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MANAGING RISK USING MULTI-STAGE STOCHASTIC OPTIMIZATION

By

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Abstract

The paper discusses the application of multi-stage stochastic optimization for managing and optimizing expected returns versus risk, and contrasts static (single-stage) versus dynamic (multi-stage) portfolio optimization. We present how to best fund a pool of similar fixed rate mortgages through issuing bonds, callable and non-callable, of various maturities using stochastic optimization. We discuss the estimation of expected net present value and risk for different funding instruments using Monte Carlo sampling techniques, and the optimization of the funding using single- and multi-stage stochastic optimization.

Using practical data we computed efficient frontiers of expected net present value versus risk for the single- and the multi-stage model, and studied the underlying funding strategies. Constraining the duration and convexity of the mortgage pool and the funding portfolios to match at any decision point, we computed delta and gamma hedged funding strategies and compared them to the ones from the multi-stage stochastic optimization model. The results for the different data assumptions demonstrate that multi-stage stochastic optimization yields significantly larger net present values at the same or at a lower level of risk, compared to single-stage optimization and delta and gamma hedging. We found that the funding strategies obtained from the multi-stage model differed significantly from those from the single-stage model and were again significantly different to funding strategies obtained from delta and gamma hedging. Using multi-stage stochastic optimization for determining the best funding of mortgage pools will lead in the average to significant profits, compared to using single-stage funding strategies, or using delta and gamma hedging.

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1. Introduction

Historically, the business of conduits, like Freddie Mac, Fannie Mae or Ginnie Mae, has been to purchase mortgages from primary lenders, pooling these mortgages into mortgage pools, securitizing some if not all of the pools by selling the resulting Participation Certificates (PCs) to Wall Street. Conduits keep a fixed markup on the interest for their profit and roll over most of the (interest rate and prepayment) risk to the PC buyers. Recently, a more active approach, with the potential for significantly higher profits, has become increasingly attractive: Instead of securitizing, funding the purchase of mortgage pools by issuing debt. The conduit firm raises the money for the mortgage purchases through a suitable combination of long and short term debt. Thereby the conduit assumes a higher level of risk due to interest rate changes and prepayment risk but gains higher expected revenues due to the higher spread between the interest on debt and mortgage rates compared to the fixed markup by securitizing the pool.

In this report we present how multi-stage stochastic optimization can be used for determining the best funding of a pool of similar fixed rate mortgages through issuing bonds, callable and non-callable, of various maturities. We show that significant profits can be obtained using multi-stage stochastic optimization compared to using a single-stage model formulation and compared to using strategies of delta and gamma hedging. For the comparison we use an implementation of Freddie Mac's interest rate model and prepayment function. We describe the basic formulations of funding mortgage pools and discuss the estimation of expected net present value and risk for different funding instruments using Monte Carlo sampling techniques in Section 2. In Section 3 we discuss the single-stage model. In Section 4 we present the multi-stage model. Section 5 discusses duration and convexity and delta and gamma hedging. In Section 6 we discuss numerical results using practical data obtained from Freddie Mac. We compare the efficient frontiers from the single-stage and multi-stage models, discuss the different funding strategies and compare them with delta and gamma hedged strategies, and evaluate the different strategies using out of sample simulations. Section 7 reports on the solution of very large models and gives model sizes and solution times.

While not explicitely discussed in this report, the problem of what fraction of the mortgage pool should be securitized, and what portion should be retained and funded through issuing debt can be addressed within the proposed framework through a minor extension of the models presented. Funding decisions for a particular pool are not independent of all other pools already in the portfolio and those to be acquired in the future. The approach can of course address also the funding of a number of pools with different characteristics. While the report focuses on funding a pool of fixed rate mortgages, the framework applies analogously to funding pools of adjustable rate mortgages.

2. Funding Mortgage Pools

2.1. Interest Rate Term Structure

Observations of the distributions of future interest rates are obtained using an implementation of the interest rate model of Luytjes (1993) [7] and its update according to the Freddie Mac document [9]. The model reflects a stochastic process based on equilibrium theory using random shocks for short rate, spread and inflation.

We do not use the inflation part of the model. To generate a possible interest rate path we feed the model with realizations of $2T$ standard normal random variables and obtain as output for each period t a possible outcome of a yield curve of interest rates based on the particular realizations of the random shocks. We denote as $i_t(m)$, $t = 1, \dots, T$ the random interest rate of a zero coupon bond of term m in period t.

2.2. The Cash Flows of a Mortgage Pool

We consider all payments of a pool of fixed-rate mortgages during its life time. Time periods t range from $t = 0, \dots, T$, where T denotes the end of the horizon; e.g., $T = 360$ reflects a horizon of 30 years considering monthly payments. We let B_t be the balance of the principal of the pool at the end of period t. The principal capital B_0 is given to the homeowners at time period $t = 0$ and is regained through payments β_t and through prepayments α_t at periods $t = 1, \dots, T$. The balance of the principal is updated periodically by

$$B_t = B_{t-1}(1 + \rho_0) - \beta_t - \alpha_t,\ t = 1, \dots, T.$$

The rate ρ_0 is the contracted interest rate of the fixed-rate mortgage at time $t = 0$. We define λ_t to be the payment factor at period $t = 1, \dots, T$. The payment factor when multiplied by the mortgage balance, yields the constant monthly payments necessary to pay off the loan over its remaining life, e.g.,

$$\lambda_t = \rho_0 / (1 - (1 + \rho_0)^{t-T-1});$$

thus,

$$\beta_t = \lambda_t B_{t-1}.$$

The payment factor λ_t depends on the interest rate ρ_0. For fixed rate mortgages the quantity ρ_0, and thus the quantities λ_t are known with certainty. However, prepayments α_t, at periods $t = 1, \dots, T$ depend on future interest rates and are therefore random parameters.

Prepayment models or functions represent the relationship between interest rates and prepayments. In order to determine α_t we use an implementation of Freddie Mac's prepayment function according to Lekkas and Luytjes (1993) [6]. Denoting the prepayment rates obtained from the prepayment function as γ_t, $t = 1, \dots, T$, we compute the prepayments α_t in period t as

$$\alpha_t = \gamma_t B_{t-1}.$$

2.3. Funding through Issuing Debt

We consider funding through issuing bonds, callable and non-callable, with various maturity. Let ℓ be a bond with maturity m_ℓ, $\ell \in L$, where L denotes the set of bonds under consideration. Let $f^\tau_{\ell t}$ be the payment factor corresponding to a bond ℓ issued at period τ at period t, $\tau \le t \le \tau + m_\ell$:

$$f^\tau_{\ell t} \equiv \begin{cases} +1 & \text{if } t - \tau = 0; \\ -(i_\tau(m_\ell) + s_{\tau\ell}) & \text{if } 0 < t - \tau < m_\ell; \\ -(1 + i_\tau(m_\ell) + s_{\tau\ell}) & \text{if } t - \tau = m_\ell; \end{cases}$$

where $i_\tau(m_\ell)$ reflects the interest rate of a zero coupon bond with maturity m_ℓ, issued at period τ, and $s_{\tau,\ell}$ denotes the spread between the zero coupon rate and the actual rate of bond ℓ issued at τ. The spread $s_{\tau,\ell}$ includes the spread of bullet bonds over zero coupon bonds (at Freddie Mac referred to as agency spread) and the spread of callable bonds over bullet bonds (at Freddie Mac referred to as agency call spread), and is computed according to the model specification given in the Freddie Mac document [8].

Let M_τ^ℓ denote the balance of a bullet bond ℓ at the time τ it is issued. The finance payments resulting from bond ℓ, are

$$d_t^\ell = f_{\ell t}^\tau M_\tau^\ell, \quad t = \tau, \ldots, \tau + m_\ell,$$

from the time of issue (τ) until the time it matures ($\tau + m_\ell$), or if callable, it is called. We consider the balance of the bullet bond from the time of issue until the time of maturity as

$$M_t^\ell = M_\tau^\ell, \quad t = \tau, \ldots, \tau + m_\ell.$$

2.4. Leverage Ratio

Regulations require that at any time t, $t = 0, \ldots, T$, equity is set aside against debt at the amount that the ratio of the difference of all assets minus all liabilities versus all assets is greater or equal to a given value μ. Let E_t be the balance of an equity account associated to the funding. The equity constraint requires that

$$\frac{B_t + E_t - M_t}{B_t + E_t} \geq \mu,$$

where the total asset balance represents the sum of the mortgage balance and the equity balance, $B_t + E_t$, and $M_t = \sum_\ell M_t^\ell$ represents the total liability balance.

At time periods $t = 0, \ldots, T$, given the mortgage balance B_t, and the liability balance M_t, we compute the equity balance that fulfills the leverage ratio constraint with equality as

$$E_t = \frac{M_t - B_t(1-\mu)}{1-\mu}, \quad t = 0, \ldots, T.$$

We assume that the equity account accrues interest according to the short rate $i_t(\text{short})$, the interest rate of the 3-month zero coupon bond. Thus, we have the following balance equation of the equity account:

$$E_t = E_{t-1}(1 + i_{t-1}(\text{short})) + e_{t-1},$$

where e_t are payments into the equity account (positive) or payments out of the equity account (negative). Using this equation we compute the payments e_t in and from the equity account necessary to maintain the equity balance E_t computed for holding the leverage ratio μ.

2.5. Simulation

Using the above specification we may perform a simulation run in order to obtain an observation of all cash flows resulting from the mortgage pool and from financing the pool through various bonds. In order to determine in advance how the funding is carried out, we need to specify certain decision rules defining what to do when a bond matures, when to call a callable bond, at what level to fund, and how to manage profits and losses. For the experiment we employed the following rules:

- Initial funding is obtained at the level of the initial balance of the mortgage pool, $M_0 = B_0$.
- Since at time $t = 0$, $M_0 = B_0$, it follows that $E_0 = \frac{\mu}{1-\mu} B_0$, which amount we assume as a given endowed initial equity balance.
- When a bond matures, refunding is carried out using short-term debt (non callable 3-month bullet bond) until the end of the planning horizon, each time at the level of the balance of the mortgage pool.
- Callable bonds are called according to the call rule specification in Freddie Mac's document [8]. Upon calling, refunding is carried out using short-term debt until the end of the planning horizon, each time at the level of the balance of the mortgage pool.
- The leverage ratio (ratio of all assets versus all liabilities) is $\mu = 0.025$
- At each time period t, after maintaining the leverage ratio, we consider a positive sum of all payments as profits and a negative sum as losses.

According to the decision rules, when funding a mortgage pool using a single bond ℓ, we assume at time $t = 0$ that $M_0^\ell = B_0$, i.e., that exactly the amount of the initial mortgage balance is funded using bond ℓ. After bond ℓ matures refunding takes place using another bond (according to the decision rules, short-term debt, say bond $\hat{\ell}$), based on the interest rate and the level of the mortgage balance at the time it is issued. If the initial bond ℓ is callable, it may be called and then funding carried out through another bond (say short-term debt $\hat{\ell}$). Financing based on bond $\hat{\ell}$ is continued until the end of the planning horizon, i.e., until $T - \tau < m_{\hat{\ell}}$, and no more bond is issued. Given the type bond being used for refunding and given an appropriate calling rule, all finance payments for the initial funding using bond ℓ and the consequent refunding using bond $\hat{\ell}$ can be determined. We denote the finance payments accruing from the initial funding based on bond ℓ and its consequent refunding based on bond $\hat{\ell}$ as

$$d_t^\ell, \quad t = 1, \ldots, T.$$

Once the funding and the corresponding liability balance M_t^ℓ is determined, the required equity balance $E_t = E_t^\ell$ and the payments $e_t = e_t^\ell$ are computed.

2.6. The Net Present Value of the Payment Stream

Finally, we define as

$$P_t^\ell = \beta_t + \alpha_t + d_t^\ell - e_t^\ell, \quad t = 1, \ldots, T, \quad P_0 = M_0 - B_0 = 0,$$

the sum of all payments in period t, $t = 0, \ldots, T$ resulting from funding a pool of mortgages (initially) using bond ℓ

Let I_t be the discount factor of period t, i.e.,

$$I_t = \prod_{k=1}^{t} (1 + i_k(\text{short})), \quad t = 1, \ldots, T, \quad I_0 = 1,$$

where we used the short rate at time t, $i_t(\text{short})$), for discounting. The net present value (NPV) of the payment stream is then is calculated as

$$r_\ell = \sum_{t=0}^{T} \frac{P_t^\ell}{I_t}.$$

So far, we considered all quantities that depended on interest rates as random parameters. In particular, P_t^ℓ is a random parameter, since β_t, α_t, d_t^ℓ, and e_t^ℓ are random parameters depending on random interest rates. Therefore also the net present value r_ℓ is a random parameter as well. In order to simplify the notation we did not label any specific outcomes of the these random parameters. A particular run of the interest rate model required $2T$ random parameter outcomes of unit normal random shocks. We now label a particular path of the interest rates obtained from one run of the interest rate model and all corresponding quantities with ω. In particular, we label a realization of the net present value based on a particular interest rate path as r_ℓ^ω.

2.7. Estimating the Expected NPV of the Payment Stream

We use Monte Carlo sampling to estimate the expected value of the NPV of a payment stream. Under a crude Monte Carlo approach to the NPV estimation, we sample N paths $\omega \in S$, $N = |S|$, using different observations of the distributions of the $2T$ random parameters as input to the interest rate model, and we compute r_ℓ^ω for each $\omega \in S$. Then, an estimate for the expected net present value (NPV) of the cash flow stream based on initial funding using bond ℓ is

$$\bar{r}_\ell = \frac{1}{N} \sum_{\omega \in S} r_\ell^\omega.$$

We do not describe in this document how we use advanced variance reduction techniques (e.g., importance sampling) for the estimation of the expected net present value of a payment stream.

2.8. The Expected NPV of a Funding Mix

Using simulation (as described above) we compute the net present value of the payment stream r_ℓ^ω for each realization $\omega \in S$ and each initial funding $\ell \in L$. The net present value of a funding mix is given by the corresponding convex combination of the net present values of the components $\ell \in L$, i.e.,

$$r^\omega = \sum r_\ell^\omega x_\ell, \quad \sum x_\ell = 1, \quad x_l \geq 0,$$

where x_l are nonnegative weights summing to one. The expected net present value of a funding mix,

$$\bar{r} = \frac{1}{N} \sum_{\omega \in S} r^\omega,$$

is also represented as the convex combination of the expected net present values of the components $\ell \in L$, i.e.,

$$\bar{r} = \sum \bar{r}_\ell x_\ell, \quad \sum x_\ell = 1, \quad x_\ell \geq 0.$$

2.9. Risk of a Funding Mix

In order to measure risk, we use as an appropriate asymmetric penalty function the negative part of the deviation of the net present value of a funding portfolio from a pre-specified target u, i.e.,

$$v^\omega = \sum_\ell (r_\ell^\omega x_\ell - u)^-$$

and consider risk as the expected value of v^ω, estimated as

$$\bar{v} = \frac{1}{N} \sum_{\omega \in S} v^\omega.$$

A detailed discussion of the particular risk measure is given in Infanger (1996) [4].

