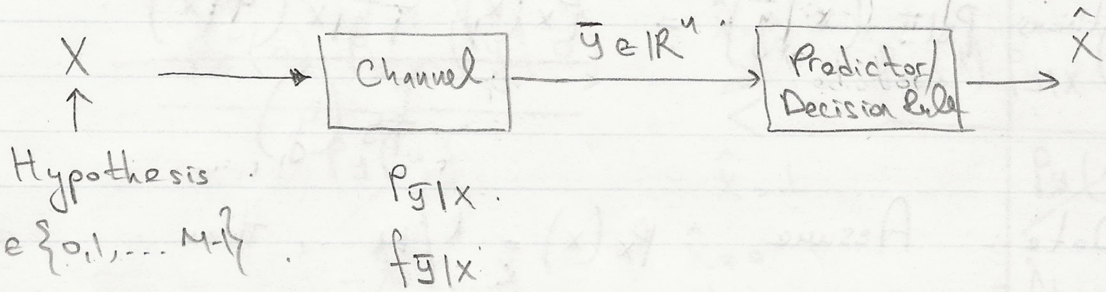
Lecture 12.

Hypothesis Testing (a.k.a. detection, decision making, decoding, classification etc.)



Probabilistic Model: $p_X(i) = P(X = i)$ for $i \in \{0,1,\dots M-1\}$

prior distribution.

$p_{\bar{y}|X}$ or $f_{\bar{y}|X}$ channel Likelihoods.

Goal: Find a decision rule $g : \mathbb{R}^n \to \{0,1,\dots M-1\}$

$\hat{x} = g(\bar{y})$: guess provided by g when $\bar{Y} = \bar{y}$

so that $\underbrace{P(X \neq g(\bar{Y}))}_{\text{probability of error, } P_e} = P(X \neq \hat{X})$ is minimized

<u>MAP Rule:</u> $\hat{x} = g_{MAP}(y)$

$= \underset{i}{\operatorname{argmax}} \; p_{X|\bar{Y}}(i|\bar{y})$. where small $\bar{y}$ is our observati

Fact: The MAP Rule maximizes the probability of correct decision, equivalently minimize P_e

$\bar{y}$ is discrete. $p_{X|\bar{y}}(x|\bar{y}) = \dfrac{p_{\bar{y}|X}(\bar{y}|x)\, p_X(x)}{p_{\bar{y}}(\bar{y})}$ Bayes Rule

$\bar{y}$ is continuous $p_{X|\bar{y}}(x|\bar{y}) = \dfrac{p_X(x)\, f_{\bar{y}|X}(\bar{y}|x)}{f_{\bar{y}}(\bar{y})}$

Note: Assume $p_X(x) = 1/M$, then

$$g_{MAP}(\bar{y}) = \underset{i\in\{0,1,..M-1\}}{\arg\max}\; p_{X|\bar{y}}(i|\bar{y})$$

$$= \underset{i\in\{0,1,..M-1\}}{\arg\max}\; \frac{\overbrace{p_X(i)}^{1/M}\, f_{\bar{y}|X}(\bar{y}|i)}{f_{\bar{y}}(\bar{y})}$$

$$= \underset{i\in\{0,1,..M-1\}}{\arg\max}\; f_{\bar{y}|X}(\bar{y}|i).$$

ML Rule: (Maximum Likelihood) $g_{ML}(\bar{y}) = \underset{i\in\{0,1,..M-1\}}{\arg\max}\; f_{\bar{y}|X}(\bar{y}|i).$

Example 1: Optical communication channel

$X \in \{0,1\}$ $p_X(0) = p_0$ $p_X(1) = p_1$. $0 \le \lambda_0 \le \lambda_1$

If $X=1$, we switch a LED on, $p_{y|X}(y|1) = e^{-\lambda_1} \lambda_1^{y} / y!$, $y \in \mathbb{N}$

If $X=0$, we switch the LED off, $p_{y|X}(y|0) = \dfrac{e^{-\lambda_0} \lambda_0^{y}}{y!}$

MAP Decision Rule: Let y be the # of photons we observe

$$p_{X|Y}(0|y) \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} p_X(1|y)$$

$$\frac{p_X(0)\, p_{Y|X}(y|0)}{\cancel{p_Y(y)}} \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} \frac{p_X(1)\, p_{Y|X}(y|1)}{\cancel{p_Y(y)}}$$

