Part I

The Finite Element Method for One-Dimensional Problems

The one-dimensional problem is useful for displaying many of the features of the finite element method without the attendant complications that necessarily arise in the multi-dimensional case. The familiar problem of an axially loaded, linear elastic bar will provide the primary motivating problem. A bar is a straight but possibly non-prismatic member which is loaded along is centroidal axis producing only axial deformation and a uniaxial state of stress. Because of the simple deformation and stress states, bars are relatively easy to analyze. In fact, development of the finite element method for the analysis of bars reveals some of the essential concepts that are required for more complex structural systems and the simplicity of the 1-d setting makes this a worthwhile exercise. In this chapter we will review the notion of equilibrium for bars and introduce two alternative but equivalent descriptions:

1. The strong form which is the boundary value problem (BVP) that directly expresses the statical equilibrium of the bar, and

2. The weak form which is the principle of virtual displacements. The weak form can be derived from the strong form or found from the principle of minimum potential energy.

Although the strong form is perhaps the most familiar description, it is the weak form that is the starting point for developing the finite element method and, for this reason, we will devote some effort to reviewing this important idea.

It is worthwhile noting that the form of the final equations for the bar problem are indentical to the governing equations for many other important one-dimensional problems, including:

- Torsion of a circular rod,
- One-dimensional heat flow,
- Deflection of a tensioned flexible string,
- Simple flow in pipes,
- Current in a conductor.

Accordingly, the methods developed in this section will also apply directly to these and many other one-dimensional problems.

Chapter 1

The Axially Loaded Bar Problem

1.1 Principle of Virtual Work for "Discrete" Systems

Before we start our study of a bar structure let us undertake a brief review of the principle of virtual work as applied to simple spring systems. We will turn to the analysis of a bar in Sect. 1.3. Our purpose in this Section is to show that the statement of equilibrium can be expressed equivalently either through the statical equations of equilibrium (strong form of the problem) or through the principle of virtual displacements (weak form). It is useful to see this equivalence in this simple setting.

1.1.1 Strong Form

Consider the simple spring system shown in Fig. 1.1.

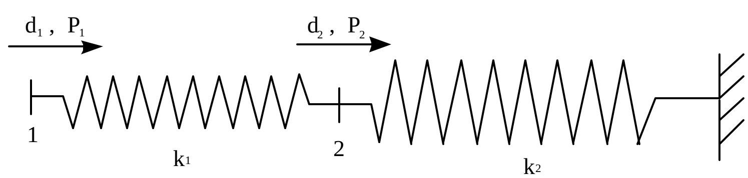

Figure 1.1: A simple spring system.

The deformation of the spring system is described in terms of two degrees of freedom, denoted d_1 and d_2. These are the displacements at joints 1 and 2. We call this structure discrete, in the sense that specification of the two degrees of freedom completely defines the deformation of the system without any approximation. Associated with degrees of freedom 1 and 2, we introduce two external loads denoted F_1 and F_2, respectively, see Fig. 1.1.

The conditions of equilibrium can be found directly using three concepts:

1. Force Equations of Equilibrium. Let the internal force (tension positive) in springs 1 and 2 be denoted N^1 and N^2, respectively. Isolating joints 1 and 2 as free bodies, the equations of equilibrium at joints 1 and 2 are,

$$F_1 + N^1 = 0 \quad (1.1)$$

$$F_2 - N^1 + N^2 = 0 \quad (1.2)$$

In order to find the displacement equations of equilibrium, we need to introduce two additional ingredients:

2. Constitutive Equation. With the spring constants k^1 and k^2, we can express the spring internal forces in terms of the spring stretches Δ^1 and Δ^2 as

$$N^1 = k^1 \Delta^1 \quad (1.3)$$
$$N^2 = k^2 \Delta^2 \quad (1.4)$$

3. Compatibility. The spring stretches are related to the displacements as

$$\Delta^1 = d_2 - d_1 \quad (1.5)$$
$$\Delta^2 = 0 - d_2 \quad (1.6)$$

Combining (1.1)-(1.6) we have finally the displacement equations of equilibrium,

$$F_1 + k^1 (d_2 - d_1) = 0$$
$$F_2 - k^1 (d_2 - d_1) + k^2 (-d_2) = 0$$

or, in matrix form,

$$\begin{bmatrix} k^1 & -k^1 \\ -k^1 & k^1 + k^2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad (1.7)$$

which can be solved for the dispacements in terms of the loads.

