Hypothesis testing:

$$X \in \{0,1,\dots M-1\} \longrightarrow \boxed{\text{Channel}} \xrightarrow{\;y\;} \boxed{\text{Decision rule}} \longrightarrow \hat{X}$$

$P_X(x)$ $\qquad$ $P_{Y|X}$, $f_{Y|X}$

Optimal Decision Rule:

MAP Rule: $\hat{X} = \underset{i \in \{0,1,\dots M-1\}}{\arg\max}\; P_{X|Y}(x|y)$

minimizes the prob of error $P_e = P\{\hat{X} = X\}$.

Example: Communicating 1-bit over the additive Gaussian channel

$$X \in \{a,b\} \longrightarrow \oplus \longrightarrow y = X + Z$$

Z: Noise

$Z \sim N(0, \sigma^2)$ independent of X.

If $B = 1$, TX sends $b \in \mathbb{R}$ $\quad (P(B=1) = p_1)$

$$f_{Y|X}(y|1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-b)^2/2\sigma^2}$$

If $B = 0$, TX sends $a \in \mathbb{R}$ $\quad (P(B=0) = p_0)$

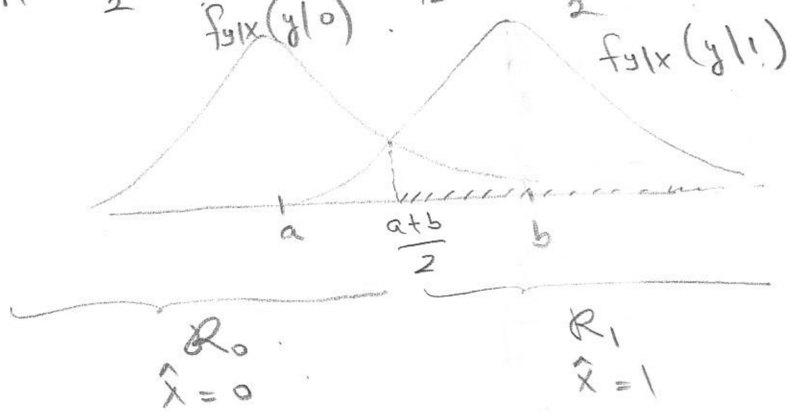
$$f_{Y|X}(y|0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-a)^2/2\sigma^2}$$

MAP Rule

$$y \underset{\hat{B}=0}{\overset{\hat{B}=1}{\gtrless}} \frac{b+a}{2} + \frac{\sigma^2 \log p_0/p_1}{b-a} = \theta$$

$p_0 = p_1 = \frac{1}{2}$ $\quad \theta_{1/2} = \frac{b+a}{2}$

ML Rule.



$p_0 > p_1 \qquad \theta > \theta_{1/2}$

$p_1 < p_0 \qquad \theta < \theta_{1/2}$

For a fixed $p_0 > p_1$ if $\sigma^2 \nearrow$. $\theta \nearrow +\infty$.

$p_0 < p_1$ if $\sigma^2 \nearrow$. $\theta \searrow -\infty$

if $\sigma^2 \to 0 \qquad \theta \to \theta_{1/2}$

$$P_e = P(\hat{B} \neq B) = p_0\, P(\hat{B} = 1 | B = 0) + p_1\, P(\hat{B} = 0 | B = 1)$$

$$\begin{aligned}
P(\hat{B} = 1 | B = 0) &= P(y \geq \theta | B = 0) \\
&= P(y \geq \theta | X = a) \\
&= P(a + z \geq \theta | X = a) \\
&= P(a + z \geq \theta) \\
&= P(z \geq \theta - a) \\
&= P\left(\frac{z}{\sigma} \geq \frac{\theta - a}{\sigma}\right) \\
&= Q\left(\frac{\theta - a}{\sigma}\right)
\end{aligned}$$