3. Single-Stage Stochastic Optimization

Based on the computation of the net present values r_ℓ^ω for all initial funding options $\ell \in L$ and all scenarios $\omega \in S$, we optimize the funding mix with respect to expected returns and risk solving the (stochastic) linear program:

$$\begin{array}{lrcrcll}
\min & & & \frac{1}{N}\sum v^\omega & = & \bar{v} & \\
\text{s.t.} & & & & & & \\
& \sum_\ell r_\ell^\omega x_\ell & + & v^\omega & \geq & u, & \omega \in S \\
& \sum_\ell \bar{r}_\ell x_\ell & & & \geq & \rho & \\
& \sum_\ell x_\ell & & & = & 1 & \\
& x_\ell & & & \geq & 0. &
\end{array}$$

The parameter ρ is a pre-specified value that the expected net present value of the portfolio should exceed to be equal to. Clearly, $\rho \leq \rho^{\max} = \max_\ell \{\bar{r}_\ell\}$. Using the model we trace out an efficient frontier starting with $\rho = \rho^{\max}$ and successively reducing ρ until $\rho = 0$, each time solving the linear program to obtain the portfolio with the minimum risk $\bar{v}$ corresponding to each value of ρ.

The single-stage stochastic optimization model optimizes funding strategies based on decision rules defined over the whole planning horizon of $T = 360$ periods, where the net present value of each funding strategy using initially bond ℓ and applying the decision rules is estimated using simulation.

4. Multi-Stage Stochastic Optimization

In the following we will relax the application of decision rules at certain decision points within the planning horizon, and optimize the funding decisions at these points. This leads to a multi-stage stochastic model formulation.

We partition the planning horizon $< 0, T >$ into n sub horizons $< T_1, T_2 >$, $< T_2, T_3 >$, $\ldots$, $< T_n, T_{n+1} >$, where $T_1 = 0$, and $T_{n+1} = T$. For the experiment, we consider $n = 3$, and partition at $T_1 = 0$, $T_2 = 12$, $T_3 = 60$, and $T_4 = 360$. We label the decision points at time $t = T_1$ as stage 1, at time $t = T_2$ as stage 2, and at time $t = T_3$ as stage 3 decisions. Funding obtained at the decision stages is labelled as $\ell_1 \in L_1$, $\ell_2 \in L_2$, and $\ell_3 \in L_3$ according to the decision stages. In between the explicit decision points, where funding is subject to optimization, we apply the design rules defined above.

Instead of interest rate paths in the single-stage model, we now use an interest rate tree with nodes at $T_1 = 0$, $T_2 = 12$, $T_3 = 60$ and $T_4 = T = 360$ months. We consider $|S_2|$ paths $\omega_2 \in S_2$ between $t = T_1$ and $t = T_2$; for each node $\omega_2 \in S_2$ we consider $|S_3|$ paths $\omega_3 \in S_3$ between $t = T_2$ and $t = T_3$; for each node $(\omega_2, \omega_3) \in \{S_2 \times S_3\}$ we consider $|S_4|$ paths $\omega_4 \in S_4$ between $t = T_3$ and $t = T_4$. Thus the tree has $|S_2 \times S_3 \times S_4|$ end points. We may interpret as $S = \{S_2 \times S_3 \times S_4\}$ and as $\omega = (\omega_2, \omega_3, \omega_4)$. Thus a particular path through the tree is now labeled as $\omega = (\omega_2, \omega_3, \omega_4)$ using an index for each partition.

The simulation runs for each partition of the planning horizon are carried out in such a way that the dynamics of the interest rate process and the prepayment function are fully carried forward from one partition to the next. Since the interest rate model and the prepayment function include many lagged terms, this requires the storing of 64 state variables. The application of dynamic programming for solving the multi-stage program is therefore not feasible.

Let $I_{\tau t}$ be the discount factor of period t, discounted to period τ, i.e.,

$$I_{\tau t} = \prod_{k=\tau+1}^{t} (1 + i_k(\text{short})), \quad t > \tau, \quad I_{\tau\tau} = 1,$$

where we used the short rate at time t, $i_t(\text{short})$), for discounting.

Let L_1 be the set of funding instruments available at time $= T_1$. Funding obtained at time $t = T_1$ may mature or be called during the first partition, i.e., before or at time $t = T_2$, during the second partition, i.e., after time $t = T_2$ and before or at time $t = T_3$, or during the third partition, i.e., after time $t = T_3$ and before or at time $t = T_4$. We denote the set of funding instruments issued at time $t = T_1$ and matured or called during the first partition of the planning horizon as $L_{11}^{\omega_2}$, the set of funding instruments issued at time $t = T_1$ and matured or called during the second partition of the planning horizon as $L_{12}^{\omega_2\omega_3}$, and the set of funding instruments issued at time $t = T_1$ and matured or called during the third partition of the planning horizon as $L_{13}^{\omega_2\omega_3\omega_4}$. Clearly, $L_1 = L_{11}^{\omega_2} \cup L_{12}^{\omega_2\omega_3} \cup L_{13}^{\omega_2\omega_3\omega_4}$, for each $(\omega_2, \omega_3, \omega_4) \in \{S_2 \times S_3 \times S_4\}$. Similarly, we denote the set of funding instruments issued at time $t = T_2$ and matured or called during the second partition of the planning horizon as $L_{22}^{\omega_2\omega_3}$, and the set of funding instruments issued at time $t = T_2$ and matured or called during the third partition of the planning horizon as $L_{23}^{\omega_2\omega_3\omega_4}$. Clearly, $L_2 = L_{22}^{\omega_2\omega_3} \cup L_{23}^{\omega_2\omega_3\omega_4}$, for each $(\omega_2, \omega_3, \omega_4) \in \{S_2 \times S_3 \times S_4\}$. Finally, we denote the set of funding instruments issued at time $t = T_3$ and matured or called during the third partition of the planning horizon as L_{33}. Clearly, $L_3 = L_{33} = L_{33}^{\omega_2\omega_3\omega_4}$, for each $(\omega_2, \omega_3, \omega_4) \in \{S_2 \times S_3 \times S_4\}$.

For all funding instruments $\ell_1 \in L_1^{\omega_2}$ initiated at time $t = 0$ that mature or are called during the first partition between $t = 0$ and $t = T_2$, we obtain the net present values

$$r^{\omega_2}_{\ell_1(11)} = \sum_{t=0}^{T_2} \frac{P^{\ell_1\omega_2}_t}{I^{\omega_2}_{0t}},$$

for all funding instruments $\ell_1 \in L^{\omega_2\omega_3}_{12}$ initiated at time $t = 0$ that mature or are called during the second partition between $t = T_2 + 1$ and $t = T_3$, we obtain the net present values

$$r^{\omega_2,\omega_3}_{\ell_1(12)} = \frac{1}{I^{\omega_2}_{0T_2}} \sum_{t=T_2+1}^{T_3} \frac{P^{\ell_1\omega_3}_t}{I^{\omega_3}_{T_2t}},$$

and all initial funding instruments $\ell_1 \in L^{\omega_2\omega_3\omega_4}_{13}$, initiated at time $t = 0$ that mature or are called during the third partition between $t = T_3 + 1$ and $t = T_4$, we obtain the net present values

$$r^{\omega_2,\omega_3,\omega_4}_{\ell_1(13)} = \frac{1}{I^{\omega_2}_{0T_2}} \frac{1}{I^{\omega_3}_{T_2T_3}} \sum_{t=T_3+1}^{T_4} \frac{P^{\ell_1\omega_4}_t}{I^{\omega_4}_{T_3t}}.$$

For all funding instruments $\ell_2 \in L^{\omega_2\omega_3}_{22}$, initiated at time $t = T_2$ that mature or are called during the second partition between $t = T_2 + 1$ and $t = T_3$ we obtain the net present values

$$r^{\omega_2,\omega_3}_{\ell_2(22)} = \frac{1}{I^{\omega_2}_{0T_2}} \sum_{t=T_2+1}^{T_3} \frac{P^{\ell_2\omega_3}_t}{I^{\omega_3}_{T_2t}}.$$

and for all funding instruments $\ell_2 \in L^{\omega_2\omega_3}_{23}$, initiated at time $t = T_2$, that mature or are called during the third partition between $t = T_3 + 1$ and $t = T_4$, we obtain the net present values

$$r^{\omega_2,\omega_3,\omega_4}_{\ell_2(23)} = \frac{1}{I^{\omega_2}_{0T_2}} \frac{1}{I^{\omega_3}_{T_2T_3}} \sum_{t=T_3+1}^{T_4} \frac{P^{\ell_2\omega_4}_t}{I^{\omega_4}_{T_3t}}.$$

We obtain for all initial funding $\ell \in L_{33}$ initiated at time $t = T_3$ the net present values

$$r^{\omega_2,\omega_3,\omega_4}_{\ell_3(33)} = \frac{1}{I^{\omega_2}_{0T_2}} \frac{1}{I^{\omega_3}_{T_2T_3}} \sum_{t=T_3+1}^{T_4} \frac{P^{\ell_3\omega_4}_t}{I^{\omega_4}_{T_3t}}.$$

Let $\omega = (\omega_2, \omega_3, \omega_4)$, $S = \{S_2 \times S_3 \times S_4\}$, and $N = |S|$. Let $R^{\omega}_{\ell_1} = r^{\omega_2}_{\ell_1(11)} + r^{\omega_2\omega_3}_{\ell_1(12)} + r^{\omega_2\omega_3\omega_4}_{\ell_1(13)}$, $R^{\omega}_{\ell_2} = r^{\omega_2\omega_3}_{\ell_2(22)} + r^{\omega_2\omega_3\omega_4}_{\ell_2(23)}$, and $R^{\omega}_{\ell_3} = r^{\omega_2\omega_3\omega_4}_{\ell_3(33)}$. Let x_{ℓ_1} be the amount of funding in instrument $\ell_1 \in L_1$ issued at time $t = T_1$, x_{ℓ_2}, be the amount of funding in instrument $\ell_2 \in L_2$ issued at time $t = T_2$, and x_{ℓ_3}, be the amount of funding in instrument $\ell_3 \in L_3$ issued at time $t = T_3$. Based on the computation of the net present values $r^{\omega_2}_{\ell_1}$, $r^{\omega_2\omega_3}_{\ell_1}$, $r^{\omega_2\omega_3\omega_4}_{\ell_1}$, $r^{\omega_2\omega_3}_{\ell_2}$, $r^{\omega_2\omega_3\omega_4}_{\ell_2}$, and

$r_{\ell_3}^{\omega_2\omega_3\omega_4}$, we optimize the funding mix solving the multi-stage (stochastic) linear program:

$$
\begin{array}{rcccccccccl}
\min & & & & & E\ v^\omega & & & = & \bar{v} \\
\text{s.t.} & & & & & & & & & \\
\sum_{\ell_1 \in L_1} x_{\ell_1} & & & & & & & & = & 1 \\
-\sum_{\ell_1 \in L_{11}^{\omega_2}} x_{\ell_1} & + & \sum_{\ell_2 \in L_2} x_{\ell_2}^{\omega_2} & & & & & & = & 0 \\
-\sum_{\ell_1 \in L_{12}^{\omega_2\omega_3}} x_{\ell_1} & - & \sum_{\ell_2 \in L_{22}^{\omega_2\omega_3}} x_{\ell_2}^{\omega_2} & + & \sum_{\ell_3 \in L_3} x_{\ell_3}^{\omega_2\omega_3} & & & & = & 0 \\
\sum_{\ell_1 \in L_1} R_{\ell_1}^\omega x_{\ell_1} & + & \sum_{\ell_2 \in L_2} R_{\ell_2}^\omega x_{\ell_2}^{\omega_2} & + & \sum_{\ell_3 \in L_3} R_{\ell_3}^\omega x_{\ell_3}^{\omega_2\omega_3} & & - & w^\omega & = & 0 \\
 & & & & & v^\omega & + & w^\omega & \geq & u \\
 & & & & & & & E\ w^\omega & \geq & \rho \\
x_{\ell_1}, & & x_{\ell_2}^{\omega_2}, & & x_{\ell_3}^{\omega_2\omega_3}, & v^\omega & & & \geq & 0,
\end{array}
$$

where $E\ w^\omega = \frac{1}{N} \sum w^\omega$ is the estimate of the expected net present value and $E\ v^\omega = \frac{1}{N} \sum v^\omega$ is the estimate of the risk. The parameter ρ is a pre-specified value for the expected net present value of the portfolio. We define as $\rho^{\max}$ the maximum value of ρ that can be assumed without the linear program becoming infeasible. Starting with $\rho = \rho^{\max}$ we trace out an efficient frontier by successively reducing ρ from $\rho = \rho^{\max}$ till $\rho = 0$ and computing for each level of ρ the corresponding value of risk $\bar{v}$ by solving the multi-stage stochastic linear program. The quantity $\rho^{\max}$, the maximum expected net present value without considering risk, can be obtained by solving the linear program

$$
\begin{array}{rcccccccl}
\max & & & & & & E\ w^\omega & = & \rho^{\max} \\
\text{s.t.} & & & & & & & & \\
\sum_{\ell_1 \in L_1} x_{\ell_1} & & & & & & & = & 1 \\
-\sum_{\ell_1 \in L_{11}^{\omega_2}} x_{\ell_1} & + & \sum_{\ell_2 \in L_2} x_{\ell_2}^{\omega_2} & & & & & = & 0 \\
-\sum_{\ell_1 \in L_{12}^{\omega_2\omega_3}} x_{\ell_1} & - & \sum_{\ell_2 \in L_{22}^{\omega_2\omega_3}} x_{\ell_2}^{\omega_2} & + & \sum_{\ell_3 \in L_3} x_{\ell_3}^{\omega_2\omega_3} & & & = & 0 \\
\sum_{\ell_1 \in L_1} R_{\ell_1}^\omega x_{\ell_1} & + & \sum_{\ell_2 \in L_2} R_{\ell_2}^\omega x_{\ell_2}^{\omega_2} & + & \sum_{\ell_3 \in L_3} R_{\ell_3}^\omega x_{\ell_3}^{\omega_2\omega_3} & - & w^\omega & = & 0 \\
x_{\ell_1}, & & x_{\ell_2}^{\omega_2}, & & x_{\ell_3}^{\omega_2\omega_3} & & & \geq & 0.
\end{array}
$$

Note that the model formulation presented above does not consider calling of callable bonds as subject to optimization at the decision stages; calling of callable bonds is handled through the calling rule as part of the simulation. However, optimizing the calling of callable bonds at the decision stages requires only a minor extension to the model formulation, but is not discussed here.

5. Duration and Convexity

Since the payments from a mortgage pool are not constant, but due to the prepayments depend on the interest rate term structure and its history since the inception of the pool, an important issue arises as to how the net present value (price) of the mortgage pool changes as a result of a small change in interest rates. The same issue arises for all funding instruments, namely how bond prices change as a result of small changes in yield. The issue is especially important in the case of callable bonds. Using methods of traditional finance funding is carried out in such a way that a change in the price of the mortgage pool is closely matched by the negative change in the price of the funding portfolio such that the change of the total (mortgage pool and funding portfolio) is close to zero. In order to calculate the changes in expected net present value due to changes in interest rates one usually resorts to first and second order approximations, where the first order (linear, or delta) approximation is called the duration and the second order (quadratic, or gamma) approximation is called the convexity. While the duration and convexity of non-callable bonds could be calculated analytically, the duration and convexity of a mortgage pool and callable bonds can only be estimated through simulation. We refer to the terms effective duration and effective convexity if estimated through simulation.

