Lecture 15: Estimation

$$X \rightarrow \boxed{\text{Channel}} \xrightarrow{y} \boxed{\text{Estimator}} \rightarrow \hat{X}$$

$$\underbrace{f_X(x)\ f_{Y|X}(y|x)}_{f_{X,Y}(x,y)}$$

MMSE Estimation: Design the estimator $g: Y \rightarrow X$ s.t $g(y) = \hat{x}$ that minimizes

$$\mathbb{E}\left[(x-\hat{x})^2\right]$$

Optimal Solution: $g(y) = \mathbb{E}[X|Y]$ MMSE Estimator

MMSE: $\mathbb{E}\left[\mathrm{Var}(X|Y)\right] = \int \mathrm{Var}(X|Y=y)\, f_Y(y)\, dy$

Example: X and Y are jointly Gaussian and zero-mean with covariance matrix

$$K_{\begin{bmatrix} X \\ Y \end{bmatrix}} = \begin{bmatrix} \sigma_X^2 & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \sigma_Y^2 \end{bmatrix}$$

We observe Y and want to estimate X.

$$\hat{X} = \mathbb{E}[X|Y] \qquad f_{X|Y}(x|y)$$

Last lecture: If X and Y are jointly Gaussian, we can always write

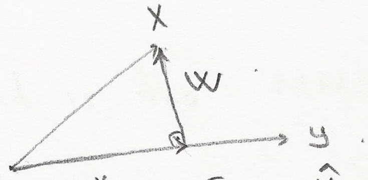
$$X = \alpha^* Y + W$$

If $\alpha^* = \frac{\mathrm{Cov}(X,Y)}{\sigma_Y^2} = \rho \frac{\sigma_X}{\sigma_Y}$ where $\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$ Correlation Coefficient

then W is independent of Y and $W \sim \mathcal{N}(0, \sigma_W^2)$

where $\sigma_w^2 = (1-\rho^2)\,\sigma_x^2$.

<u>Interpretation</u>:



$\alpha^* y = \rho \frac{\sigma_x}{\sigma_y} y = \hat{X}$

$$\sigma_x^2 = \rho^2 \frac{\sigma_x^2}{\sigma_y^2} \sigma_y^2 + \sigma_w^2$$

$$\sigma_w^2 = (1-\rho^2)\,\sigma_x^2 .$$

$$\hat{X} = \mathbb{E}[X|y] = \rho \frac{\sigma_x}{\sigma_y} y$$

$$\Rightarrow X|Y=y \sim N\left(\rho \frac{\sigma_x}{\sigma_y} y,\ (1-\rho^2)\,\sigma_x^2\right)$$

$$\mathbb{E}[X|Y=y] = \rho \frac{\sigma_x}{\sigma_y} y .$$

$$\begin{aligned} MSE &= \mathbb{E}\left[\mathrm{Var}(X|Y)\right] \\ &= \int \mathrm{Var}(X|Y=y)\, f_Y(y)\, dy \\ &= (1-\rho^2)\,\sigma_x^2 . \end{aligned}$$

Alternatively,

$$MSE = \mathbb{E}\left[(X-\hat{X})^2\right] = \mathbb{E}\left[W^2\right] = \sigma_w^2 = (1-\rho^2)\,\sigma_x^2$$

$$X = \underbrace{\alpha^* y}_{=\hat{X}} + W$$

...ension to non-zero mean

X and y are jointly Gaussian with means μ_x and μ_y respectively and covariance $K_{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} \sigma_x^2 & Cov(x,y) \\ Cov(x,y) & \sigma_y^2 \end{bmatrix}$.