1.1.2 Weak Form

We will derive the principle of virtual displacements (PVD) for the model spring system directly from the force equilibrium equations. This will result in an alternative but equivalent description of equilirium of the spring system that has much practical use.

Proposition 1 The force equilibrium equations (1.1) and (1.2) are implied by requiring that

$$\delta_1 \left[F_1 + N^1 \right] + \delta_2 \left[F_2 - N^1 + N^2 \right] = 0 \quad (1.8)$$

hold for all δ_1 and δ_2.

Proof. Let

$$R = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} F_1 + N^1 \\ F_2 - N^1 + N^2 \end{Bmatrix}$$

and

$$\delta = \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

Then (1.8) may be written as

$$\delta \cdot R = 0$$

This orthogonality condition must hold for all $\delta \in \mathbb{R}^2$. The only possible vector R which can satisfy this condition is $R = 0$ which gives

$$\begin{Bmatrix} F_1 + N^1 \\ F_2 - N^1 + N^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

∎

Thus the weak form of the problem (1.8) is an equivalent means of writing the strong form.

Remark 2 In the orthogonality condition $\delta \cdot R = 0$, the vector R is a vector of "equation residuals" and the vector δ is a vector of arbitrary "weights," and the approach is called the "method of weighted residuals."

It is interesting to rearrange (1.8) as follows:

$$N^1(\delta_2 - \delta_1) + N^2(-\delta_2) = F_1\delta_1 + F_2\delta_2 \tag{1.9}$$

If we think of δ_1 and δ_2 as "virtual displacements," we can define corresponding compatible "virtual stretches (strains)" as follows

$$\delta\Delta^1 = \delta_2 - \delta_1 \tag{1.10}$$
$$\delta\Delta^2 = 0 - \delta_2 \tag{1.11}$$

In this case (1.9) becomes

$$N^1\delta\Delta^1 + N^2\delta\Delta^2 = F_1\delta_1 + F_2\delta_2 \tag{1.12}$$

which, in this form, is called the principle of virtual displacements[1]:

internal virtual work = external virtual work, for all virtual displacements.

The point is, (1.12) is equivalent to the force equations of equilibrium (1.1) and (1.2), providing that:

1. (1.12) holds for all δ_1 and δ_2; and
2. The virtual strains and virtual displacements are comptatible as given by (1.10) and (1.11).

Remark 3 The PVD provides the force equations of equilibrium – the unknown displacements d_1 and d_2 have not appeared in the development! We will introduce these as a second and subsidiary step.

PVD(1): The PVD can be used to obtain the displacement equations of equilibrium as follows:

1. Write the PVD
$$N^1\delta\Delta^1 + N^2\delta\Delta^2 = F_1\delta_1 + F_2\delta_2$$
2. Require the virtual displacements to be compatible with the virtual stretches
$$\delta\Delta^1 = \delta_2 - \delta_1$$
$$\delta\Delta^2 = -\delta_2$$

[1] The PVD may be formally stated in the language of structural mechanics as follows: among all kinematically admissible configurations of a structure, those that satisfy the equations of equilibrium make the virtual work done by the internal stresses (internal virtual work) equal to the virtual work done by the applied loads (external virtual work) for all kinematically admissible virtual displacements.

resulting in,

$$(\delta_2 - \delta_1)N^1 + (-\delta_2)N^2 = \delta_1 F_1 + \delta_2 F_2$$

which, after collecting terms, may be re-written as (our starting point),

$$\delta_1(-N^1 - F_1) + \delta_2(N^1 - N^2 - F_2) = 0$$

Since this must hold for all virtual displacements δ_1 and δ_2, it implies that,

$$\begin{aligned} -N^1 - F_1 &= 0 \\ N^1 - N^2 - F_2 &= 0 \end{aligned}$$

3. Finally, using the stiffness relationships (1.3) and (1.4) and compatibility conditions (1.5) and (1.6) leads to,

$$\begin{bmatrix} k^1 & -k^1 \\ -k^1 & k^1 + k^2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

as found in the strong form.

