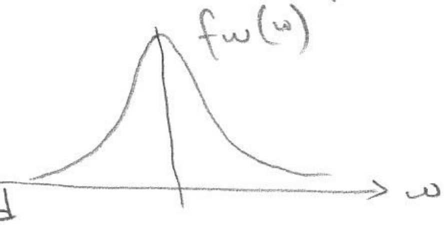
Gaussian random variable

Standard Gaussian r.v: $W \sim N(0,1)$ if

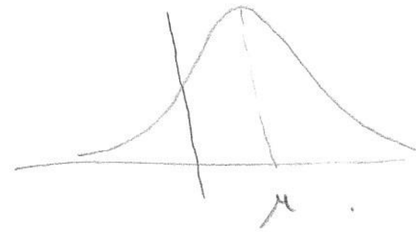
$$f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \quad w \in \mathbb{R}$$



Gaussian random variable: X is called a Gaussian r.v if there exist $\sigma \in \mathbb{R}$ and $\mu \in \mathbb{R}$ s.t $X = \sigma W + \mu$ where $W \sim N(0,1)$.

$$f_X(x) = \frac{1}{\sigma} f_W\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$



$$X \sim N(\mu, \sigma^2)$$

Gaussian random vector:

Standard Gaussian random vector: A random vector $\overline{W}$

$\overline{W} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix}$ is called a standard Gaussian random vector if its entries $W_1, \dots W_n \sim N(0,1)$ and independent

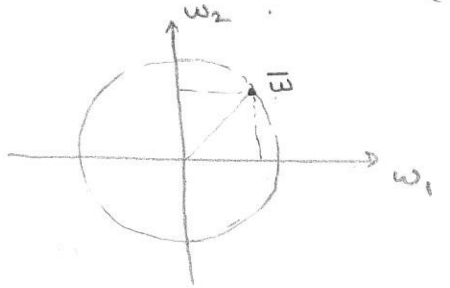
$$\overline{W} \sim N(0, I_n)$$

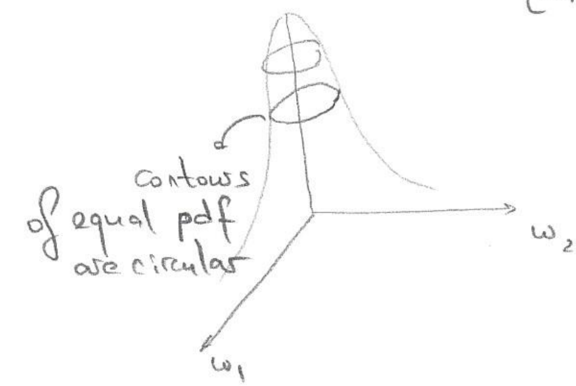
$$f_{\overline{W}}(\overline{w}) = f_{W_1, W_2 \dots W_n}(w_1, \dots w_n) = f_{W_1}(w_1) \dots f_{W_n}(w_n)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-w_1^2/2} \dots \frac{1}{\sqrt{2\pi}} e^{-w_n^2/2}$$

$$= \frac{1}{(\sqrt{2\pi})^n} e^{-\sum_{i=1}^{n} w_i^2/2}$$

$$= \frac{1}{(\sqrt{2\pi})^n} e^{-\|\bar{\omega}\|^2/2} . \qquad \bar{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}$$





<u>General Gaussian random vector:</u>

<u>Def:</u> A random vector $\bar{X} \in \mathbb{R}^n$ is called a Gaussian random vector (or equivalently its entries $X_1, X_2, \dots X_n$ are called jointly Gaussian) iff there exists a matrix $A \in \mathbb{R}^{n \times m}$ and a vector $\bar{\mu} \in \mathbb{R}^n$ s.t $\bar{X}$ can be expressed as

$$\bar{X} = A\bar{W} + \bar{\mu} \quad \text{where } \bar{W} \sim N(0, I_m) .$$

Observations: $\mathbb{E}[\bar{X}] = \bar{\mu}$. $K_{\bar{X}} = AA^T$.