What is the MMSE estimate of X from y?

$$\tilde{X} = X - \mu_x \qquad \tilde{y} = y - \mu_y$$

$$\tilde{X} = \alpha^* \tilde{y} + W \quad \text{where } \alpha^* = \frac{Cov(\tilde{X},\tilde{y})}{\sigma_{\tilde{y}}^2} = \frac{Cov(x,y)}{\sigma_y^2} = \rho\frac{\sigma_x}{\sigma_y}$$

$$\text{where } \rho = \frac{Cov(x,y)}{\sigma_x \sigma_y}$$

$$\text{and } W \sim \mathcal{N}(0, (1-\rho^2)\sigma_x^2)$$

$$X - \mu_x = \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y) + W$$

$$X = \underbrace{\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)} + W$$

$$\hat{X} = E[x|y] = E\left[\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y) \,\Big|\, y\right] + \underbrace{E[W|y]}_{=0}$$

$$= \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$$

$$X|y=y \sim \mathcal{N}\left(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y), \sigma_\omega^2\right)$$

$$E[x|y] = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$$

$$MSE = \sigma_\omega^2 = (1-\rho^2)\sigma_x^2$$

...mple 2 (Special case of Example 1)

$$X \sim \mathcal{N}(0, P) \longrightarrow \bigoplus \longrightarrow y = X + Z, \qquad Z \sim \mathcal{N}(0, \sigma_z^2) \text{ added at } \bigoplus$$

where Z is independent of X.

y and X are jointly Gaussian because $\begin{bmatrix} y \\ X \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}$

$$\hat{X}_{MMSE} = \mathbb{E}[X|y] = \rho \frac{\sigma_x}{\sigma_y} y = \frac{\sigma_x^2}{\sigma_y^2} y = \frac{P}{P + \sigma_z^2} y .$$

$$\rho = \frac{\mathrm{Cov}(X, y)}{\sigma_x \sigma_y} = \frac{\mathrm{Cov}(X, X+Z)}{\sigma_x \sigma_y} = \frac{\sigma_x^2}{\sigma_x \sigma_y} = \frac{\sigma_x}{\sigma_y} .$$

$$\mathrm{MMSE} = (1 - \rho^2)\sigma_x^2 = \left(1 - \frac{\sigma_x^2}{\sigma_y^2}\right)\sigma_x^2 = \frac{\sigma_y^2 - \sigma_x^2}{\sigma_y^2} \sigma_x^2$$

$$= \frac{\sigma_z^2 P}{P + \sigma_z^2}$$

if $\sigma_z^2 \to 0$ $\quad \hat{X}_{MMSE} \to y$. $\quad$ MSE $\to 0$

$\sigma_z^2 \to \infty$. $\quad \hat{X}_{MMSE} \to 0$. $\quad$ MSE $\to P$.

The MMSE estimate balances between prior knowledge and observation.

Non-zero mean. $X \sim \mathcal{N}(\mu_x, P)$ and $Z \sim \mathcal{N}(\mu_z, \sigma_z^2)$

$$\hat{X}_{MMSE} = \mu_x + \rho \frac{\sigma_x}{\sigma_y}(y - \mu_y)$$

$$= \mu_x + \frac{P}{P + \sigma_z^2}(y - \mu_x - \mu_z)$$

-xample 3:

$$X \to y_1 = h_1 X + Z_1$$
$$X \to y_2 = h_2 X + Z_2$$
$$X \to y_n = h_n X + Z_n$$

$X \sim N(0, P)$

$Z_1, Z_2, \dots Z_n \sim N(0, \sigma_z^2)$ independent of X.

$$\hat{X}_{MMSE} = \mathbb{E}[X \mid y_1, y_2, \dots y_n].$$

$$\hat{x}_{MMSE} = \mathbb{E}[X \mid y_1 = y_1, \dots y_n = y_n].$$

HW 7 pr 1. $\bar{X} \mid \bar{Y} = \bar{y} \sim N\left(\mu_{\bar{X}} + K_{\bar{X}\bar{Y}} K_y^{-1} (\bar{y} - \bar{\mu}_y), K_x - K_{xy} K_y^{-1} K_x^{T}\right)$

$$\mathbb{E}[\bar{X} \mid \bar{Y}] = \mu_{\bar{X}} + K_{\bar{X}\bar{Y}} K_{\bar{Y}}^{-1} (\bar{Y} - \bar{\mu}_y)$$

where $K_{\bar{X}\bar{Y}} = \mathbb{E}[\bar{X}\bar{Y}^T]$

Next setting: Estimate X recursively from $y_1, y_2, \dots y_n$.