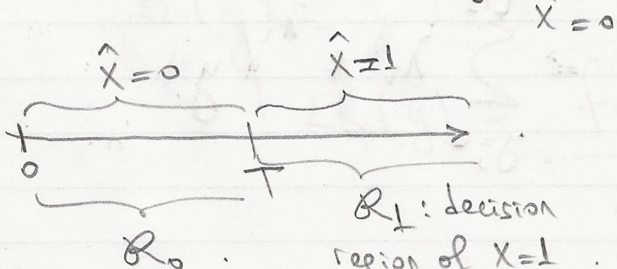
$$\frac{p_0 \cdot \cancel{e^{-\lambda_0}} \dfrac{\lambda_0^{y}}{\cancel{y!}}}{\cancel{p_0 e^{-\lambda_0}}\, \lambda_1^{y}} \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} \frac{p_1\, e^{-\lambda_1} \dfrac{\lambda_1^{y}}{\cancel{y!}}}{p_0\, e^{-\lambda_0}\, \cancel{\lambda_1^{y}}}$$

$$\left(\frac{\lambda_0}{\lambda_1}\right)^{y} \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} \frac{p_1}{p_0}\, e^{\lambda_0-\lambda_1}$$

$$y \log\left(\frac{\lambda_0}{\lambda_1}\right) \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} (\lambda_0-\lambda_1) + \log p_1/p_0$$

$$y \underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} \frac{(\lambda_0-\lambda_1) + \log p_1/p_0}{\log(\lambda_0/\lambda_1)}$$

Threshold rule $\quad y \underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} \dfrac{(\lambda_1-\lambda_0) + \log p_0/p_1}{\log(\lambda_1/\lambda_0)}$ $\quad T$



if $p_0 = p_1 = 1/2$ $\quad T_{1/2} = \dfrac{\lambda_1 + \lambda_0}{\log \lambda_1/\lambda_0}$.

if $p_1 > p_0$. $\quad T \nearrow$. $\quad$ if $p_1 \to 1$.

if $p_0 < p_1$ $\quad T \searrow$.

if $\lambda_1 = \lambda_0$. $\quad$ if $p_1 > p_0$. $\quad T = -\infty$.

if $p_1 < p_0$ $\quad T = +\infty$.

$$P_e = P(\hat{X} \neq X) = P(\{X=0, \hat{X}=1\} \cup \{X=1, \hat{X}=0\})$$
$$= P(X=0, \hat{X}=1) + P(X=1, \hat{X}=0)$$
$$= \underbrace{P(X=0)}_{p_0} P(\hat{X}=1 \mid X=0) + \underbrace{P(X=1)}_{p_1} P(\hat{X}=0 \mid X=1).$$

$$P(\hat{X}=1 \mid X=0) = P(y \geq T \mid X=0)$$
$$= \sum_{y=\lceil T \rceil}^{\infty} p_{y|x}(y \mid 0)$$
$$= \sum_{y \geq T}^{\infty} \lambda_0^{y} e^{-\lambda_0} / y!$$

$$P(\hat{X}=0 \mid X=1) = P(y < T \mid X=1).$$
$$= \sum_{y=0}^{\lfloor T \rfloor} \lambda_1^{y} e^{-\lambda_1} / y!.$$

Example 2: Communicate 1-bit over a Gaussian model.

$$b \geq a.$$

$$X \in \{a,b\} \longrightarrow \oplus \longrightarrow y = X + Z \qquad Z \sim \mathcal{N}(0,\sigma^2)$$

independent of X

$$\uparrow$$

Z: Noise

$P_X(0) = P_0 \quad P_X(1) = P_1$

If $B=1$, TX sends $b \in \mathbb{R}$ $\quad y \sim \mathcal{N}(b,\sigma^2)$.

$$f_{Y|X}(y|1) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(y-b)^2/2\sigma^2}$$

If $B=0$, TX sends $a \in \mathbb{R}$, $\quad y \sim \mathcal{N}(a,\sigma^2)$

$$f_{Y|X}(y|0) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(y-a)^2/2\sigma^2}$$

MAP Rule:

$$p_{X|Y}(b|y) \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} p_{X|Y}(a|y).$$

$$\frac{P_X(b)\, f_{Y|X}(y|b)}{\cancel{f_Y(y)}} \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \frac{P_X(a)\, f_{Y|X}(y|a)}{\cancel{f_Y(y)}}$$

