function that takes proofs of propositions as arguments and produces a proof of a proposition as a result

- Define a new type Proof

- And-Intro: Proof → Proof → Proof
- And-Elim-Left: Proof → Proof
- And-Elim-Right: Proof → Proof

Review

So now we can:

- Define new types
- Define new type combinators (and, or, not ...)
- Define new inference rules (and-intro, ...)

- All using a uniform system based on types
- Note the system also checks type functions and inference rules are correctly used
  - E.g., we can only build valid proofs

Are We Done?

- Not yet

- There are three more important features of type theories:
  - Type stratification
  - Inductively defined data types
  - Pi types

Type Stratification

- Recall we ``Introduce a new type Type that contains all types''
  - Type = { Int, Bool, Int → Int, ... }

- So is Type ∈ Type ?

And Now ... A Little Set Theory

- Recall in the early 20^{th} century there was a systematic effort to understand the foundations of logic
  - As part of the goal of formalizing mathematics
- *Set theory* was recognized as a potential foundation

Why Set Theory?

- A function f can be represented as a set of (input,output) pairs:

$$\{(x_i,y_i) \mid f(x_i) = y_i\}$$

- Natural numbers:

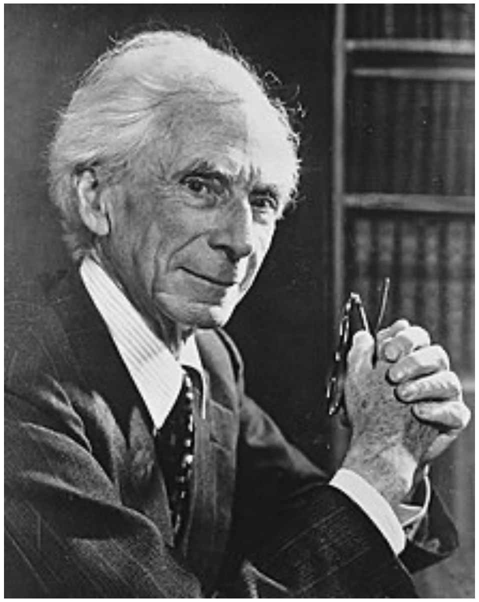
  $0 \cong \emptyset$

  $\mathrm{Succ}(n) \cong n \cup \{n\}$

- And so on ...

Russell's Paradox

Consider $R = \{ x \mid x \notin x \}$



Now we can easily show:

$$R \notin R \Rightarrow R \in R$$
$$R \in R \Rightarrow R \notin R$$

So we conclude:

$$R \in R \Leftrightarrow R \notin R$$

Implications

- Russell's paradox showed naïve set theory is inconsistent
  - Can prove ``false is true'' and so can prove anything
  - Not a great foundation for mathematics!
- Led to a reconsideration of the foundations of set theory
  - Over a couple of decades
- One conclusion: No set could be an element of itself
  - Set theory should be *well-founded*

What Does Well-Founded Mean?

- There is no set of all sets
- Instead, there is an infinite hierarchy of stratified sets

- We define ``small'' sets at stratum 0
- The set of all level 0 sets is a stratum 1 set
- The set of all level 1 sets is a stratum 2 set
- ...

- In this way no set can be an element of itself
  - Stratum *n* sets can only contain small sets of stratum *n* and sets of strata less than *n*
- Similar to the definition of ordinals

Back To Types ...

- Recall that types are sets
  - So Russell's paradox applies to types as well
- Implies we will need a type hierarchy
  - In a consistent type system
  - The set of all types lives at a higher level in the hierarchy than ordinary types

Ordinary Types

$0 : Int$

$succ : Int \rightarrow Int$

$add: Int \rightarrow Int \rightarrow Int$

$true: Bool$

$false: Bool$

$and: Bool \rightarrow Bool \rightarrow Bool$

Next Level …

- What are Int, Bool, $\alpha \rightarrow \beta$, …?

- They are types
  - Int : Type
  - Bool: Type
  - Int → Int: Type

- Int, Bool, etc. are at level 0 of the type hierarchy
- Type is at level 1

Next Level ...

- What are $\rightarrow$ and and?

- They are functions of types that produce types
  - $\rightarrow$: Type $\rightarrow$ Type $\rightarrow$ Type
  - and: Type $\rightarrow$ Type $\rightarrow$ Type

- These are functions that operate on elements of type level 1

Inductively Defined Data Types

- Dependent type theories generally include inductively defined data types as a primitive concept
  - So users can define natural numbers, lists, trees, etc.
  - With constructors of the appropriate types

- We have already talked about how to represent inductively defined data types as lambda terms in previous lectures.
  - Nothing new here ...

Pi Types

- What we have discussed so far is still missing an important feature

- We can't express type functions that depend on their arguments

- Example cons: $\alpha \rightarrow \text{List}(\alpha) \rightarrow \text{List}(\alpha)$
  - What is the type of cons?
  - Explanation 1: cons has a family of types indexed by a parameter α
  - Explanation 2: cons has many types, one for each α
    - a product or intersection of an infinite set of types

Pi Types

Defining the List data type :

List: Type $\rightarrow$ Type

Cons: $\Pi\ \alpha$: Type. $\alpha \rightarrow$ List(α) $\rightarrow$ List(α)

Nil: $\Pi\ \alpha$: Type. List(α)

Polymorphic types are an example of *dependent types:* The type depends on a parameter. Note how Π functions like $\forall$.

There is also a corresponding sum type Σ that functions like $\exists$

Pi Types

The parameter in a Pi type doesn't have to range over Type.

A polymorphic array that includes its length in the type:

Array: Type → Int → Type
mkarray: $\Pi\ \alpha$: Type. $\Pi\ \beta$: Int. $\alpha \rightarrow \beta \rightarrow$ Array(α, β)

Here β is an integer – which could be any expression of type Int!

Discussion

- Without Pi types, type theory is very limited
  - E.g., simply typed lambda calculus
- Pi types are extremely powerful
  - The construct for creating infinite families of types
  - The signature feature of dependent type theories
  - Play a somewhat similar role to set comprehension in set theory
- Dependent type systems are often undecidable
  - Performing computation as part of type checking is bound to quickly run into computability issues!

Type Theory

- A foundation for all mathematics
  - Especially constructive mathematics
  - Sufficiently powerful to prove anything we can think of proving
  - And thus also a foundation for verifying the correctness of software

- Key features
  - Isomorphism of programs/types with proofs/propositions
  - Type hierarchy allows uniform definition of types, type operations, proofs, ...
  - Dependent types allow very expressive (even to the point of undecidability) types to be constructed

Type Theory in the Real World

- Type theory has been used to verify the correctness of real systems

- CompCert
  - A formally verified (subset of) C compiler

- Sel4
  - A formally verified OS microkernel
  - Has many but not all features of a real OS

State of Practice

- Compcert and Sel4 show that formal verification of significant systems using type theory-based proof assistants is possible

- Compcert and Sel4 have very high levels of assurance
  - Debugging is not an issue
  - Guaranteed, for example, to be extremely secure

- But Compcert and Sel4 have shown the software engineering costs of full formal verification are still high
  - Sel4 has over 1M lines of proofs
  - Modifications may require much more reproving than recoding

- The biggest barrier for most systems, though, is having the specification
  - To use a theorem prover, you first have to state a theorem to prove!