PVD(2): Instead of postponing the substitution of the stiffness relationships (1.3) and (1.4) and compatibility conditions (1.5) and (1.6) until Step 3, we can introduce them directly into the PVD as follows. This variant of the approach is the most commonly used.

1. Write the PVD

$$N^1 \delta\Delta^1 + N^2 \delta\Delta^2 = F_1 \delta_1 + F_2 \delta_2$$

2. Introduce the the stiffness relationships (1.3) and (1.4) and compatibility conditions (1.5) and (1.6), giving

$$k^1(d_2 - d_1)\delta\Delta^1 + k^2(-d_2)\delta\Delta^2 = F_1\delta_1 + F_2\delta_2$$

3. Require the virtual displacements to be compatible with the virtual stretches

$$\begin{aligned} \delta\Delta^1 &= \delta_2 - \delta_1 \\ \delta\Delta^2 &= -\delta_2 \end{aligned}$$

resulting in,

$$k^1(d_2 - d_1)(\delta_2 - \delta_1) + k^2(-d_2)(-\delta_2) = F_1\delta_1 + F_2\delta_2$$

which, after collecting terms, may be re-written as

$$\delta_1\left[-k^1(d_2 - d_1) - F_1\right] + \delta_2\left[k^1(d_2 - d_1) - k^2(-d_2) - F_2\right] = 0$$

Since this must hold for all virtual displacements δ_1 and δ_2, it implies that,

$$\begin{aligned} -k^1(d_2 - d_1) - F_1 &= 0 \\ k^1(d_2 - d_1) - k^2(-d_2) - F_2 &= 0 \end{aligned}$$

or

$$\begin{bmatrix} k^1 & -k^1 \\ -k^1 & k^1 + k^2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

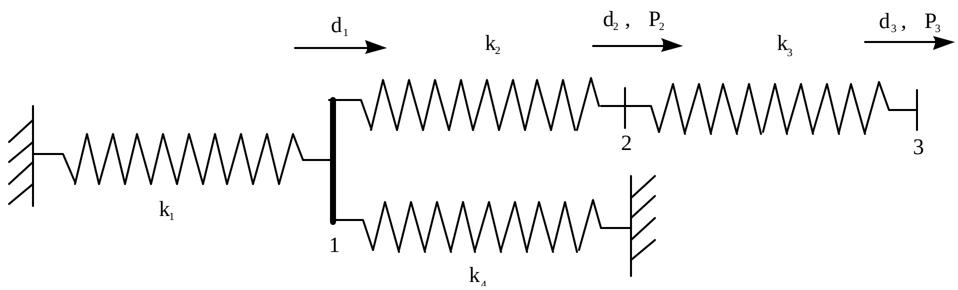

Figure 1.2: A statically indeterminate spring system.

Remark 4 The PVD applies to both statically determinate and indeterminate systems – this makes the PVD a very practical tool for analysis of complex systems.

Example 5 Consider the one-dimensional, statically indeterminate spring system shown in Fig. 1.2. Use PVD(1) to obtain the equations of equilibrium in terms of displacements in matrix form.