5.1. Effective Duration and Convexity

Let

$$p = \sum_{t=0}^{T} \frac{P_t}{I_t},$$

be the net present value of the payments from the mortgage pool, where $P_t = \alpha_t + \beta_t$ and I_t the discount factor using the short rate for discounting. We compute p^ω, for scenarios $\omega \in S$, using Monte Carlo simulation runs, and we calculate the expected net present value (price) of the payments of the mortgage pool as

$$\bar{p} = \frac{1}{N} \sum_{\omega \in S} p^\omega.$$

Note that the payments $P_t = P_t(i_k,\ k = 0, \ldots, t)$ depend on the the interest rate term structure and its history up to period t, where i_t denotes the vector of interest rates for different maturities at time t. Writing explicitely the dependency,

$$\bar{p} = \bar{p}(i_t, \quad t = 0, \ldots, T).$$

We now define

$$\bar{p}_+ = \bar{p}(i_t + \Delta)), \quad t = 0, \ldots, T),$$

the net present value of the payments of the mortgage pool for an upward shift of all interest rates by Δ% and

$$\bar{p}_- = \bar{p}(i_t - \Delta), \quad t = 0, \ldots, T),$$

the net present value of the payments of the mortgage pool for a downward shift of all interest rates by Δ%, where Δ is a shift of, say, 1% point of the entire term structure at all periods $t = 1, \ldots, T$.

Using the three points $\bar{p}_-$, $\bar{p}$, $\bar{r}_+$, and the corresponding interest rate shifts $-\Delta$, 0, $+\Delta$, we compute the effective duration of the mortgage pool as

$$\text{dur} = \frac{\bar{p}_- - \bar{p}_+}{2\Delta\bar{p}},$$

and the effective convexity of the mortgage pool as

$$\text{con} = \frac{\bar{p}_- + \bar{p}_+ - 2\bar{p}}{100\Delta^2\bar{p}}.$$

The quantities of the effective duration and effective convexity represent a local first (duration) and second order (duration and convexity) Taylor approximation of the net present value of the payments of the mortgage pool as a function of yield. The approximation considers the effects on a constant shift of the entire yield curve at every point t, $t = 1, \dots, T$. The way it is computed, we expect a positive value for the duration, meaning that decreasing interest rates result in a larger expected net present value and increasing interest rates result in a smaller expected net present value. We also expect a negative value for the convexity, meaning that the function of price versus yield is locally concave.

In analogous fashion we compute the duration dur_ℓ and the convexity con_ℓ for all funding instruments ℓ. Let

$$p_\ell = \sum_{t=0}^{m_\ell} \frac{d_t^\ell}{I_t}$$

be the net present value of the payments of the bond ℓ, where d_t^ℓ represents the payments of bond ℓ till maturity or until it is called. We compute p_ℓ^ω, using Monte Carlo simulation runs $\omega \in S$, and we calculate the expected net present value (price) of the payments of the bond ℓ as

$$\bar{p}_\ell = \frac{1}{N} \sum_{\omega \in S} p_\ell^\omega.$$

We calculate

$$\bar{p}_{\ell+} = \bar{p}_\ell(i_t + \Delta)), \quad t = 0, \dots, T),$$

and

$$\bar{p}_{\ell-} = \bar{p}_\ell(i_t - \Delta)), \quad t = 0, \dots, T),$$

the expected net present values for an upwards and downwards shift of interest rates, respectively. Analogously to the mortgage pool we obtain the duration of bond ℓ as

$$\mathrm{dur}_\ell = \frac{\bar{p}_{\ell-} - \bar{p}_{\ell+}}{2\Delta \bar{p}_\ell},$$

and its convexity as

$$\mathrm{con}_\ell = \frac{\bar{p}_{\ell-} + \bar{p}_{\ell+} - 2\bar{p}_\ell}{100\Delta^2 \bar{p}_\ell}.$$

Since in the case of non-callable bonds the payments d_t^ℓ are fixed, changes in expected net present value due to changes in interest rates are influenced by the discount factor only. For non-callable bonds we expect a positive value for duration meaning that increasing interest rates imply a smaller bond value, and decreasing interest rates imply a larger bond value. We expect a positive value for convexity meaning that the function of expected net present value versus interst rates ins locally convex. In the case of callable bonds the behavior of the function of expected net present value verus interest rates is influenced besides the discount rate by the calling rule. If interest rates decrease the bond may be called and the principal returned. The behavior of callable bonds is similiar to that of mortgage pools in that we expect a positive value for duration and a negative value for convexity.

5.2. Duration and Convexity in the Single-Stage Model

In the single-stage case we add the following pair of constraints to the linear program

$$\sum_\ell \mathrm{dur}_\ell x_\ell - \mathrm{dg} = \mathrm{dur}$$

$$\sum_{\ell} \mathrm{con}_\ell x_\ell - \mathrm{cg} = \mathrm{con},$$

where

$$-\mathrm{dg}^{\max} \leq \mathrm{dg} \leq \mathrm{dg}^{\max}, \quad -\mathrm{cg}^{\max} \leq \mathrm{cg} \leq \mathrm{cg}^{\max},$$

where the variable dg accounts for the duration gap, cg accounts for the convexity gap, $\mathrm{dg}^{\max}$ represents a predefined upper bound on the absolute value of the duration gap, and $\mathrm{cg}^{\max}$ a predefined upper bound on the absolute value of the convexity gap. The formulation allows one to constrain the absolute value of the duration- and convexity gap to any specified level, to the extent as the single-stage stochastic program remains feasible.

5.3. Duration and Convexity in the Multi-Stage Model

In the multi-stage model we wish to constrain the duration and convexity gap not only in the first stage, but at any decision stage, and in any scenario. Thus, in the four-stage model discussed above we have one pair of constraint for the first stage, $\omega_2 \in S_2$ paits of constraints in the second stage, and $(\omega_2, \omega_3) \in \{S_2 \times S_3\}$ pairs of constraints in the third stage. Accordingly, we need to compute the duration and the convexity for the mortgage pool and all funding instruments at any decision point in all stages from one to three.

Let dur_1, and con_1 be the duration and convexity of the mortgage pool in the first stage, and dur_{ℓ_1} and con_{ℓ_1} be the duration and convexity of the funding instruments $\ell_1 \in L_1$ available at the first period. Let $\mathrm{dur}_2^{\omega_2}$, and $\mathrm{con}_2^{\omega_2}$ be the duration and convexity of the mortgage pool in the second stage, $\mathrm{dur}_{\ell_2}^{\omega_2}$ and $\mathrm{con}_{\ell_2}^{\omega_2}$ be the duration and convexity of the funding instruments $\ell_2 \in L_2$ in the second stage, and $\mathrm{dur}_{\ell_1(12)}^{\omega_2}$ and $\mathrm{con}_{\ell_1(12)}^{\omega_2}$ be the duration and convexity of the funding instruments $\ell_1 \in L_{12}^{\omega_2}$ from the first stage that are still available in the second stage. Let $\mathrm{dur}_3^{\omega_2,\omega_3}$, and $\mathrm{con}_3^{\omega_2,\omega_3}$ be the duration and convexity of the mortgage pool in the third stage, $\mathrm{dur}_{\ell_3}^{\omega_2,\omega_3}$ and $\mathrm{con}_{\ell_3}^{\omega_2,\omega_3}$ be the duration and convexity of the funding instruments $\ell_3 \in L_3$ in the third stage, $\mathrm{dur}_{\ell_2(23)}^{\omega_2,\omega_3}$ and $\mathrm{con}_{\ell_2(23)}^{\omega_2,\omega_3}$ be the duration and convexity of the funding instruments $\ell_2 \in L_{23}^{\omega_2,\omega_3}$ issued in second stage that are still available in the third stage, and $\mathrm{dur}_{\ell_1(13)}^{\omega_2,\omega_3}$ and $\mathrm{con}_{\ell_1(13)}^{\omega_2,\omega_3}$ be the duration and convexity of the funding instruments $\ell_1 \in L_{13}^{\omega_2,\omega_3}$ from the first stage that are still available in the third stage.

In the multi-stage case, considering up to four stages, we additionally consider in the first stage

$$\sum_{\ell_1 \in L_1} \mathrm{dur}_{\ell_1} x_{\ell_1} - \mathrm{dg}_1 = \mathrm{dur}_1$$

$$\sum_{\ell_1 \in L_1} \mathrm{con}_{\ell_1} x_{\ell_1} - \mathrm{cg}_1 = \mathrm{con}_1,$$

where

$$-\mathrm{dg}^{\max} \leq \mathrm{dg}_1 \leq \mathrm{dg}^{\max}, \quad -\mathrm{cg}^{\max} \leq \mathrm{cg}_1 \leq \mathrm{cg}^{\max},$$

in the second stage for each $\omega_2 \in S_2$,

$$\sum_{\ell_2 \in L_2} \mathrm{dur}_{\ell_2}^{\omega_2} x_{\ell_2}^{\omega_2} + \sum_{\ell_1 \in L_{12}^{\omega_2}} \mathrm{dur}_{\ell_1(12)}^{\omega_2} x_{\ell_1} - \mathrm{dg}_2^{\omega_2} = \mathrm{dur}_2^{\omega_2}$$

$$\sum_{\ell_2 \in L_2} \mathrm{con}_{\ell_2}^{\omega_2} x_{\ell_2}^{\omega_2} + \sum_{\ell_1 \in L_{12\omega_2}} \mathrm{con}_{\ell_1(12)}^{\omega_2} x_{\ell_1} - \mathrm{cg}_2^{\omega_2} = \mathrm{con}_2^{\omega_2},$$

where

$$-\mathrm{dg}^{\max} \le \mathrm{dg}_2^{\omega_2} \le \mathrm{dg}^{\max}, \quad -\mathrm{cg}^{\max} \le \mathrm{cg}_2^{\omega_2} \le \mathrm{cg}^{\max},$$

and in the third stage for each $(\omega_2, \omega_3) \in \{S_2 \times S_3\}$,

$$\sum_{\ell_3 \in L_3} \mathrm{dur}_{\ell_3}^{\omega_2,\omega_3} x_{\ell_3}^{\omega_2,\omega_3} + \sum_{\ell_2 \in L_{23}^{\omega_2,\omega_3}} \mathrm{dur}_{\ell_2(23)}^{\omega_2,\omega_3} x_{\ell_2}^{\omega_2} + \sum_{\ell_1 \in L_{13}^{\omega_2,\omega_3}} \mathrm{dur}_{\ell_1(13)}^{\omega_2,\omega_3} x_{\ell_1} - \mathrm{dg}_3^{\omega_2,\omega_3} = \mathrm{dur}_3^{\omega_2,\omega_3}$$

$$\sum_{\ell_3 \in L_3} \mathrm{con}_{\ell_3}^{\omega_2,\omega_3} x_{\ell_3}^{\omega_2,\omega_3} + \sum_{\ell_2 \in L_{23}^{\omega_2,\omega_3}} \mathrm{con}_{\ell_2(23)}^{\omega_2,\omega_3} x_{\ell_2}^{\omega_2} + \sum_{\ell_1 \in L_{13}^{\omega_2,\omega_3}} \mathrm{con}_{\ell_1(13)}^{\omega_2,\omega_3} x_{\ell_1} - \mathrm{cg}_3^{\omega_2,\omega_3} = \mathrm{con}_3^{\omega_2,\omega_3},$$

where

$$-\mathrm{dg}^{\max} \le \mathrm{dg}_3^{\omega_2,\omega_3} \le \mathrm{dg}^{\max}, \quad -\mathrm{cg}^{\max} \le \mathrm{cg}_3^{\omega_2,\omega_3} \le \mathrm{cg}^{\max}.$$

The variables dg_1, cg_1, $\mathrm{dg}_2^{\omega_2}$, $\mathrm{cg}_2^{\omega_2}$, $\mathrm{dg}_3^{\omega_2,\omega_3}$, and $\mathrm{cg}_3^{\omega_2,\omega_3}$ account for the duration and convexity gap, respectively, in the decsision stages one, two and three, and in each of the scenarios $\omega_2 \in S_2$ and $(\omega_2, \omega_3) \in \{S_2 \times S_3\}$. At each decision node the absolute values of the duration and convexity gap are constrained by $\mathrm{dg}^{\max}$ and $\mathrm{cg}^{\max}$, respectively. The formulation allows one to constrain the duration and convexity gap to any specified level, to the extent as the multi-stage stochstic linear program remains feasible.

6. Computational Results

6.1. Data Assumptions

For the experiment we used three data sets, based on different initial yield curves, labelled as

- Normal,
- Flat, and
- Steep.

The data represent assumptions about the initial yield curve, the model parameters of the interest rate model, and the prepayment function, assumptions about the funding instruments, assumptions about refinancing and the calling rule, and the planning horizon and its partitioning.

The following Table 1 represents the initial yield curve (corresponding to a zero coupon bond) for each data set. For each data set the mortgage contract rate is assumed 1 percent point above the 10-year rate (labelled as "y10").

For the experiment we consider 16 different funding instruments. The following Table 2 represents the maturity, the time after the instrument may be called, and the initial spread over the corresponding zero coupon bond (of the same maturity) for each instrument and for each of the data sets. For example, "y03nc1" refers to a callable bond with a maturity of 3 years (36 months) and callable after 1 year (12 months); based on the data set "Normal", it could be issued initially (at time $t = 0$) at a rate of 6.12% + 0.36% = 6.48%.

Table 1: Initial yield curves

Label	Maturity (months)	Interest Rate (%)		
		Normal	Flat	Steep
m03	3	5.18	5.88	2.97
m06	6	5.31	6.38	3.16
y01	12	5.55	6.76	3.36
y02	24	5.97	7.06	4.18
y03	36	6.12	7.36	4.58
y05	60	6.33	7.56	5.56
y07	84	6.43	7.59	5.97
y10	120	6.59	7.64	6.36
y30	360	6.87	7.76	7.20

Parameters for the interest rate model, and the prepayment function can be found in the appendix.

We represented the results for a pool of $100M. As the target for risk u we used the maximum expected net present value obtained using single-stage optimization, i.e., we considered risk as the expected net present value below this target, respectively for each data set. For example, the target for risk equals $u = 10.5$M for the "Normal" data set, $u = 11.3$M for the "Flat" data set, and $u = 18.6$M for the "Steep" data set.

We first used single-stage optimization using $N = 300$ interest rate paths. These results are not documented. Then, in order to more accurately compare single-stage and multi-stage optimization we used a tree with $N = 4000$ paths, where the sample sizes in each stage are $|S_2| = 10$, $|S_3| = 20$ and $|S_4| = 20$. For this tree the multi-stage linear program has 8213 rows, 11377 columns and 218512 non-zero elements. The program can easily be solved on a modern personal computer in fast time. Also the simulation runs to obtain the coefficients for the linear program can easily be carried out on a modern personal computer.

6.2. Results Single-Stage Optimization

As a base case for the experiment we computed the efficient frontier for each of the data sets using the single-stage model. Figure 1 represents the result for the "Normal", Figure 2 for the "Flat", and Figure 3 for the "Steep" data set in comparison to the efficient frontiers obtained from the multi-stage model. The results closely resemble the efficient frontiers obtained from using single-stage optimization and 300 interest rate paths. While the efficient frontiers for data sets "Normal" and "Flat" have the typical shape one would expect, i.e., steep at low levels of risk and bending more flat with increasing risk, it is interesting to note that the efficient frontier for data set "Steep" is very steep at all levels of risk.