$X = \sigma W + \mu$.

<u>Fact 0:</u> $$f_X(x) = \frac{1}{|\sigma|} f_W\left(\frac{x-\mu}{\sigma}\right)$$

$$f_{\bar{X}}(\bar{x}) = \frac{1}{|\det(A)|} f_{\bar{W}}\left(A^{-1}(\bar{x} - \bar{\mu})\right) \quad \text{if } \underline{A \text{ is invertible and } A \in \mathbb{R}^{n \times n}} .$$

$$f_{\bar{X}}(\bar{x}) = \frac{1}{|\det(A)|} \frac{1}{(\sqrt{2\pi})^n} e^{-\|A^{-1}(\bar{x} - \bar{\mu})\|^2/2} .$$

Facts from linear algebra. $\det(AB) = \det(A)\det(B)$

$$\det(K_{\bar{X}}) = \det(AA^T) = (\det(A))^2 .$$

$$\det(A) = \sqrt{\det(K_{\bar{X}})} .$$

$$\|A^{-1}(\bar{x}-\bar{\mu})\|^2 \doteq (A^{-1}(\bar{x}-\bar{\mu}))^T (A^{-1}(\bar{x}-\bar{\mu})).$$
$$= (\bar{x}-\bar{\mu})^T (A^{-1})^T A^{-1} (\bar{x}-\bar{\mu}).$$

$$(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = K_{\bar{X}}^{-1}.$$

Fact: $(AB)^{-1} = B^{-1}A^{-1}$. assuming B^{-1} and A^{-1} exist.

$$= (\bar{x}-\bar{\mu})^T K_{\bar{X}}^{-1} (\bar{x}-\bar{\mu}).$$

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n \det(K_{\bar{X}})}} e^{-(\bar{x}-\bar{\mu})^T K_X^{-1} (\bar{x}-\bar{\mu})} \circledast$$

<u>Observation:</u> The density of a Gaussian random vector $\bar{X}$ is fully specified by its mean and covariance matrix.

$$\bar{X} \sim \mathcal{N}(\bar{\mu}, K_{\bar{X}}).$$

<u>Fact 1:</u> $\bar{X}$ is a Gaussian random vector, then each $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ is Gaussian.

<u>Proof:</u> $\bar{X} = A\bar{W} + \bar{\mu}$. $A \in \mathbb{R}^{n\times m}$ $\bar{W} = \begin{bmatrix} w_1 \\ w_m \end{bmatrix}$

$$X_i = a_{i1}W_1 + a_{i2}W_2 + \dots a_{im}W_m + \mu_i.$$

Moment Generating Function.

$$M_{X_i}(s) = \mathbb{E}\left[e^{sX_i}\right]$$
$$= \mathbb{E}\left[e^{s(a_{i1}W_1 + \dots a_{im}W_m + \mu_i)}\right]$$
$$= \mathbb{E}\left[e^{sa_{i1}W_1} e^{sa_{i2}W_2} \dots e^{sa_{im}W_m} e^{s\mu_i}\right].$$

$$= \mathbb{E}\left[e^{s a_{i1} W_1}\right] \mathbb{E}\left[e^{s a_{i2} W_2}\right] \cdots \mathbb{E}\left[e^{s a_{in} W_n}\right] e^{s\mu_i}.$$

If $W \sim \mathcal{N}(0,1)$ $\quad \mathbb{E}\left[e^{sW}\right] = e^{s^2}$

$$= e^{s^2 a_{i1}^2} \, e^{s^2 a_{i2}^2} \cdots e^{s^2 a_{in}^2} \, e^{s\mu_i}$$

$$= e^{s\mu_i + s^2 \underbrace{(a_{i1}^2 + a_{i2}^2 + \dots a_{in}^2)}_{= \sigma_i^2 = K_{\bar{X}, ii}}} \qquad K_{\bar{X}} = AA^T.$$

$$\Rightarrow X_i \sim \mathcal{N}(\mu_i, K_{\bar{X}, ii})$$

<u>Fact 2:</u> If $\bar{X}$ is a Gaussian random vector and

$$\bar{y} = B\bar{X} + \bar{c}$$

then $\bar{y}$ is also a Gaussian random vector.

$$\bar{X} = A\bar{W} + \bar{\mu}$$

$$\bar{y} = B(A\bar{W} + \bar{\mu}) + \bar{c} = (BA)\bar{W} + (B\bar{\mu} + \bar{c})$$

$\Rightarrow$ $\bar{y}$ is a Gaussian random vector.