Let the actual and virtual displacements at node i be denoted d_i and $\pm_i$, respectively. Let the internal force (tension positive) in member i be denoted N^i and let the virtual deformation (stretching positive) of member i be denoted $\pm\text{¢}^i$, then the PVD states that the equilibrium of the spring system is expressed as,

$$N^1\pm\text{¢}^1 + N^2\pm\text{¢}^2 + N^3\pm\text{¢}^3 + N^4\pm\text{¢}^4 = F_1\pm_1 + F_2\pm_2 + F_3\pm_3$$

which must hold for all compatible virtual displacements such that,

$$\begin{aligned} \pm\text{¢}^1 &= \pm_1 \\ \pm\text{¢}^2 &= \pm_2 \text{ ¡ } \pm_1 \\ \pm\text{¢}^3 &= \pm_3 \text{ ¡ } \pm_2 \\ \pm\text{¢}^4 &= \text{¡ } \pm_1 \end{aligned}$$

Using these conditions of compatibility in the PVD now gives,

$$N^1\pm_1 + N^2(\pm_2 \text{ ¡ } \pm_1) + N^3(\pm_3 \text{ ¡ } \pm_2) + N^4(\text{¡ }\pm_1) = F_1\pm_1 + F_2\pm_2 + F_3\pm_3$$

Collecting terms we …nd,

$$\pm_1(N^1 \text{ ¡ } N^2 \text{ ¡ } N^4 \text{ ¡ } F_1) + \pm_2(N^2 \text{ ¡ } N^3 \text{ ¡ } F_2) + \pm_3(N^3 \text{ ¡ } F_3) = 0$$

Since this must hold for all virtual displacements $\pm_i$, we have,

$$\begin{aligned} N^1 \text{ ¡ } N^2 \text{ ¡ } N^4 \text{ ¡ } F_1 &= 0 \\ N^2 \text{ ¡ } N^3 \text{ ¡ } F_2 &= 0 \\ N^3 \text{ ¡ } F_3 &= 0 \end{aligned}$$

which may easily be veri…ed to be the equations of equilibrium (using free body diagrams of the joints). To obtain the equilibrium equations in terms of the displacements, we must now express the internal forces N^i in terms of the member stretches

$$N^i = k^i\text{¢}^i$$

and use compatibility to relate the stretches to the displacements

$$\begin{aligned}\Delta^1 &= d_1\\ \Delta^2 &= d_2 - d_1\\ \Delta^3 &= d_3 - d_2\\ \Delta^4 &= -d_1\end{aligned}$$

giving

$$\begin{aligned}N^1 &= k^1(d_1)\\ N^2 &= k^2(d_2 - d_1)\\ N^3 &= k^3(d_3 - d_2)\\ N^4 &= k^4(-d_1)\end{aligned}$$

Using these stiffness relationships in the equilibrium equations above results in:

$$\begin{bmatrix} k^1 + k^2 + k^4 & -k^2 & 0 \\ -k^2 & k^2 + k^3 & -k^3 \\ 0 & -k^3 & k^3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}. \tag{1.13}$$

In summary, if we consider a general system of N interconnected springs involving M unknown displacements, the weak form (i.e. PVD) describing the equilibrium of the system may be stated as follows:

Find the unknown displacement vector $d \in R^M$ such that

$$\sum_{i=1}^{N} \delta\Delta^i k^i \Delta^i = \sum_{i=1}^{M} \delta_i F_i \tag{1.14}$$

holds for all δ and such that both Δ^i and $\delta\Delta^i$ are compatible with d and δ, respectively.

1.2 Principle of Minimum Potential Energy for "Discrete" Systems

With every statement of the principle of virtual work it is possible to associate a quadratic functional called the potential energy such that the exact solution corresponding to the to PVD minimizes the potential energy. We define the potential energy Π as

$$\Pi(d) = U(d) + V(d) \tag{1.15}$$

where the strain energy U and the potential of the external load V are given by

$$U(d) = \sum_{i=1}^{N} \frac{1}{2} k^i \left(\Delta^i\right)^2 \tag{1.16}$$

$$V(d) = -\sum_{i=1}^{M} d_i F_i \tag{1.17}$$

Let us now show that the exact solution corresponding to the to PVD, denoted d_{ex}, minimizes the potential energy Π.