As a base case we look at the optimal funding strategy at the point of 95% maximum expected net present value. In the graphs of the efficient frontiers this is the second point starting from the maximum expected net present value point. The funding strategies are represented in Table 6 for the "Normal" data set, in Table 7 for the "Flat" data set, and in Table 8 for the "Steep" data set.

In the "Normal" case, the optimal initial funding mix consists of 85.3% six month non-

Table 2: Spreads for different funding instruments

Label	Maturity (months)	Callable after (Months)	Spread (%)		
			Normal	Flat	Steep
m03n	3		0.17	0.15	0.19
m06n	6		0.13	0.10	0.13
y01n	12		0.02	0.08	0.08
y02n	24		0.00	0.11	0.1
y03n	36		0.04	0.18	0.12
y03nc1	36	12	0.36	0.61	0.22
y05n	60		0.08	0.21	0.13
y05nc1	60	12	0.65	0.84	0.22
y05nc3	60	36	0.32	0.41	0.18
y07n	84		0.17	0.22	0.15
y07nc1	84	12	0.90	0.95	0.45
y07nc3	84	36	0.60	0.73	0.35
y10n	120		0.22	0.29	0.22
y10nc1	120	12	1.10	1.28	0.57
y10nc3	120	36	0.87	0.94	0.48
y30n	360		0.30	0.32	0.27

callable debt, and 14.7% seven year non-callable debt. The risk level of this strategy is at 3.1M NPV.

In the "Flat" case, the optimal initial funding mix consists of 91.1% three months non-callable debt, and 8.9% seven year non-callable debt. The risk level of this strategy is at 3.1M NPV.

In the "Steep" case, the initial funding mix consists of 100.0% three months non-callable debt (short-term debt). The risk level of this strategy is at 2.6M NPV.

6.3. Results Multi-Stage

Figure 1 represents the result for the "Normal", Figure 2 for the "Flat", and Figure 3 for the "Steep" data set. In addition to the multi-stage efficient frontier, each graph also contains the corresponding single-stage efficient frontier for better comparison. The results show impressive differences in the risk and expected net present value profile of multi-stage versus single-stage funding strategies. For any of the three data sets, the efficient frontier obtained from the multi-stage model is significantly north-west of the one of the single-stage model, i.e., multi-stage optimization yields a larger expected net present value at the same or a smaller level of risk.

In the "Normal" case, the minimum risk of the single-stage curve is about 2.7M NPV. Since the efficient frontier is very steep at low levels of risk we use the point of 85% maximum risk as the one with the lowest risk, even if the risk could be further decreased by a very small insignificant amount. The maximum expected net present value point of the multi-stage curve has a risk of about 2.4M NPV. At this level of risk (noting that the risk at the minimum risk point of the single stage model is larger than the risk at the maximum risk point of the multi-stage model), the expected net present value on the single stage efficient frontier is about 8.9M NPV, versus the expected net present value of the multi-stage curve is about 13.6M NPV, which represents an improvement of 52.8%.

In the "Flat" case, we cannot compare the expected net present values from the single-stage and multi-stage model at the same level of risk, because each point on the efficient frontier of the multi-stage model has a smaller level of risk than the point with the smallest risk on the efficient frontier of the single-stage model. We compare the point with the smallest risk of 3.0M NPV on the single-stage efficient frontier with the one with the largest risk of 2.5M NPV on the multi-stage efficient frontier. The difference in the net present value of the two points is 13.6M NPV (multi-stage) versus 9.6M NPV (single-stage), which represents an improvement of 41.7%.

In the "Steep" case, the situation is similar to the one in the "Flat" case as there is no point with the same level of risk on the multi-stage versus the single-stage efficient frontier. Again we compare the point with the smallest risk of 2.6M NPV on the single-stage efficient frontier with the one with the largest risk of 1.9M NPV on the multi-stage efficient frontier. The difference in the net present value of the two points is 20.4M NPV (multi-stage) versus 18.6M NPV (single-stage), which represents an improvement of 9.7%.

Again we look at the funding strategies at the point of 95% maximum expected return (on the efficient frontier the second point starting from the maximum expected return point). The funding strategies are represented in Table 9 for the "Normal", in Table 10 for the "Flat", and in Table 11 for the "Steep" data set.

In the "Normal" case, the initial funding mix consists of 78.5% six month non-callable debt, and 21.5% five year debt callable after one year. After one year the 78.5% six month debt (that according to the decision rules was re-financed through short term debt and is for disposition in the second stage) and, if called in certain scenarios, also the five year callable debt are refunded through various mixes of short-term debt, three year, five year and ten year callable and non-callable debt. In one scenario, labeled "w20001", when interest rates fell to a very low level, the multi-stage model resorted to funding with 30 year non-callable debt in order to secure the very low rate for the future. The risk associated to this strategy is 1.85M NPV and the expected net present value is 12.9M NPV. The corresponding (95% maximum net present value) strategy of the single-stage model, discussed above, had a risk of 3.1M NPV and a net present value of 10.0M NPV. Thus the multi-stage strategy exhibits 57.4% of the risk of the single-stage strategy, and a 29% larger expected net present value.

In the "Flat" case, the initial funding mix consists of 50.9% three months non-callable debt, and 49.1% six month non callable debt. After one year the entire portfolio is refunded through various mixes of short-term, three year, and ten year callable and non-callable debt. Like in the "Normal" case above, in one scenario, labeled "w20001", when interest rates were very low, the multi stage model used thirty year non-callable debt to secure the low rates, however here at the extent of 60.3% in combination with 39.7% ten year non-callable debt. The risk associated to this strategy is at 2.1M NPV, the expected net present value is 12.9M NPV. The corresponding single-stage strategy had a risk of 3.1M NPV and an expected net present value of 10.7M NPV. Thus the multi-stage strategy exhibits 68% of the risk of the single-stage strategy, and a 20.6% larger expected net present value.

In the "Steep" case, the initial funding consists of 100% three month non-callable debt. After one year the portfolio is refunded through various mixes of short-term, three year and five year noon-callable, and ten year callable and non-callable debt. Again in one scenario, labeled "w20001", when interest rates are very low, thirty year non-callable debt is used, here at the extent of 12.2% in combination with a mix of three year, five year and ten year non-callable debt. The risk associated with this strategy is 1.6M NPV, the expected net present value is 19.4M NPV. The corresponding single-stage strategy had a risk of 2.6M NPV and an expected net present value of 18.6M NPV. Thus the multi-stage strategy exhibits 62% of the risk of the single-stage strategy, and a 4.3% larger expected net present value.

Summarizing, the results demonstrate that multi-stage stochastic optimization yields significantly larger net present values at the same or at a lower level of risk and significantly different funding strategies, compared to single-stage optimization. Using multi-stage stochastic optimization for determining the funding of mortgage pools will lead in the average to significant profits, compared to using single-stage funding strategies.

6.4. Results Duration and Convexity

Funding a mortgage pool by a portfolio of bonds that matches the (negative) value of duration and convexity, the expected net present value of the total of mortgage pool and bonds is invariant to small changes in interest rates. However, duration and convexity give only a local approximation, and the portfolio needs to be updated as time goes on and interest rates change. The duration and convexity hedge is one-dimensional, since it considers only changes of the whole yield curve at the same amount and does not consider different shifts for different maturities. The mult-stage stochastic optimization model takes into account multi-dimensional changes of interest rates and considers the entire distribution of possible yield curve developments. In this Section we quantify the difference between duration and convexity hedging versus hedging using the single- and multi-stage stochastic optimization model. The following Table 3 gives the initial (first stage) values for duration and convexity (as obtained from the simulation runs) for the mortgage pool and for all funding instruments for each of the three yield curve cases "Normal", "Flat", and "Steep". In each of the three yield curve cases the mortgage pool exhibits a positive value for duration and a negative value for convexity. All funding instruments have positive values for duration, non-callable bonds exhibit a positive value for convexity and callable bonds show a negative value for convexity. Note as an exception the positive value for convexity of bond "y07nc1".

To both the single-stage and multi-stage stochastic optimization model we added constraints that at any decision point the duration and the convexity of the mortgage pool and the funding portfolio are as close as possible. Maximizing expected returns we successively reduced the gap in duration and convexity between the mortgage pool and the funding portfolio. We started from the unconstrained case (the maximum expected return – maximum risk case from the efficient frontiers discussed above) and reduced first the duration gap and consequently the convexity gap, where we understand as duration gap the absolute value of the difference in duration between the mortgage pool and the funding portfolio, and as convexity gap the absolute value of the difference in convexity between the mortgage pool and the funding portfolio. We will discuss the results with respect to the downside risk measure (expected value of returns below a certain target) as discussed in Section 2.9, and also with respect to the standard deviation of the returns.

Table 3: Initial Duration and Convexity

Label	normal		flat		steep	
	dur.	conv.	dur.	conv.	dur.	conv.
mortg	3.442	-1.880	2.665	-1.548	2.121	-2.601
m03n	0.248	0.001	0.247	0.001	0.249	0.001
m06n	0.492	0.003	0.491	0.003	0.495	0.003
y01n	0.971	0.010	0.964	0.010	0.982	0.011
y02n	1.882	0.038	1.861	0.038	1.917	0.039
y03n	2.737	0.081	2.686	0.079	2.801	0.084
y03nc1	1.596	-0.384	1.376	-0.186	1.338	-0.235
y05n	4.278	0.205	4.160	0.197	4.378	0.212
y05nc1	1.914	-0.810	1.613	-0.700	1.478	-0.632
y05nc3	3.274	-0.012	3.068	-0.111	3.114	-0.241
y07n	5.622	0.367	5.448	0.351	5.769	0.380
y07nc1	2.378	0.128	1.671	0.959	0.898	0.032
y07nc3	3.406	0.269	3.117	-0.298	3.122	-0.230
y10n	7.335	0.656	7.083	0.624	7.538	0.681
y10nc1	1.555	0.246	1.397	-0.052	0.593	-0.119
y10nc3	3.409	-0.160	3.265	-0.346	3.086	-0.446
y30n	13.711	2.880	13.306	2.743	14.204	3.011

6.4.1. Duration and Convexity, Multi-Stage Model

Figure 4 and Figure 5 give the risk-return profile for the case "Normal" with respect to downside risk and standard deviation, respectively. We compared the efficient frontier (already depicted in Figure 1) obtained from minimizing downside risk for different levels of expected returns (labeled "Downside") with risk-return profile obtained from restricting the duration and convexity gap (labeled "Delta Gamma"). For the different levels of duration and convexity gap we maximized expected returns. The unconstraint case with respect to duration and convexity is identical with the point on the efficient frontier with the maximum expected returns. Figure 4 shows that the downside risk increased when decreasing the duration and convexity gap and was significantly larger than the minimized downside risk of the efficient frontier. We are especially interested in comparing the point with the smallest duration and convexity gap with the point with minimum risk from the efficient frontier. The point with the smallest downside risk on the efficient frontier exhibits expected returns of 9.5M NPV and a downside risk of 1.5M NPV. The point with the smallest duration and convexity gap has expected returns of 7.9M NPV and a downside risk of 3.1 M NPV. The point is characterized with a maximum duration gap in the first and second stage of 0.5 and of 1.0 in the third stage, and by a maximum convexity gap of 2.0 in the first and second stage and of 4.0 in the third stage. A further decrease of the convexity gap was not possible and led to an infeasible problem. The actual duration gap in the first stage was -0.5, where the negative value indicates that the duration of the funding portfolio was smaller than the one of the mortgage pool, and the actual convexity gap in the first stage was 1.57, where the positive value indicates that the convexity of the funding portfolio was larger than the one of the mortgage pool. Comparing the points with regard to their performance, restricting the duration and convexity gap led to a decrease of expected returns by 15% and an increase of downside risk by a factor of 2. It is interesting to note that for the point on the efficient frontier with the smallest risk the first-stage duration gap was -0.97 and the first-stage convexity gap was 1.75. Looking at the risk in terms of standard deviation of returns, both minimizing downside risk and controlling the duration and convexity gap led to smaller values of standard deviation. In the unconstrained case, the smallest standard deviation was obtained at the minimum downside risk point at 3.3M NPV and the smallest standard deviation in the constrained case was obtained when the duration and convexity gap was smallest, 3.8M NPV.

Table 12 gives the funding strategy for the minimum downside risk portfolio and Table 13 the funding strategy for the duration and convexity constrained case. Both strategies use six month non-callable debt, five year non-callable debt and five year debt callable after one year for the initial funding. The minimum downside risk portfolio used in addition five year debt callable after three years, while the duration and convexity constrained case used seven year non-callable debt and ten year non-callable debt. In the second stage funding differed significantly in the different scenarios for both funding strategies. In the duration and convexity constrained case the multi-stage optimization model resorted to calling five year debt callable after one year in scenario "w20004" at a fraction and in scenario "w20010" at the whole amount. In the minimum risk case calling of debt other than by applying the decision rules did not happen. In the second stage interest rates were low in scenario "w20001" and "w20008" and were high in scenarios "w20004", 'w20006", and "w20010". The minimum downside risk strategy tended towards more long-term debt when interest rates were low and towards more short-term debt when interests were high. The amounts for each of the scenarios depended on the dynamics of the process and the interest rate distributions. The duration and convexity constrained strategy was less in the position to take advantage of the level of interest rates, and funding was balanced to match the duration and convexity of the mortgage pool.

The results for the case of the "Flat" initial yield curve are very similar. Figure 6 and Figure 7 give the risk-return profile with respect to downside risk and standard deviation, respectively. Again we compared the efficient frontier (already depicted in Figure 2) obtained from minimizing downside risk for different levels of expected returns with the risk-return profile obtained from restricting the duration and convexity gap. Figure 6 shows that the downside risk increased when decreasing the duration and convexity gap and was significantly larger than the minimized downside risk of the efficient frontier. Comparing the point with the smallest duration and convexity gap with the point with minimum risk from the efficient frontier, the point with the smallest downside risk on the efficient frontier has expected returns of 10.2M NPV and a downside risk of 1.9M NPV, and the point with the smallest duration and convexity gap has expected returns of 8.4M NPV and a downside risk of 3.5M NPV. The maximum duration gap in the first and second stage was set to 0.5 and to 1.0 in the third stage, and the maximum convexity gap was set to 2.0 in the first and second stage and to 4.0 in the third stage. The duration and convexity gap could not be decreased further since the problem became infeasible. The actual duration gap in the first stage was -0.5 and the actual convexity gap in the first stage was 1.5. The portfolio with the constrained duration and convexity gap led to a decrease of expected returns by 17.6% and an increase of downside risk by 84%. The point on the efficient frontier with the smallest risk had a the first-stage duration gap of -1.4 and a first-stage convexity gap of 1.5.

Evaluating the risk in terms of standard deviation of returns, both minimizing downside risk and controlling the duration and convexity gap led to smaller values of standard deviation. In the unconstrained case, the smallest standard deviation was obtained at the minimum downside risk point at 4.3M NPV and the smallest standard deviation for the duration and convexity constrained case was 4.2M NPV.