<u>Corollary:</u> If $X_1, X_2, \dots X_n$ are <u>jointly Gaussian</u> random variables and

$$y = b_{11} X_1 + b_{12} X_2 + \dots b_{1n} X_n + c = \bar{b}^T \bar{X} + c$$

then y is Gaussian, i.e. $y \sim \mathcal{N}(\mu, \sigma^2)$. $\quad \bar{b} = \begin{bmatrix} b_{11} \\ \vdots \\ b_{1n} \end{bmatrix}$

s.t. $\mu = b_{11}\mathbb{E}[X_1] + b_{12}\mathbb{E}[X_2] + \dots b_{1n}\mathbb{E}[X_n] + c$

$$\sigma^2 = \bar{b}^T K_{\bar{X}} \bar{b}.$$

__Remark:__ If X_1 and X_2 are marginally Gaussian then $X_1 + X_2$ is not always Gaussian.

__Fact 3:__ If $K_{\bar{X}} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \; \sigma_n^2 \end{bmatrix}$, i.e. diagonal and $\bar{X}$ is a Gaussian vector, then $X_1, X_2, \ldots X_n$ are independent.

In words, if $\bar{X}$ is a Gaussian random vector and its entries are uncorrelated then they are independent.

Equivalent, if $X_1, \ldots X_n$ are jointly Gaussian and uncorrelated, then they are independent.

$$f_{\bar{X}}(\bar{x}) = f_{X_1, \ldots X_n}(x_1, \ldots x_n)$$

$$= \frac{1}{(\sqrt{2\pi})^n} \frac{1}{\sqrt{\det(K_X)}} e^{-(\bar{x}-\bar{\mu})^T K_X^{-1} (\bar{x}-\bar{\mu})/2}.$$

$$K_{\bar{X}} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} \qquad K_{\bar{X}}^{-1} = \begin{bmatrix} 1/\sigma_1^2 & & 0 \\ 0 & 1/\sigma_2^2 & \\ & & 1/\sigma_n^2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}\sqrt{2\pi\sigma_2^2}\cdots\sqrt{2\pi\sigma_n^2}} e^{-\sum_{i=1}^{n} \frac{1}{\sigma_i^2}(x_i-\mu_i)^2/2}.$$

$$= \quad e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} \cdots e^{-\frac{(x_n-\mu_n)^2}{2\sigma_n^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-(x_1-\mu_1)^2/2\sigma_1^2} \cdots \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-(x_n-\mu_n)^2/2\sigma_n^2}$$

$$= f_{X_1}(x_1) \cdots f_{X_n}(x_n).$$

Fact 4: If $\overline{X}$ is Gaussian with mean $\overline{\mu}$ and covariance matrix $K_{\overline{X}}$ and $K_{\overline{X}}$ is invertible, then $\overline{X}$ has a density and its density is given by (*)

Proof: $K_{\overline{X}} = SS^T$

If $\det(K_{\overline{X}}) \neq 0$, i.e $K_{\overline{X}}$ is invertible, then $\det(S) \neq 0$, i.e S is also invertible

Let $\overline{Z} = S\overline{W} + \overline{\mu}$, where S is invertible.

Then $\overline{Z}$ is a Gaussian random vector with mean $\overline{\mu}$ and covariance matrix $K_{\overline{X}}$, and we know from fact 0 that it has a density given by (*) (Fact 0 applies because S is invertible)

But if $\overline{X}$ and $\overline{Z}$ are both Gaussian with the same mean and same covariance matrix: then they should have the same distribution/density. This follows from HW5 - problem 3, where you will show that they have the same MGF.