Proposition 6 The exact solution corresponding to the to PVD, denoted d_{ex}, minimizes the potential energy Π; that is

$$\Pi(d_{ex}) = \min_{d \in R^M} \Pi(d)$$

Proof. Select any vector $\delta \in R^M$ and consider

$$\begin{aligned}
\Pi(d_{ex} + \delta) &= U(d_{ex} + \delta) + V(d_{ex} + \delta) \\
&= \sum_{i=1}^{N} \frac{1}{2} k^i \left(\Delta^i_{ex} + \delta\Delta^i \right)^2 - \sum_{i=1}^{N} ((d_{ex})_i + \delta_i) F_i \\
&= \sum_{i=1}^{N} \frac{1}{2} k^i \left(\Delta^i_{ex} \right)^2 + \sum_{i=1}^{N} \frac{1}{2} k^i \left(\delta\Delta^i \right)^2 + \sum_{i=1}^{N} k^i \Delta^i_{ex} \delta\Delta^i - \sum_{i=1}^{N} (d_{ex})_i F_i - \sum_{i=1}^{N} \delta_i F_i \\
&= \Pi(d_{ex}) + \sum_{i=1}^{N} \frac{1}{2} k^i \left(\delta\Delta^i \right)^2 + \sum_{i=1}^{N} \delta\Delta^i k^i \Delta^i_{ex} - \sum_{i=1}^{N} \delta_i F_i \qquad (1.18)
\end{aligned}$$

However, by the PVD, (1.14),

$$\sum_{i=1}^{N} \delta\Delta^i k^i \Delta^i_{ex} - \sum_{i=1}^{N} \delta_i F_i = 0$$

for any choice of δ. Also

$$\sum_{i=1}^{N} \frac{1}{2} k^i \left(\delta\Delta^i \right)^2 > 0$$

for any choice of δ. It follows that

$$\Pi(d_{ex} + \delta) < \Pi(d_{ex})$$

and therefore d_{ex} is the minimizing vector of $\Pi(d)$ ■

This is called the principle of minimum potential energy (PMPE). Since the exact solution minimizes Π, we conclude that Π must be stationary at d_{ex}.

Proposition 7 The condition of stationarity of Π,

$$\lim_{\delta \to 0} [\Pi(d_{ex} + \delta) - \Pi(d_{ex})] = 0$$

is identical to the PVD.

Proof. From (1.18), it follows that

$$\Pi(d_{ex} + \delta) - \Pi(d_{ex}) = \sum_{i=1}^{N} \delta\Delta^i k^i \Delta^i_{ex} - \sum_{i=1}^{N} \delta_i F_i + O\left(\left(\delta\Delta^i \right)^2 \right)$$

where

$$O\left(\left(\delta\Delta^i \right)^2 \right) = \sum_{i=1}^{N} \frac{1}{2} k^i \left(\delta\Delta^i \right)^2$$

Hence

$$\lim_{\delta \to 0} [\Pi(d_{ex} + \delta) - \Pi(d_{ex})] = \sum_{i=1}^{N} \delta\Delta^i k^i \Delta^i_{ex} - \sum_{i=1}^{N} \delta_i F_i \qquad (1.19)$$

and so the condition of stationarity returns the PVD. ■

Observe also that the potential energy is negative, as follows.

Proposition 8 The potential energy is negative, that is $\Pi(d_{ex}) < 0$

Proof. Recall that

$$\Pi(d_{ex}) = \sum_{i=1}^{M} \frac{1}{2} k^i \left(\Delta^i_{ex}\right)^2 - \sum_{i=1}^{M} (d_{ex})_i F_i$$

Since the PVD must hold for all $\delta \in \mathbb{R}^M$, it must hold also for the choice of $\delta = d_{ex}$. In this special case, the PVD reads

$$\sum_{i=1}^{M} k^i \left(\Delta^i_{ex}\right)^2 = \sum_{i=1}^{M} (d_{ex})_i F_i$$