Table 14 gives the initial funding and the second-stage updates for the minimum downside risk portfolio and Table 15 the funding strategy for the duration and convexity constrained case. The minimum downside risk strategy used six month non-callable debt and five year debt callable after three years for the initial funding and the initial funding of the duration and convexity constrained strategy used six months, five years and seven years non-callable debt, and five year debt callable after one year. In the second stage funding differed significantly depending on the scenario. Like in the case "Normal" discussed above, the duration and convexity constrained strategy resorted to calling five year debt callable after one year in scenario "w20010" at the whole amount and calling of debt other than due to the application of the decision rules did not happen in the minimum risk case. Comparing scenario "w20001" (low interest rates) with scenario 'w20004" and "w20010" (high interest rates) the minimum downside risk strategy tended towards longer-term debt when interest rates were low and towards shorter-term debt when interests were high.

For the case "Steep" Figure 8 and Figure 9 give the risk-return profile with respect to downside risk and standard deviation, respectively. In Figure 8 the efficient frontier (from Figure 2) is compared with the risk-return profile obtained from restricting the duration and convexity gap. Downside risk increased when decreasing the duration and convexity gap and was significantly larger than the minimized downside risk of the efficient frontier. The point with the smallest downside risk on the efficient frontier has expected returns of 17.4M NPV and a downside risk of 1.6M NPV, and the point with the smallest duration and convexity gap has expected returns of 15M NPV and a downside risk of 4.3M NPV. The maximum duration gap in the first and second stage was set to 0.5 and was set to 1.0 in the third stage, and the maximum convexity gap was set to 3.0 in the first and second stage and to 6.0 in the third stage. Further decrease made the problem become infeasible. The actual duration gap in the first stage was -0.5 and the actual convexity gap in the first stage was 1.6. Thus constraining the duration and convexity gap led to a decrease of expected returns by 13.8% and an increase of downside risk by a factor of 2.7 or by 172%. The point on the efficient frontier with the smallest downside risk had a the first-stage duration gap of -1.87 and the first-stage convexity gap of 2.6. Evaluating the risk in terms of standard deviation of returns in Figure 9, both minimizing downside risk and controlling the duration and convexity gap lead to smaller values of standard deviation. In the unconstrained case, the smallest standard deviation was obtained at the minimum downside risk point at 4.4M NPV, and the smallest standard deviation for the case of the constrained duration and convexity gap was 5.5M NPV. Expected returns, compared at the same level of standard deviation decreased significantly when constraining the duration and convexity gap. For example, the maximum duration and convexity constrained case had a standard deviation of 5.5M NPV and expected returns of 15.0. At this level of standard deviation the expected returns when minimizing downside risk were 19.4M NPV (second point from above on the minimum downside risk curve), a decrease of 22.7% when constraining the duration and convexity gap.

Table 16 gives the initial funding and the second-stage updates for the minimum downside risk portfolio and Table 17 the funding strategy for the duration and convexity constrained case. The minimum downside risk strategy used three month non-callable debt for the initial funding. The initial funding of the duration and convexity constrained strategy used one, three and ten year non-callable debt. The second stage funding differed significantly in the different scenarios. Comparing scenario "w20001" (low interest rates) with scenario "w20006" (high interest rates) the minimum downside risk strategy used one, five, ten and thirty years non-callable debt in scenario "w20001" and three month non-callable debt in scenario 'w20006". The duration and convexity gap constrained strategy used three month and thirty year non-callable debt in scenario "w20001" and one year ane three year non-callable debt in scenario "w20006". Again one could see in the minimum downside risk case a tendency of using longer-term debt when interest rates were low and using shorter-term debt when interests were high, and in the duration and convexity constrained case funding was balanced to match the duration and convexity of the mortgage pool.

6.5. Out of Sample Simulations

In order to evaluate the performance of the different strategies in an unbiased way, true out of sample evaluation runs need to be performed. Any solution at any node in the tree obtained by optimization must be evaluated using an independently sampled set of observations.

For the single-stage model this evaluation is rather straight-forward. Having obtained an optimal solution from the single stage model (using sample data set one), we run simulations again with a different seed. Using the new independent sample (data set two), we start the optimizer again, however, now with the optimal solution fixed at the values as obtained from the optimization based on sample data set one. Using the second set of observations of data set two we calculate risk and expected returns.

For independently evaluating the results obtained from an N-stage model we need N independent sets of N-stage trees of observations. Using data set one we solve the multi-stage optimization problem and obtain an optimal first-stage solution. We simulate again (using a different seed) over all stages to obtain data set two. Fixing the optimal first-stage solution at the value obtained from the optimization based on data set one, we optimize based on data set two and obtain a set of optimal second-stage solutions. We simulate again (with a different seed) to obtain independent realizations for stages three and four, thereby keeping the observations for stage two the same as in data set two, and obtain data set three. Fixing the first-stage decision at the level obtained from the optimization using data set one and all the second-stage decisions at the level obtained from the optimization based on data set two, we optimize again to obtain a set of optimal third stage decisions. We simulate again (using a different seed) to obtain independent outcomes for stage four, thereby keeping the observations for stage two and three the same as in data set two and three, respectively, and obtain data set four. Fixing the first-stage decision, all second-stage decisions, and all third-stage decisions at the level obtained from the optimization based on data sets one, two and three, respectively, we finally calculate risk and returns based on data set four. Quite an effort.

The out of sample evaluations resemble how the model could be used in practice. Solving the multi-stage model (based on data set one) an optimal first-stage solution (initial portfolio) would be obtained and implemented. Then one would follow the strategy (applying the decision rules) for 12 months until decision-stage two arrives. At this point one would re-optimize, given that the initial portfolio had been implemented and that particular interest rates and prepayments had occured (according to data set two). The optimal solution for the second stage would be implemented, and one would follow the strategy for four years until decision stage three arrives. At this point one would re-optimize, given that the initial portfolio and a second stage update had been implemented and that particular interest rates and prepayments had occured (according to data set three). The optimal solution for the third stage would be implemented, and one would follow the strategy (applying the decision rules) until the end of the horizon (according to data set four). Now one possible path of using the model has been evaluated. Decisions had no information about particular outcomes of future interest rates and prepayments, and were computed based on model runs using data independent from the observed realization of the evaluation simulation. Alternatively one could simulate a strategy involving reoptimization every month, but this would take significantly more computational effort with likely only little to be gained.

Using the out of sample evaluation procedure we obtain N out of sample simulations of using the model as discussed in the above paragraph and we are now in the position to derive statistics about out of sample expected returns and risk.

Figure 10 represents the out of sample efficient frontiers for the "Normal" data set and downside risk. The Figure gives the out of sample efficient frontier for the multi-stage model without duration and convexity constraints, the out of sample efficient frontier when the duration gap was constrained to be less than or equal to 1.5%, and for comparison the out of sample efficient frontier for the single-stage model. The out of sample evaluations demonstrate clearly that the multi-stage model gives significantly better results than the single-stage model. E.g., the point with the maximum expected returns of the multi-stage model gave expected returns of 10.2M NPV and a downside risk of 3.5M NPV. The minimum risk point on the single-stage out of sample efficient frontier gave expected returns of 7.6M NPV, and a downside risk of also 3.5M NPV. Thus, at the same level of downside risk the multi-stage model gave 34% higher expected returns. The minimum downside risk point of the multi-stage model gave expected returns of 9.0M NPV and a downside risk of 2.1M NPV. Comparing the minimum downside risk point of the multi-stage model with the one of the single-stage model, the multi-stage model had 19.2% larger expected returns at 61% of the downside risk of the single-stage model. The efficient frontier of the duration constrained strategy was slightly below the one without duration and convexity constraints. Figure 11 gives the out of sample risk-return profiles when measuring risk in terms of standard deviation. In this setting the multi-stage strategy performed significantly better than the single-stage strategy at every level of risk, where the difference was between 17.6% and 23.6%. The duration constrained strategy exhibited smaller risk (in standard deviations) at the price of slightly smaller expected returns.

The out of sample evaluations for the "Flat" data set gave very similar results. Figure 12 represents out of sample efficient frontiers for downside risk and Figure 13 the risk-return profile for risk in standard deviations. The multi-stage model gave significantly better results, the downside risk was smaller and expected returns were higher. E.g., the minimum downside risk point of the multi-stage model gave expected returns of 9.1M NPV and a downside risk of 2.9M NPV, whereas the minimum downside risk point of the single-stage model gave expected returns of 8.0M NPV and a downside risk of 4.5M NPV. I.e., at 64% of the risk of the single-stage model the multi-stage model gave 13.8% larger expected returns. Looking at risk in standard deviation, the multi-stage model gave 18.8% larger expected returns at every level of risk. The duration constrained strategy reduced risk in standard deviations at the expense of reducing also expected returns.

Figure 14 represents out of sample efficient frontiers for downside risk and Figure 15 the risk-return profile for risk in standard deviations for the data set "Steep". The multi-stage model gave significantly better results, the downside risk was smaller and expected returns were larger. At 83% of the downside risk of the single-stage strategy the multi-stage strategy gave 4.1% larger expected returns. The duration constrained strategy gave smaller expected returns at larger risk. When looking at risk in standard deviation the duration constrained strategy resulted in smaller expected returns compared to the unconstrained case.

All results discussed so far were obtained from solving a model with a relatively small number of scenarios at each stage, i.e. they were $|S_2| = 10$, $|S_3| = 20$ and $|S_4| = 20$ with a total number of scenarios at the end of the forth stage of $N = 4000$. This served well for analyzing and understanding the behavior of the multi-stage model in contrast to the single-stage model and in comparison to Gamma and Delta hedging. Choosing a larger sample size will improve the obtained strategies (initial portfolio and future revisions) and therefore result in better (out of sample simulation) results. Of course, also the accuracy of the prediction of the models will be improved. In order to show the effect of using an increased sample size we solved and evaluated the models using a sample size of $N = 24000$ i.e., $|S_2| = 40$, $|S_3| = 30$, and $|S_4| = 20$. The results for the data set "Normal"are represented in Figure 16 for downside risk and in Figure 17 for risk as standard deviation . Indeed one can see improved performance in both smaller risk and larger expected returns compared to the smaller sample size.

The point with the maximum expected returns of the multi-stage model gave expected returns of 12.9M NPV and a downside risk of 2.4M NPV. The minimum risk point from the single-stage model had expected returns of 9.0M NPV and a downside risk of 2.7M NPV. At slightly smaller downside risk the multi-stage model gave expected returns that were 43% higher. The minimum downside risk point of the multi-stage model had expected returns of 10.9M NPV and a downside risk of 1.5M NPV. Comparing the minimum downside risk point of the multi-stage and single-stage model, the multi-stage model has 18.4% larger expected returns at 57% of the downside risk of the single-stage model. Again, the efficient frontier of duration constrained strategy was slightly below the one without duration and convexity constraints. Measuring risk in terms of standard deviation, like before when using the small sample size, the multi-stage strategy performed significantly better than the single-stage strategy at every level of risk, where the difference was between 23.3% and 24.0%. Again, the duration constrained strategy exhibited smaller risk (in standard deviations) at the price of slightly smaller expected returns.

Figure 18 gives a comparison of the multi-stage efficient frontiers predicted versus evaluated out of sample. One can see that when using the larger sample size of $N = 24000$ the predicted and the out of sample evaluated efficient frontier look almost identical, thus validating the model. It is evident that using larger sample sizes will result in both better performance and more accurate prediction. However, using a sample size of $N = 24000$ gave already reasonably accurate results.

7. Large-Scale Results

For the actual practical application of the proposed model we need to consider a large number of scenarios to have only small estimation errors regarding expected returns and risk, and very stable results. We will now explore the feasibility of solving large models regarding problem size and computation time. The following Table 4 gives results from models with larger numbers of scenarios. For example, Models L4 has 80,000 scenarios at the forth stage, composed of $|S_2| = 40$, $|S_3| = 40$ and $|S_4| = 50$, and model L5 has 100,000 scenarios at the forth stage composed of $|S_2| = 50$, $|S_3| = 40$ and $|S_4| = 50$; both models have a sufficiently large sample size in each stage.

Table 4: Large Problems

Model	Scenarios				Problem size		
	Stage 2	Stage 3	Stage 4	Total	rows	columns	nonzeros
L1	10	40	50	20000	54034	49918	1249215
L2	20	40	50	40000	108064	99783	2496172
L3	30	40	50	60000	162094	149623	3756498
L4	40	40	50	80000	216124	199497	5021486
L5	50	40	50	100000	270154	249277	6291014
L20	200	40	50	400000	808201	931271	14259216

In the case of problem L5 with 100,000 scenarios the corresponding linear program had 270,154 constraints, 249,277 variables, and 6,201,014 non-zero coefficients. Table 5 gives the elapsed times for simulation and solution, obtained on a Silicon Graphics Origin 2000 workstation. While the Origin 2000 at our disposition is a multi-processor machine with 32 processors, we did not use the parallel feature, and all results were obtained using single processor computations. For the direct solution of the linear programs we used CPLEX [2] as the linear programming optimizer. We contrasted the results to using DECIS [5], a system for solving stochastic programs, developed by the author. DECIS exploited the special structure of the problems and used dual decomposition for their solution. The problems were decomposed into two stages, breaking between the first and the second stage. The simulation runs for model L5 took less than an hour. The elapsed solution time solving the problem directly was 13 hours and 28 minutes. Solving the problem using DECIS took significantly less time, the problem was solved in 2 hours and 33 minutes. Encouraged by the quick solution time using DECIS we generated problem L20 with 400,000 scenarios composed of $|S_2| = 200$, $|S_3| = 40$ and $|S_4| = 50$, and solved it in 11 hours and 34 minutes using DECIS. Problem L20 had 808,201 constraints, 931,271 variables, and 14,259,216 non-zero coefficients.

Using parallel processing, the simulation times and the solution times could be reduced significantly. Based on our experiences with parallel DECIS, using 6 processors we would expect the solution time for the model L5 with 100,000 scenarios to be less than forty

Table 5: Solution Times

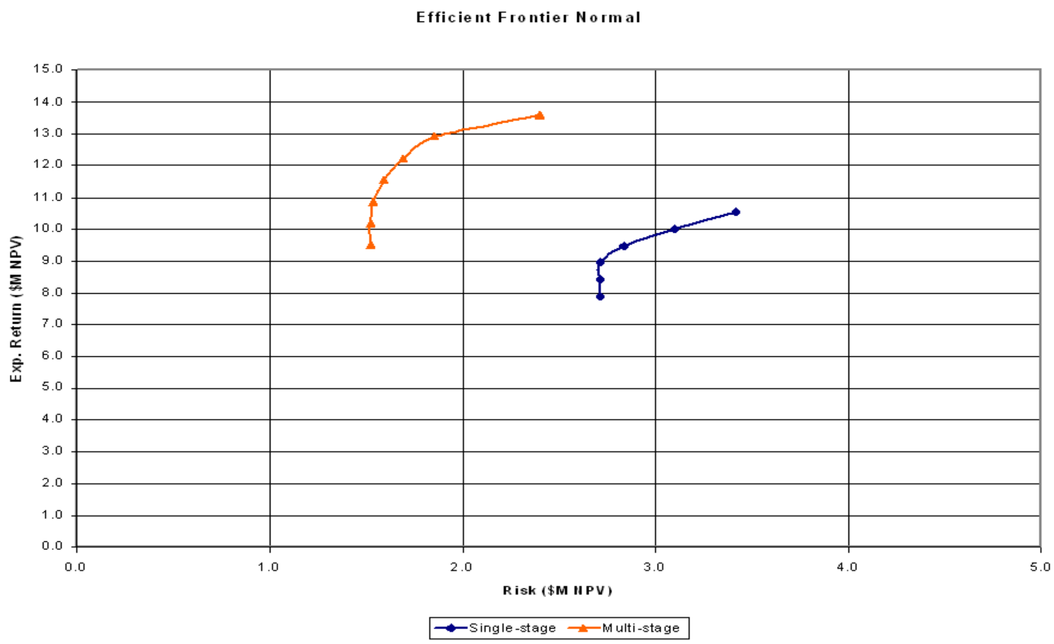
Model	Scenarios	Simul. Time (s)	Direct Sol. Time(s)	Decomp. Sol. Time (s)
L1	20000	688	1389.75	1548.91
L2	40000	1390	6144.56	
L3	60000	2060	14860.54	5337.76
L4	80000	2740	28920.69	
L5	100000	3420	48460.02	9167.47
L20	400000			41652.28

minutes, and using 16 processors one could solve model L20 with 400,000 scenarios in about one hour.