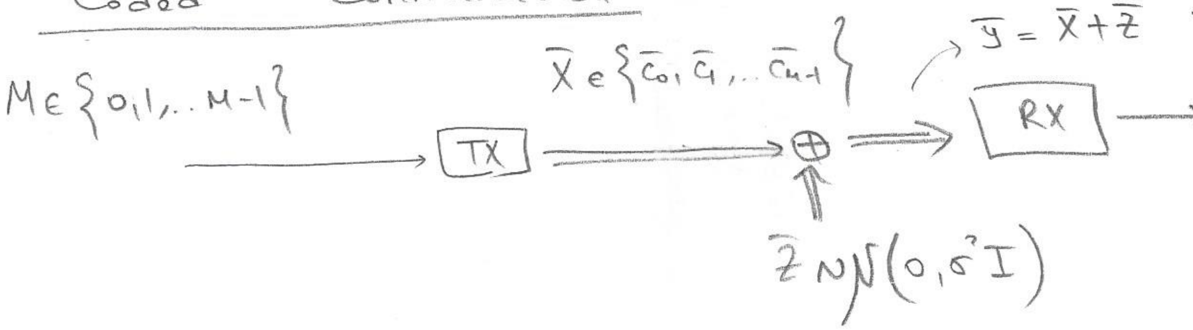
$$P(\hat{B} = 0 | B = 1) = P(y < \theta | X = b) = Q\left(\frac{b - \theta}{\sigma}\right)$$

$\frac{d}{\sigma}$: SNR

$p_0 = p_1 = \frac{1}{2} \qquad \theta = \frac{a+b}{2}$

$$P_e = \frac{1}{2} Q\left(\frac{b-a}{2\sigma}\right) + \frac{1}{2} Q\left(\frac{b-a}{2\sigma}\right) = Q\left(\frac{b-a}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

Coded Communication:



If the message $M = i$, TX sends $\bar{X} = \bar{c}_i \in \mathbb{R}^n$; the RX observes

$$\bar{y} = \bar{X} + \bar{Z} \qquad p(M=i) = p_i$$

or equivalently,

$$f_{\bar{y}|M}(\bar{y}|i) = \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n} e^{-\|\bar{y}-\bar{c}_i\|^2/2\sigma^2}$$

$$\hat{M} = \arg\max_i \; p_{M|\bar{y}}(i|\bar{y})$$

$$= \arg\max_i \; \frac{p_i f_{\bar{y}|M}(\bar{y}|i)}{\cancel{f_{\bar{y}}(\bar{y})}}$$

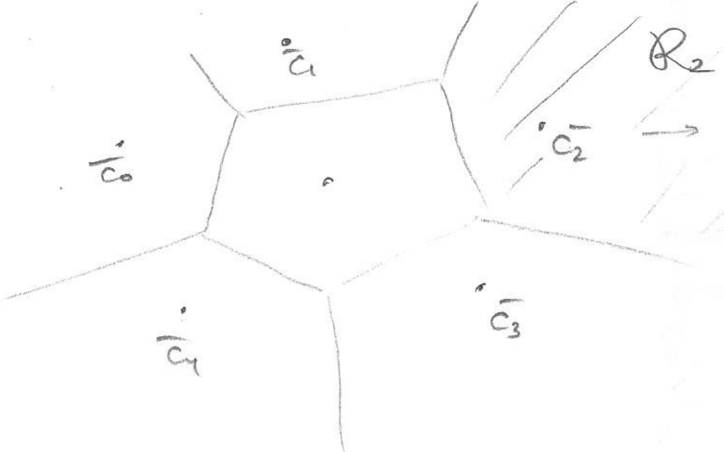
$$= \arg\max_i \; p_i \cancel{\frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n}} e^{-\|\bar{y}-\bar{c}_i\|^2/2\sigma^2}$$

$$= \arg\max_i \; -\frac{\|\bar{y}-\bar{c}_i\|^2}{2\sigma^2} + \log p_i$$

$$= \arg\min_i \; \|\bar{y}-\bar{c}_i\|^2 - 2\sigma^2 \log p_i$$

Let $p(M=i) = 1/M$

$$\hat{M} = \arg\min_i \; \|\bar{y}-\bar{c}_i\|^2 \qquad \text{Minimum distance decoding}$$



$\mathcal{R}_2$: Voronoi region of $\bar{c}_2$: The set of points in $\mathbb{R}^n$ that are closer to $\bar{c}_2$ than any other points in the constellation.

Binary case:

$M = 2$

$$\|\bar{y} - \bar{c}_0\|^2 \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} \|\bar{y} - \bar{c}_1\|^2$$