Hence

$$\begin{aligned} \Pi(d_{ex}) &= \sum_{i=1}^{M} \frac{1}{2} k^i \left(\Delta^i_{ex}\right)^2 - \sum_{i=1}^{M} (d_{ex})_i F_i \\ &= -\sum_{i=1}^{M} \frac{1}{2} k^i \left(\Delta^i_{ex}\right)^2 \\ &< 0 \end{aligned}$$

■

This condition is depicted in Fig. 1.3. The PMPE states that the exact solution is the solution which minimizes the potential energy. This requires the potential energy to be stationary. The condition of stationarity was computed above by evaluating $\lim_{\delta \to 0} [\Pi(d_{ex} + \delta) - \Pi(d_{ex})]$ and setting the result to zero. This operation is called the first variation of Π and is denoted $\delta\Pi$, so that

$$\delta\Pi := \lim_{\delta \to 0} [\Pi(d_{ex} + \delta) - \Pi(d_{ex})]$$

A convenient approach for computing the first variation is to use the directional derivative, as follows

$$\delta\Pi = \left[\frac{d}{d\epsilon} \Pi(d_{ex} + \epsilon\delta) \right]_{\epsilon=0}$$

In fact, let us re-evaluate $\delta\Pi$ using the directional derivative,

$$\begin{aligned} \delta\Pi &= \left[\frac{d}{d\epsilon} \Pi(d_{ex} + \epsilon\delta) \right]_{\epsilon=0} \\ &= \underbrace{\left[\frac{d}{d\epsilon} U(d_{ex} + \epsilon\delta) \right]_{\epsilon=0}}_{\delta U} + \underbrace{\left[\frac{d}{d\epsilon} V(d_{ex} + \epsilon\delta) \right]_{\epsilon=0}}_{\delta V} \\ &= \left[\frac{d}{d\epsilon} \sum_{i=1}^{M} \frac{1}{2} k^i (\Delta^i_{ex} + \epsilon\delta\Delta^i)^2 \right]_{\epsilon=0} - \left[\frac{d}{d\epsilon} \sum_{i-1}^{M} ((d_{ex})_i + \epsilon\delta_i) F_i \right]_{\epsilon=0} \\ &= \left[\sum_{i=1}^{M} k^i (\Delta^i_{ex} + \epsilon\delta\Delta^i) \frac{d}{d\epsilon} (\Delta^i_{ex} + \epsilon\delta\Delta^i) \right]_{\epsilon=0} - \left[\sum_{i-1}^{M} \frac{d}{d\epsilon} ((d_{ex})_i F_i + \epsilon\delta_i F_i) \right]_{\epsilon=0} \\ &= \sum_{i=1}^{M} \delta\Delta^i k^i \Delta^i_{ex} - \sum_{i-1}^{M} \delta_i F_i \end{aligned}$$