8. Efficient Frontiers

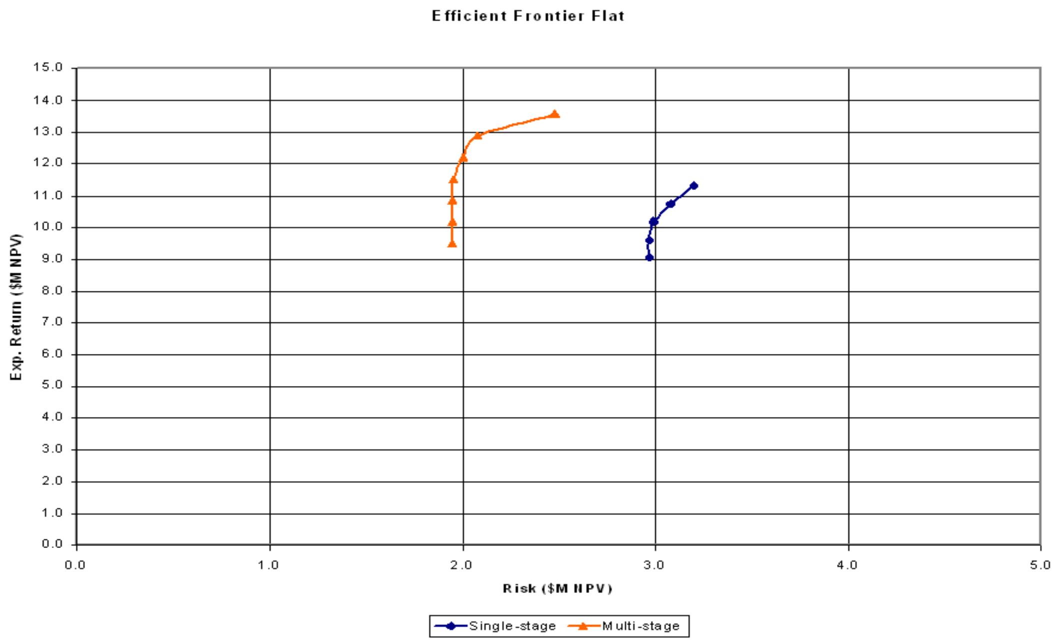
8.1. Model Normal, Single- versus Multi-Stage

Figure 1: Model Normal, Single- versus Multi-Stage



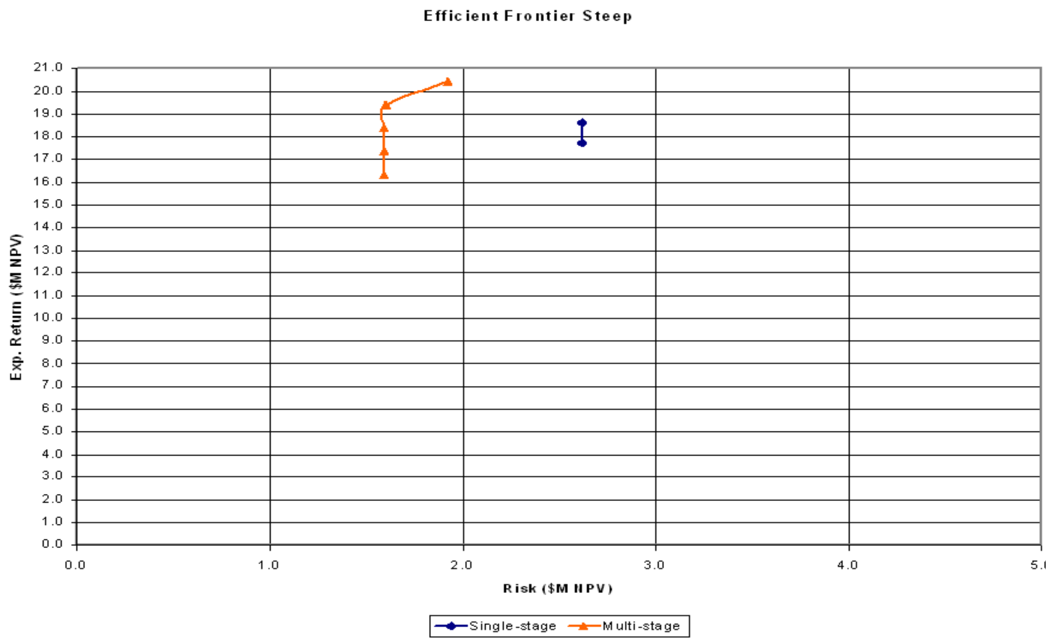
8.2. Model Flat, Single- versus Multi-Stage

Figure 2: Model Flat, Single- versus Multi-Stage



8.3. Model Steep, Single- versus Multi-Stage

Figure 3: Model Steep, Single- versus Multi-Stage



8.4. Model Normal, Multi-Stage, Duration and Convexity Hedged

Figure 4: Model Normal, Multi-Stage, Downside Risk

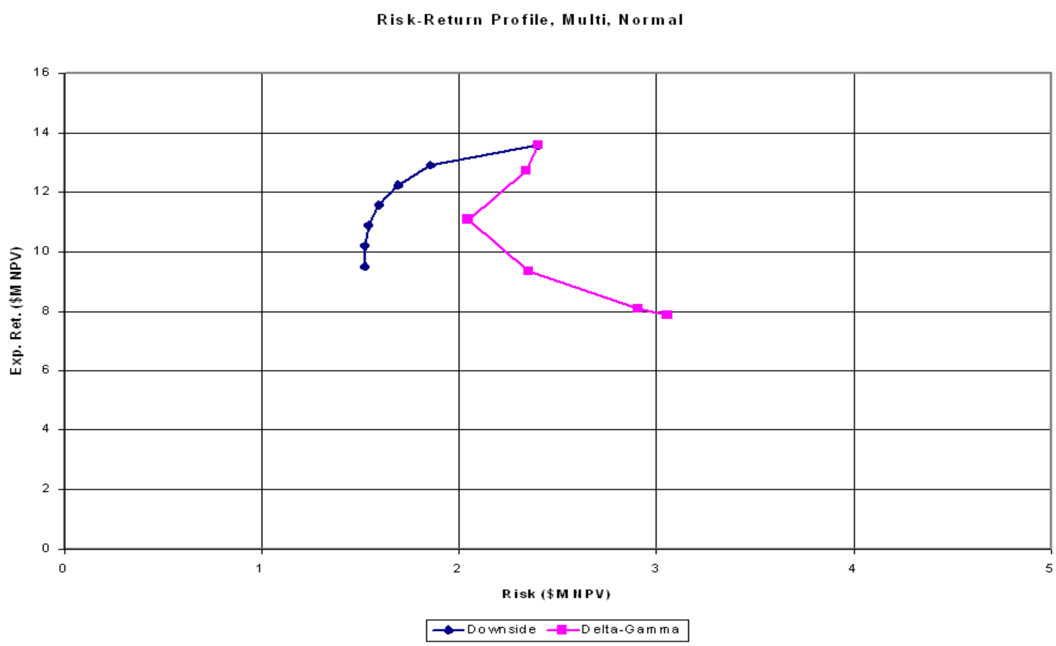
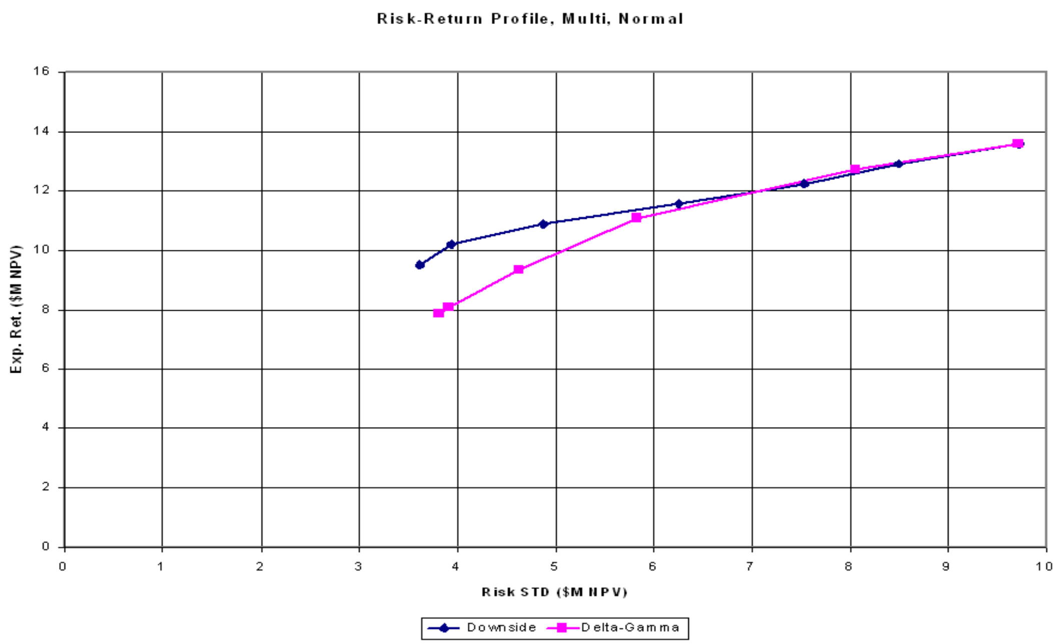


Figure 5: Model Normal, Multi-Stage, Risk Standard Deviation



8.5. Model Flat, Multi-Stage, Duration and Convexity Hedged

Figure 6: Model Flat, Multi-stage, Downside Risk

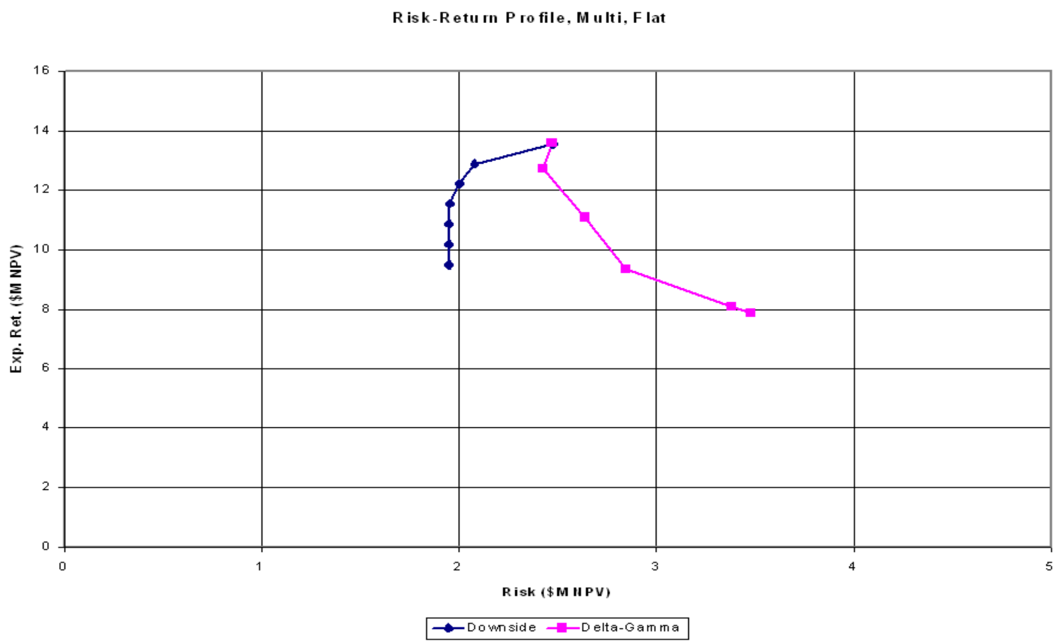
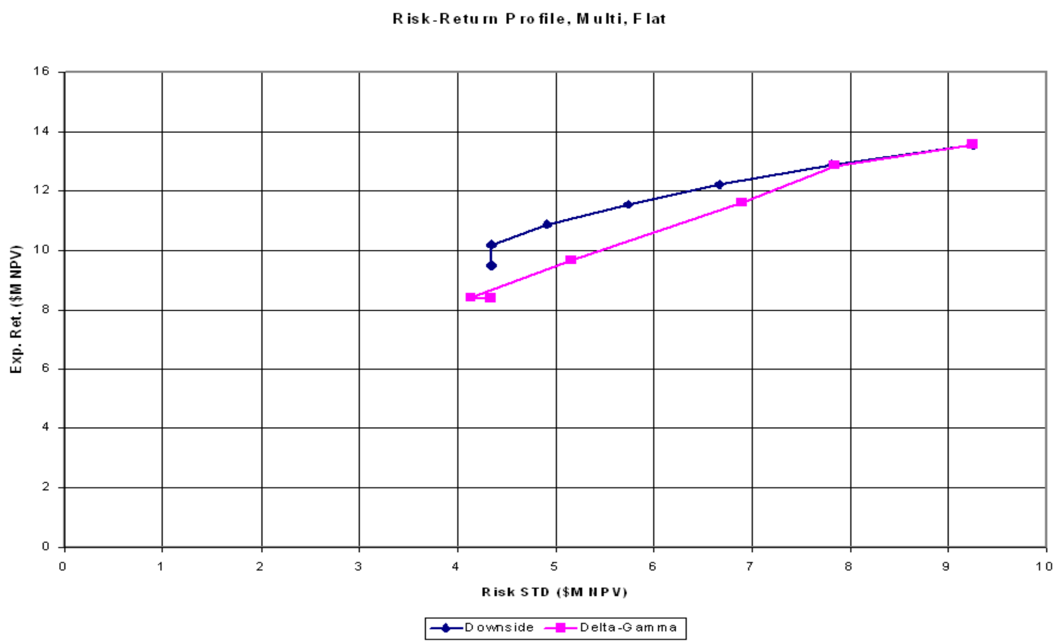