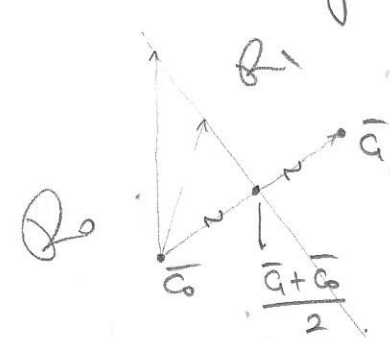
$$(\bar{y}-\bar{c}_0)^T(\bar{y}-\bar{c}_0) \gtrless (\bar{y}-\bar{c}_1)^T(\bar{y}-\bar{c}_1)$$

$$\cancel{\bar{y}^T\bar{y}} - \bar{y}^T\bar{c}_0 - \bar{c}_0^T\bar{y} + \bar{c}_0^T\bar{c}_0 \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} \cancel{\bar{y}^T\bar{y}} - \bar{y}^T\bar{c}_1 - \bar{c}_1^T\bar{y} + \bar{c}_1^T\bar{c}_1$$

$$-2\bar{y}^T\bar{c}_0 + 2\bar{y}^T c_1 \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} \|\bar{c}_1\|^2 - \|\bar{c}_0\|^2$$

$$\bar{y}^T(\bar{c}_1 - \bar{c}_0) \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} \frac{\|\bar{c}_1\|^2 - \|\bar{c}_0\|^2}{2}$$

$$\text{Boundary} = \left\{ \bar{y} \in \mathbb{R}^n : \bar{y}^T(\bar{c}_1 - \bar{c}_0) = \frac{\|\bar{c}_1\|^2 - \|\bar{c}_0\|^2}{2} \right\}$$



The boundary is a hyperplane perpendicular to $(\bar{c}_1 - \bar{c}_0)$ that passes through the midpoint $\frac{\bar{c}_0+\bar{c}_1}{2}$

Remark: Knowing $\bar{y}^T(\bar{c}_1 - \bar{c}_0)$ is sufficient to compute the MAP Rule.

$\Longrightarrow$ $\bar{y}^T(\bar{c}_1 - \bar{c}_0)$ is called sufficient statistic

$$P_e = p_0\, P\left(\hat{M}=0 \mid M=1\right) + p_1\, P\left(\hat{M}=1 \mid M=0\right). \qquad p_0 = p_1 = \frac{1}{2}$$

$$P\left(\hat{M}=0 \mid \acute{M}=1\right) = P\left(\bar{y}^T(\bar{c}_1-\bar{c}_0) < \frac{\|\bar{c}_1\|^2-\|\bar{c}_0\|^2}{2} \;\middle|\; M=1\right)$$

$$= P\left((\bar{c}_1+\bar{z})^T(\bar{c}_1-\bar{c}_0) < \frac{\|\bar{c}_1\|^2-\|\bar{c}_0\|^2}{2} \right)$$

$$= P\left(\bar{c}_1^T(\bar{c}_1-\bar{c}_0) + \bar{z}^T(\bar{c}_1-\bar{c}_0) < \frac{\|\bar{c}_1\|^2-\|\bar{c}_0\|^2}{2} \right)$$

$$= P\left(\bar{z}^T(\bar{c}_1-\bar{c}_0) < \frac{\|\bar{c}_1\|^2-\|\bar{c}_0\|^2 - 2\bar{c}_1^T(\bar{c}_1-\bar{c}_0)}{2} \right)$$

$$(\bar{c}_1-\bar{c}_0)^T(\bar{c}_1-\bar{c}_0) - 2\bar{c}_1^T(\bar{c}_1-\bar{c}_0) = \bar{c}_1^T(\bar{c}_1-\bar{c}_0) - \bar{c}_0^T(\bar{c}_1-\bar{c}_0) - 2\bar{c}_1^T(\bar{c}_1-\bar{c}_0)$$

$$= -\bar{c}_1^T(\bar{c}_1-\bar{c}_0) - \bar{c}_0^T(\bar{c}_1-\bar{c}_0)$$

$$= -(\bar{c}_1-\bar{c}_0)^T(\bar{c}_1-\bar{c}_0)$$

$$= -\|\bar{c}_1-\bar{c}_0\|^2$$

$$= P\left(\bar{z}^T \overset{\bar{v}}{\overset{\|}{\frac{(\bar{c}_1-\bar{c}_0)}{\sigma\|\bar{c}_1-\bar{c}_0\|}}} \le -\frac{\|\bar{c}_1-\bar{c}_0\|}{2\sigma} \right)$$

$$= W = \bar{z}^T\bar{v} = \bar{v}^T\bar{z}$$

$$W \sim \mathcal{N}\left(0, \bar{v}^T K_z \bar{v}\right) = \mathcal{N}(0,1)$$

$$= Q\left(\frac{\|\bar{c}_1-\bar{c}_0\|}{2\sigma}\right)$$

$$= Q\left(\frac{d}{2\sigma}\right)$$

$d = \|\bar{c}_1-\bar{c}_0\|$ distance between the two codewords.