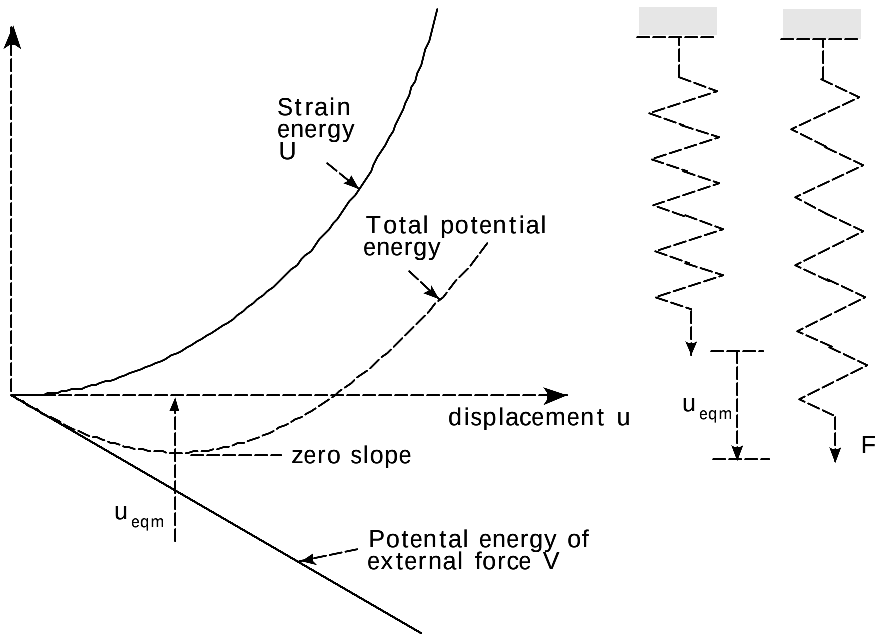

Figure 1.3: Principle of minumum potential energy: the equilibrium displacement minimizes the total potential energy.

which is the same as (1.19). Let us consider some examples which apply the PMPE to finding the equilibrium equations.

Example 9 Consider once again the spring system shown in Fig. 1.1. We will use the principle of minimum potential energy to obtain the equations of equilibrium. For this spring system we can immediately write the strain energy of the two springs from (1.16) as,

$$U(d) = \sum_{i=1} \frac{1}{2}k_i \left(\Delta_i\right)^2 = \frac{1}{2}k_1 \left(\Delta_1\right)^2 + \frac{1}{2}k_2 \left(\Delta_2\right)^2 \qquad (1.20)$$

The potential of the external loads can be written from (1.17) as,

$$V(d) = -\sum_{i=1} d_i F_i = -F_1 d_1 - F_2 d_2 \qquad (1.21)$$

The potential energy of the spring system is then,

$$\begin{aligned} \Pi(d) &= U(d) + V(d) \\ &= \frac{1}{2}k_1 \left(\Delta_1\right)^2 + \frac{1}{2}k_2 \left(\Delta_2\right)^2 - F_1 d_1 - F_2 d_2 \end{aligned} \qquad (1.22)$$

The PMPE states that the spring system is in equilibrium with displacement values d_1 and d_2 when $\delta\Pi$ is stationary with respect to all virtual displacements δ_1 and δ_2, that is $\delta\Pi = 0$. Now,

$$\delta\Pi(d) = \delta U(d) + \delta V(d)$$

and

$$
\begin{aligned}
\delta U(d) &= \left[\frac{d}{d\varepsilon}U(d+\varepsilon\delta)\right]_{\varepsilon=0} \\
&= \left[\frac{d}{d\varepsilon}\left(\frac{1}{2}k^1\left(\Delta^1+\varepsilon\delta\Delta^1\right)^2+\frac{1}{2}k^2\left(\Delta^2+\varepsilon\delta\Delta^2\right)^2\right)\right]_{\varepsilon=0} \\
&= \delta\Delta^1k^1\Delta^1+\delta\Delta^2k^2\Delta^2
\end{aligned}
$$

$$
\begin{aligned}
\delta V(d) &= \left[\frac{d}{d\varepsilon}V(d+\varepsilon\delta)\right]_{\varepsilon=0} \\
&= \left[\frac{d}{d\varepsilon}\left(-F_1(d_1+\varepsilon\delta_1)-F_2(d_2+\varepsilon\delta_2)\right)\right]_{\varepsilon=0} \\
&= -F_1\delta_1-F_2\delta_2
\end{aligned}
$$

Setting $\delta\Pi=0$ we have

$$
\delta\Pi=\delta\Delta^1k^1\Delta^1+\delta\Delta^2k^2\Delta^2-F_1\delta_1-F_2\delta_2=0
$$

which is the PVD. Using compatibility we have

$$
\begin{aligned}
\Delta^1 &= d_2-d_1 \\
\Delta^2 &= -d_2
\end{aligned}
$$

and

$$
\begin{aligned}
\delta\Delta^1 &= \delta_2-\delta_1 \\
\delta\Delta^2 &= -\delta_2
\end{aligned}
$$

and the PMPE now gives,

$$
\begin{aligned}
0 &= (\delta_2-\delta_1)k^1(d_2-d_1)+(-\delta_2)k^2(-d_2)-F_1\delta_1-F_2\delta_2 \\
&= (\delta_1)\left(-k^1d_2+k^1d_1-F_1\right)+(\delta_2)\left(k^1d_2-k^1d_1+k^2d_2-F_2\right)
\end{aligned}
$$