Figure 7: Model Flat, Multi-stage, Risk Standard Deviation



8.6. Model Steep, Multi-Stage, Duration and Convexity Hedged

Figure 8: Model Steep, Multi-Stage, Downside Risk

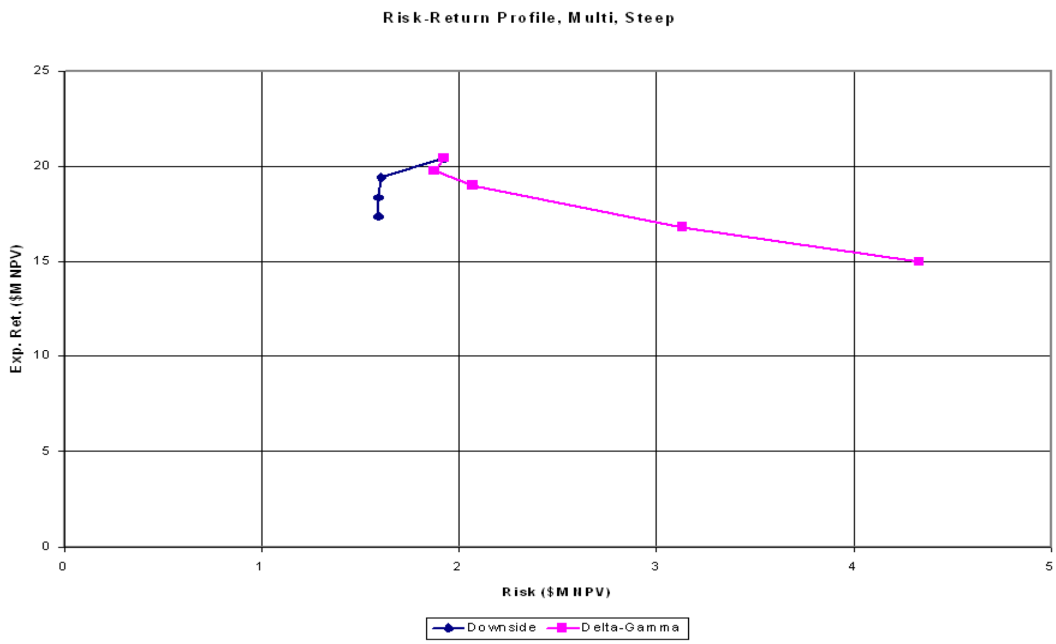
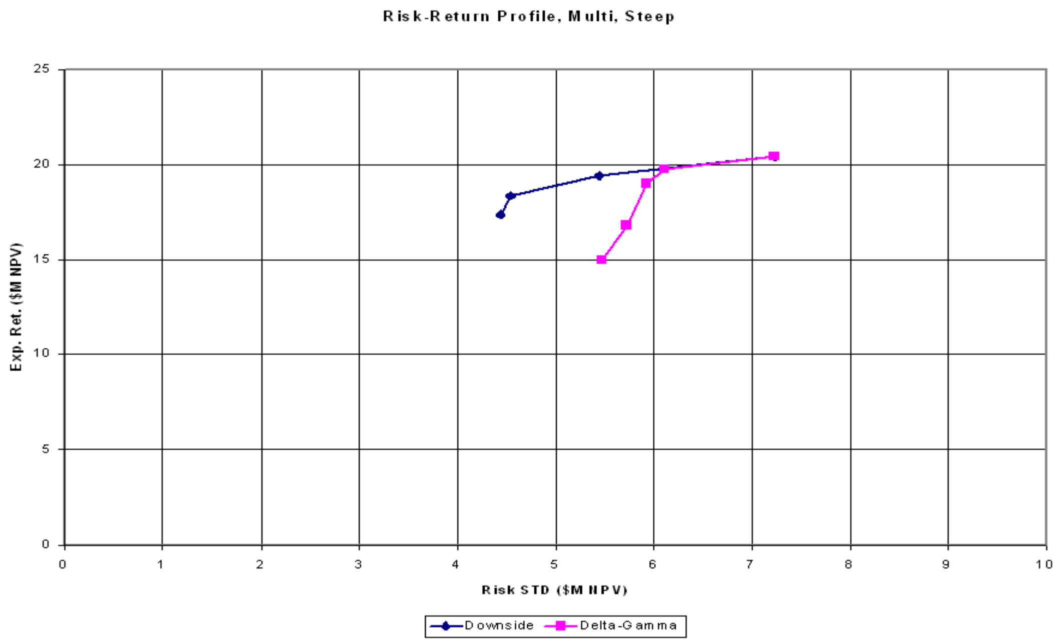


Figure 9: Model Steep, Multi-Stage, Risk Standard Deviation



8.7. Model Normal, Multi-Stage, Out of Sample Evaluation

Figure 10: Model Normal, Multi-Stage, Downside Risk

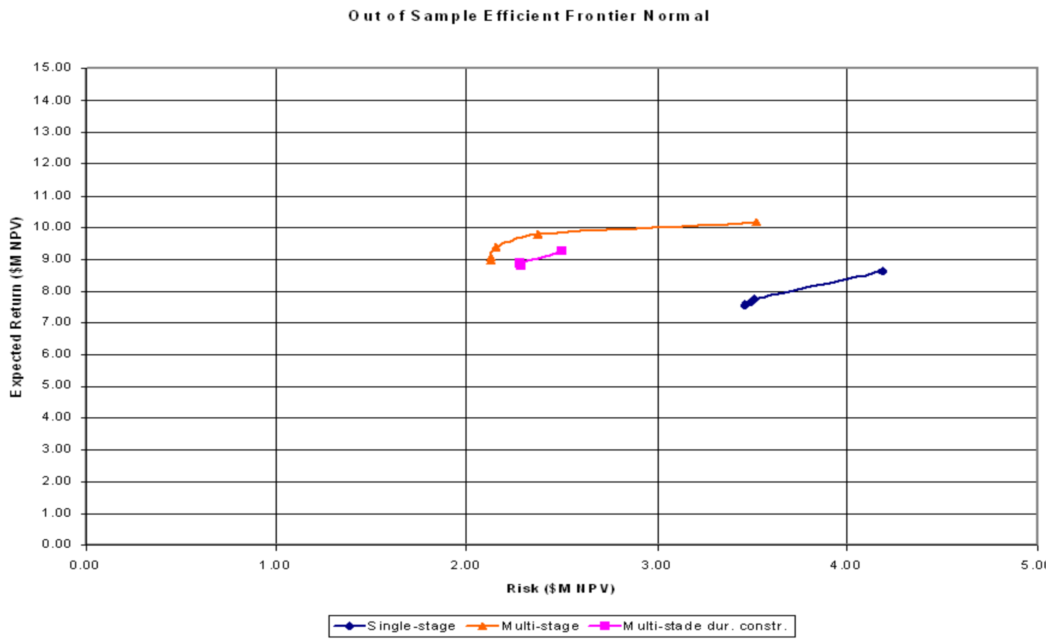
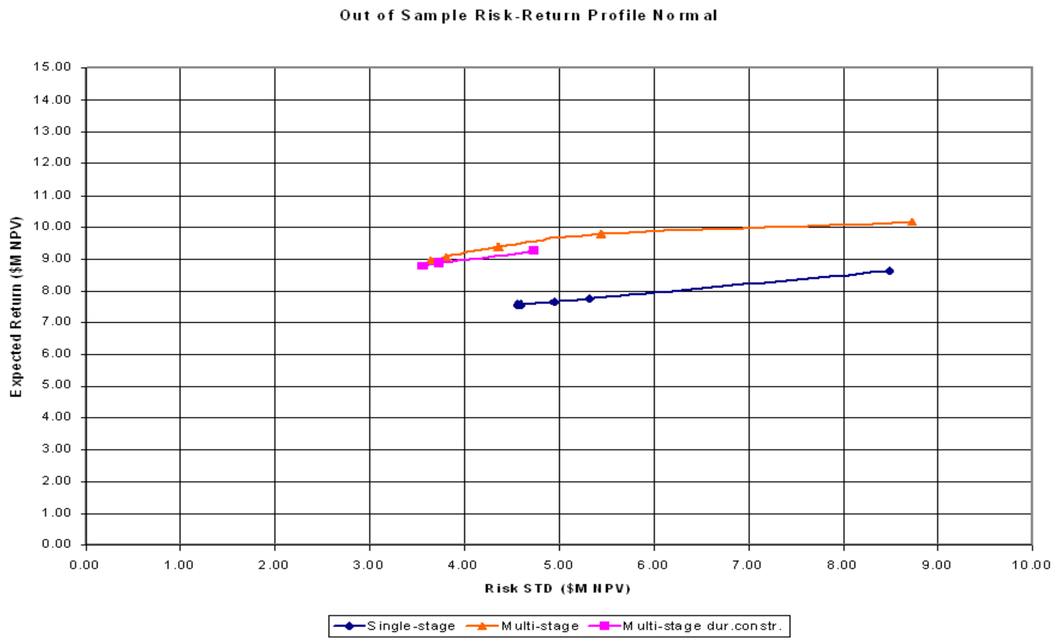


Figure 11: Model Normal, Multi-Stage, Risk Standard Deviation



8.8. Model Flat, Multi-Stage, Out of Sample Evaluation

Figure 12: Model Flat, Multi-stage, Downside Risk

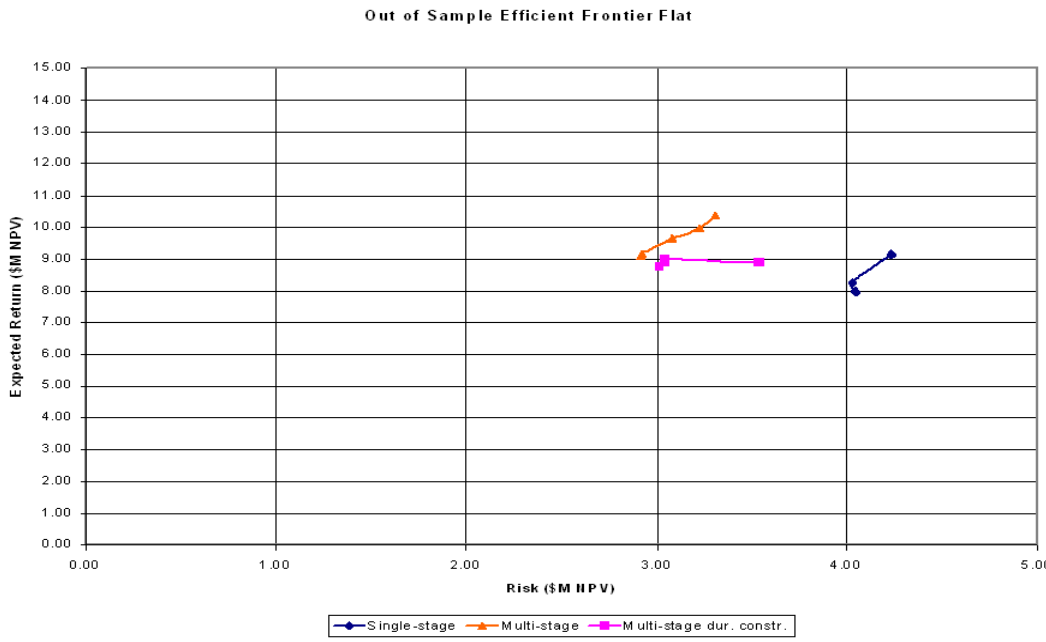
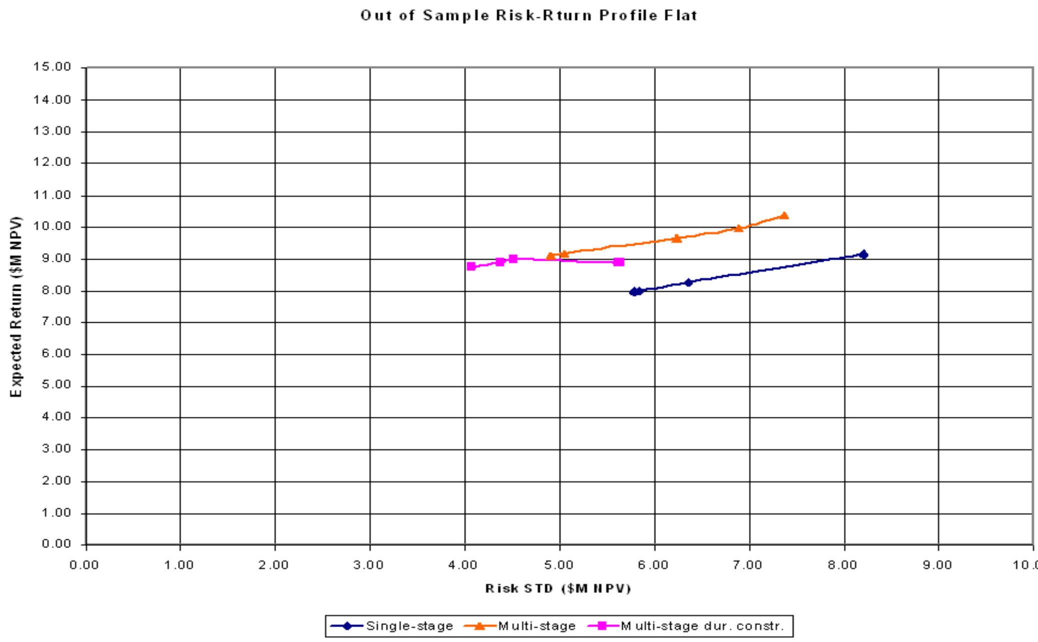