Likelihood Ratio.

$$\underbrace{\frac{f_{Y|X}(y|b)}{f_{Y|X}(y|a)}}_{LR(y)} \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \frac{P_X(0)}{P_X(1)}$$

$$LR(y) \triangleq \frac{f_{Y|X}(y|b)}{f_{Y|X}(y|a)}$$

$$\frac{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}}\, e^{-(y-b)^2/2\sigma^2}}{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}}\, e^{-(y-a)^2/2\sigma^2}} \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \frac{p_0}{p_1}.$$

$$e^{-(y-b)^2/2\sigma^2 + \frac{(y-a)^2}{2\sigma^2}} \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \frac{p_0}{p_1}$$

$$-(y-b)^2/2\sigma^2 + \frac{(y-a)^2}{2\sigma^2} \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \log p_0/p_1$$

$$\frac{\cancel{-y^2} + 2yb - b^2 + \cancel{y^2} - 2ya + a^2}{2\sigma^2} \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \log p_0/p_1$$

$$\frac{2y(b-a)}{2\sigma^2} + \frac{a^2-b^2}{2\sigma^2} \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \log p_0/p_1$$

$$\frac{\cancel{2}y(b-a)}{\cancel{2}\sigma^2} \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} + \frac{b^2-a^2}{2\sigma^2} + \log p_0/p_1.$$

$$y \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \frac{b^2-a^2}{2(b-a)} + \frac{\sigma^2 \log p_0/p_1}{(b-a)}$$

$$y \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \frac{b+a}{2} + \frac{\sigma^2 \log p_0/p_1}{b-a}.$$

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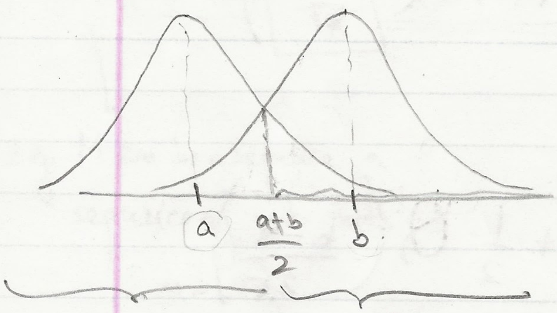
Observations:

$p_0 = p_1 = 1/2$

$$y \underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} \frac{b+a}{2}.$$



$\mathcal{R}_0$ $\hat{X}=0$. $\mathcal{R}_1$ $\hat{X}=1$

$$\boxed{|y-a| \underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} |y-b|.}$$

Minimum distance decoding

$p_0 > p_1 \quad \theta \nearrow. \qquad p_0 = 1 \quad \theta = \infty.$

$p_1 < p_0 \quad \theta \searrow.$

For a fixed $p_0 > p_1$ if $\sigma^2 \nearrow$ $\theta \nearrow +\infty$.

$p_0 < p_1$ if $\sigma^2 \nearrow$ $\theta \searrow -\infty$

if $\sigma^2 \searrow 0$ then $\theta \to \frac{b+a}{2}$.

$$P_e = P(\hat{B} \neq B) = p_0 P(\hat{B}=1 | B=0) + p_1 P(\hat{B}=0 | B=1)$$

$$\begin{aligned} P(\hat{B}=1 | B=0) &= P(y \geq \theta | X=a). \\ &= P(a+z \geq \theta | X=a) \\ &= P(a+z \geq \theta) \\ &= P(z \geq \theta - a) \\ &= Q\left(\frac{\theta - a}{\sigma}\right). \end{aligned}$$

$$P(\hat{X}=1 \mid X=0) = Q\left(\frac{b-\theta}{\sigma}\right)$$

$$P_e = P_0\, Q\left(\frac{\theta-a}{\sigma}\right) + P_1\, Q\left(\frac{b-\theta}{\sigma}\right)$$

$P_0 = P_1 = \frac{1}{2}$ $\quad \theta = \frac{a+b}{2}$

$$P_e = \frac{1}{2} Q\left(\frac{b-a}{2\sigma}\right) + \frac{1}{2} Q\left(\frac{b-a}{2\sigma}\right)$$

$$= Q\left(\frac{b-a}{2\sigma}\right) \qquad b-a = d$$

$$= Q\left(\frac{d}{2\sigma}\right) \qquad \frac{d}{\sigma} : \text{Signal to noise ratio}$$