Since this must hold for all arbitrary δ_1 and δ_2, we have,

$$
\begin{aligned}
-k^1d_2+k^1d_1-F_1 &= 0 \\
k^1d_2-k^1d_1+k^2d_2-F_2 &= 0
\end{aligned}
$$

or

$$
\begin{bmatrix} k^1 & -k^1 \\ -k^1 & k^1+k^2 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}
$$

which the same as (1.7).

Remark 10 Notice that the first variation can be used like an ordinary derivative. For example, in the above example we can write

$$
\begin{aligned}
U(d) &= \frac{1}{2}k^1\left(\Delta^1\right)^2+\frac{1}{2}k^2\left(\Delta^2\right)^2 \\
\delta U(d) &= k^1\Delta^1\delta\Delta^1+k^2\Delta^2\delta\Delta^2
\end{aligned}
$$

and

$$\begin{aligned} V(d) &= -F_1 d_1 - F_2 d_2 \\ \delta V(d) &= -F_1 \delta_1 - F_2 \delta_2 \end{aligned}$$

so that $\delta\Pi$ can be immediately written as

$$\delta\Pi = \delta\Delta^1 k^1 \Delta^1 + \delta\Delta^2 k^2 \Delta^2 - F_1\delta_1 - F_2\delta_2$$

which is a very convenient approach in practice.

Example 11 Return now to the one-dimensional, statically indeterminate, spring system shown in Fig. 1.2 and analyzed using the PVD in Section 5. We will now derive the equilibrium conditions using the PMPE. The strain energy of the springs is,

$$U = \frac{1}{2}k^1\left(\Delta^1\right)^2 + \frac{1}{2}k^2\left(\Delta^2\right)^2 + \frac{1}{2}k^3\left(\Delta^3\right)^2 + \frac{1}{2}k^4\left(\Delta^4\right)^2$$

and the potential of the external loads is,

$$V = -F_1 d_1 - F_2 d_2 - F_3 d_3$$

The PMPE requires,

$$\begin{aligned} 0 &= \delta\Pi \\ &= \delta U + \delta V \\ &= \left(k^1\Delta^1\right)\delta\Delta^1 + \left(k^2\Delta^2\right)\delta\Delta^2 + \left(k^3\Delta^3\right)\delta\Delta^3 + \left(k^4\Delta^4\right)\delta\Delta^4 - F_1\delta_1 - F_2\delta_2 - F_3\delta_3 \end{aligned}$$

Introducing the compatibility conditions between displacements and stretches,

$$\begin{aligned} \Delta^1 &= d_1 \\ \Delta^2 &= d_2 - d_1 \\ \Delta^3 &= d_3 - d_2 \\ \Delta^4 &= -d_1 \end{aligned}$$

and similarly between virtual displacements and virtual stretches,

$$\begin{aligned} \delta\Delta^1 &= \delta_1 \\ \delta\Delta^2 &= \delta_2 - \delta_1 \\ \delta\Delta^3 &= \delta_3 - \delta_2 \\ \delta\Delta^4 &= -\delta_1 \end{aligned}$$

leads to the same results as Eq. (1.13), i.e.,

$$\begin{bmatrix} k^1 + k^2 + k^4 & -k^2 & 0 \\ -k^2 & k^2 + k^3 & -k^3 \\ 0 & -k^3 & k^3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}.$$

With this introduction to the PVD and the PMPE applied to spring models complete, let us now turn to the analysis of bars.