Figure 13: Model Flat, Multi-stage, Risk Standard Deviation



8.9. Model Steep, Multi-Stage, Out of Sample Evaluation

Figure 14: Model Steep, Multi-Stage, Downside Risk

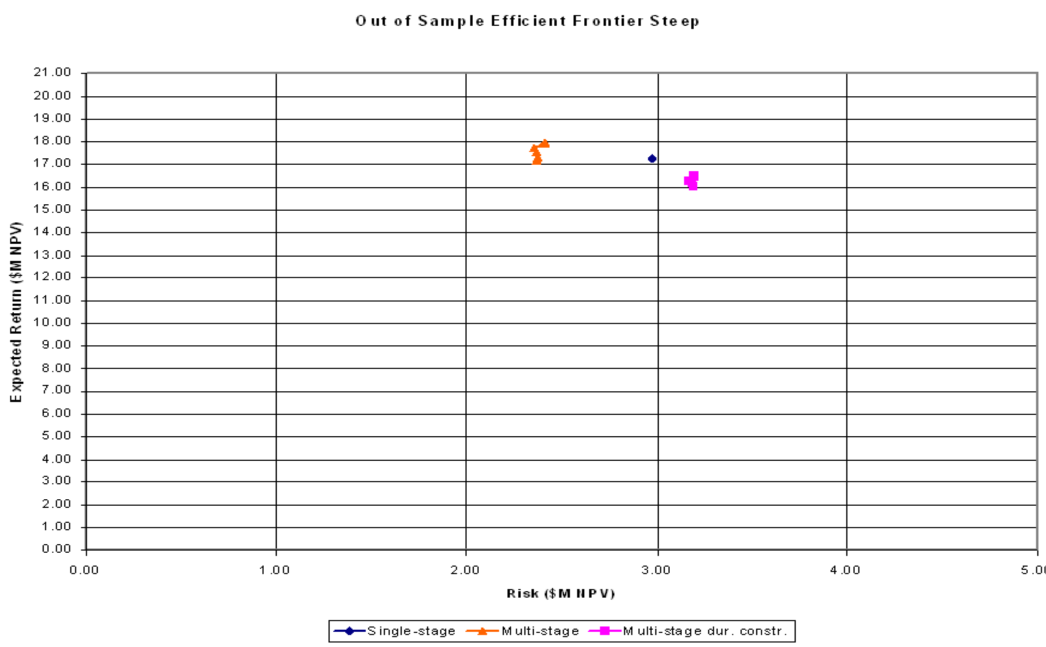
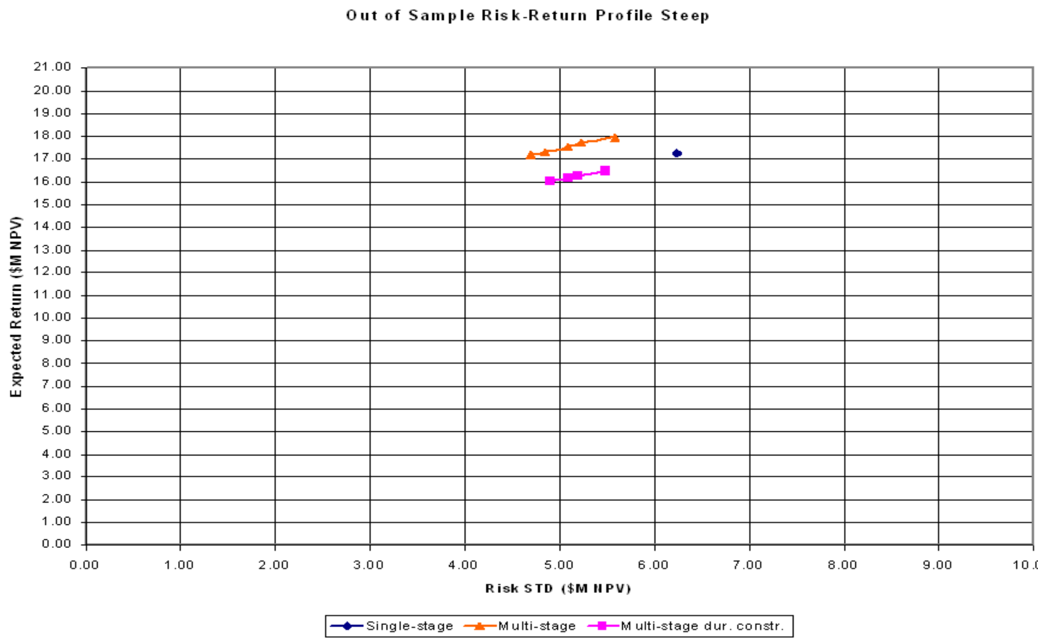


Figure 15: Model Steep, Multi-Stage, Risk Standard Deviation



8.10. Model Normal, Multi-Stage, Out of Sample Evaluation, Larger Sample

Figure 16: Model Normal, Multi-Stage, Downside Risk

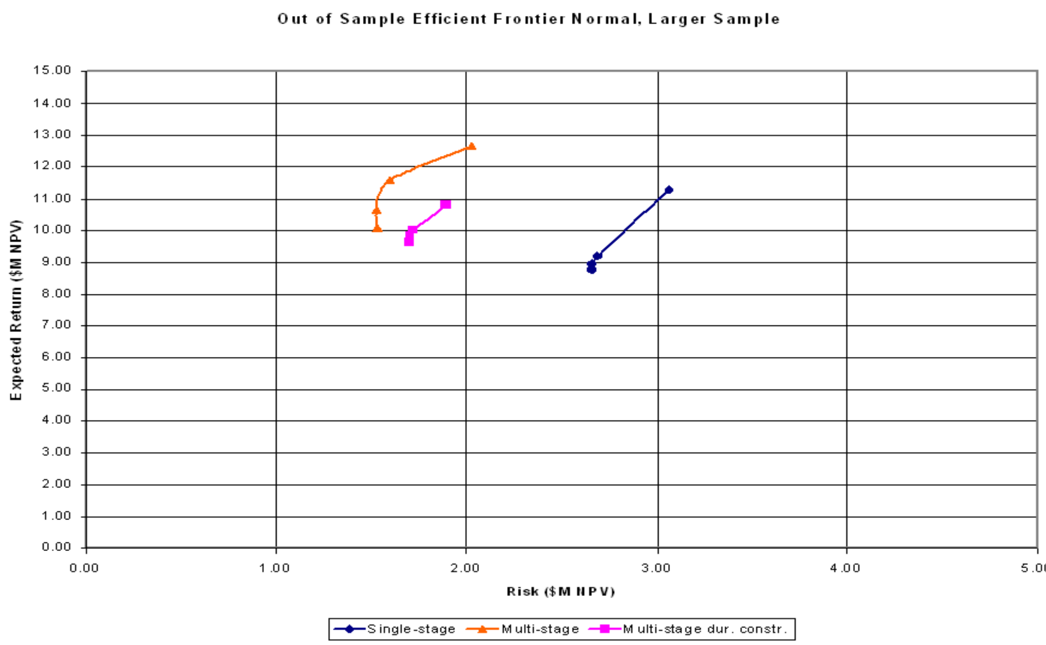


Figure 17: Model Normal, Multi-Stage, Risk Standard Deviation

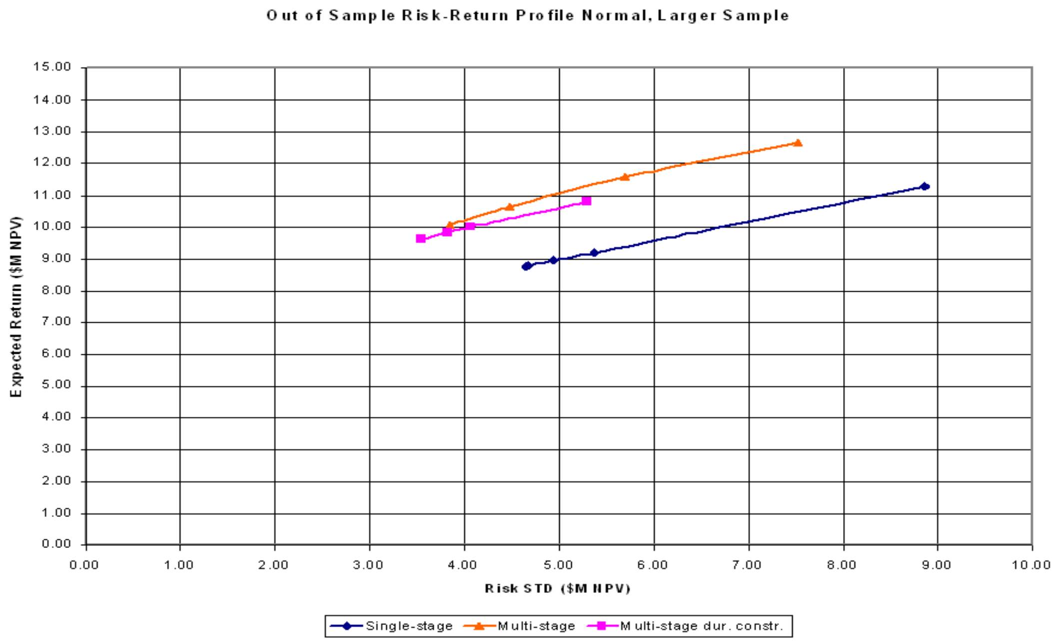
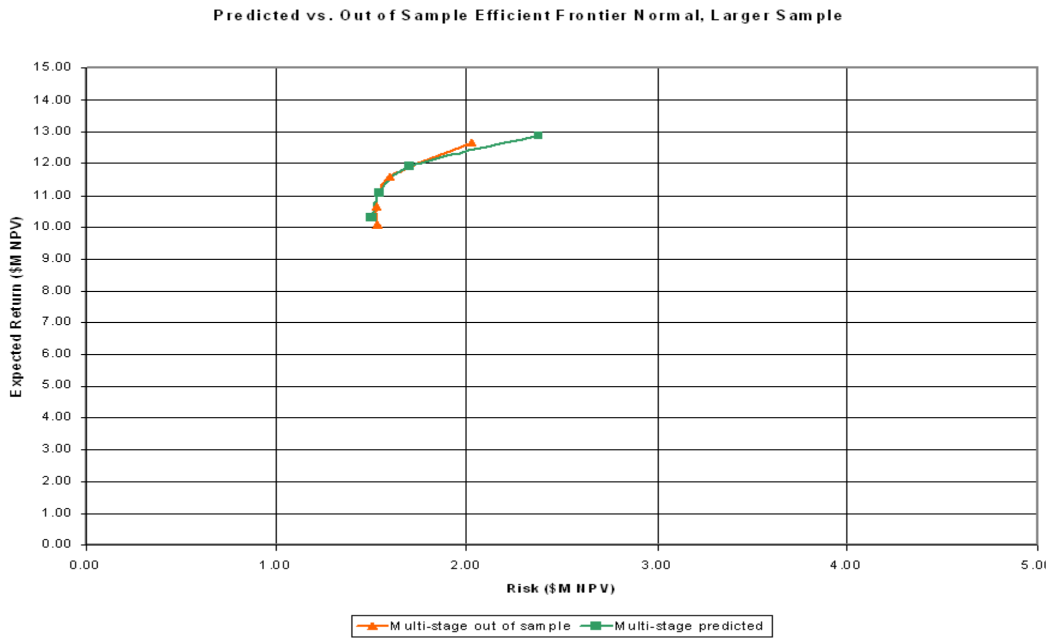


Figure 18: Model Normal, Multi-Stage, Predicted versus Out of Sample



9. Funding Examples

9.1. Model Normal, Single Stage

Table 6: Model Normal, Single-Stage

```
**** OBJECTIVE VALUE             3095.0510

---- 125805 VARIABLE  X1.L          amount product j1

j1m06n 0.853,    j1y07n 0.147


---- 125805 VARIABLE  X2.L          amount product j2

            j2m03n

w20001       0.853
w20002       0.853
w20003       0.853
w20004       0.853
w20005       0.853
w20006       0.853
w20007       0.853
w20008       0.853
w20009       0.853
w20010       0.853
```

9.2. Model Flat, Single Stage

Table 7: Model Flat, Single-Stage

```
**** OBJECTIVE VALUE             3076.4521

---- 122706 VARIABLE  X1.L          amount product j1

j1m03n 0.911,    j1y07n 0.089


---- 122706 VARIABLE  X2.L          amount product j2

            j2m03n

w20001       0.911
w20002       0.911
w20003       0.911
w20004       0.911
w20005       0.911
w20006       0.911
w20007       0.911
w20008       0.911
w20009       0.911
w20010       0.911
```

9.3. Model Steep, Single Stage

Table 8: Model Steep, Single-Stage

```
**** OBJECTIVE VALUE             2614.7548

---- 125297 VARIABLE  X1.L          amount product j1

j1m03n 1.000


---- 125297 VARIABLE  X2.L          amount product j2

            j2m03n

w20001       1.000
w20002       1.000
w20003       1.000
w20004       1.000
w20005       1.000
w20006       1.000
w20007       1.000
w20008       1.000
w20009       1.000
w20010       1.000
```

9.4. Model Normal, Multi-Stage

Table 9: Model Normal, Multi-Stage

```
**** OBJECTIVE VALUE            1854.7689

---- 125774 VARIABLE  X1.L          amount product j1

j1m06n   0.785,    j1y05nc1 0.215


---- 125774 VARIABLE  X2.L          amount product j2
```

	j2m03n	j2y03n	j2y03nc1	j2y05nc1	j2y10n	j2y10nc1
w20002						0.597
w20003		0.125	0.420		0.240	
w20004	0.785					
w20005			0.508		0.423	0.069
w20006	0.785					
w20007						0.785
w20008	0.251				0.749	
w20009	0.301			0.035	0.261	0.188
w20010	0.785					

+	j2y10nc3	j2y30n
w20001		1.000
w20002	0.188	

9.5. Model Flat, Multi-Stage

Table 10: Model Flat, Multi-Stage

**** OBJECTIVE VALUE 2077.9901

---- 122675 VARIABLE X1.L amount product j1

j1m03n 0.509, j1m06n 0.491

---- 122675 VARIABLE X2.L amount product j2

	j2m03n	j2y03n	j2y03nc1	j2y10n	j2y10nc1	j2y10nc3
w20001				0.397		
w20002					1.000	
w20003	0.251	0.590		0.011		0.148
w20004	1.000					
w20005			0.514	0.248	0.238	
w20006	1.000					
w20007	0.984				0.016	
w20008	0.620			0.380		
w20009	0.521			0.151	0.328	
w20010	1.000					

+	j2y30n
w20001	0.603

9.6. Model Steep, Multi-Stage

Table 11: Model Steep, Multi-Stage

```
**** OBJECTIVE VALUE            1602.6202

---- 125266 VARIABLE  X1.L          amount product j1

j1m03n 1.000


---- 125266 VARIABLE  X2.L          amount product j2
```

	j2m03n	j2y03n	j2y03nc1	j2y05n	j2y10n	j2y10nc1
w20001		0.038		0.296	0.544	
w20002	0.717		0.283			
w20003			1.000			
w20004	1.000					
w20005			0.745		0.085	0.170
w20006	1.000					
w20007	1.000					
w20008	0.609				0.272	0.119
w20009	0.973				0.027	
w20010	1.000					

+	j2y30n
w20001	0.122

9.7. Model Normal, Multi-Stage, Duration and Convexity Hedged

Table 12: Model Normal, Multi-Stage, Minimum Downside Risk

**** OBJECTIVE VALUE 1518.8879

---- 125967 VARIABLE X1.L amount product j1

j1m06n 0.266, j1y05n 0.235, j1y05nc1 0.218, j1y05nc3 0.281

---- 125967 VARIABLE X2.L amount product j2

	j2m03n	j2y01n	j2y03nc1	j2y05n	j2y10n	j2y10nc1
w20001		0.017		0.148	0.240	
w20002						0.102
w20003			0.117		0.149	
w20004	0.266					
w20005	0.031		0.032		0.237	0.185
w20006	0.266					
w20007					0.014	0.252
w20008	0.118				0.300	0.019
w20009	0.012				0.201	0.053
w20010	0.266					

+	j2y10nc3	j2y30n
w20001		0.079
w20002	0.165	
w20008		0.047

Table 13: Model Normal, Multi-Stage, Duration and Convexity Constrained

**** OBJECTIVE VALUE -7871.4913

---- 131553 VARIABLE X1.L amount product j1

j1m06n 0.122, j1y05n 0.083, j1y05nc1 0.534, j1y07n 0.239
j1y10n 0.022

---- 131553 VARIABLE X2.L amount product j2

	j2m03n	j2m06n	j2y01n	j2y03nc1	j2y05n	j2y05nc1
w20001	0.466					
w20003				0.064		
w20005				0.506	0.043	0.044
w20006			0.064			
w20008	0.567					
w20009	0.072					
w20010		0.316	0.340			

+	j2y07nc1	j2y10n	j2y10nc1	j2y10nc3
w20001	0.117			0.073
w20002		0.011	0.112	
w20003		0.058		
w20004		0.152		
w20005		0.058	0.004	
w20006		0.059		
w20007		0.122		
w20008		0.089		
w20009				0.051

---- 131553 VARIABLE CX12.L call amount product j1

	j1y05nc1
w20004	0.029
w20010	0.534

9.8. Model Flat, Multi-Stage, Duration and Convexity Hedged

Table 14: Model Flat, Multi-Stage, Minimum Downside Risk

```
**** OBJECTIVE VALUE             1946.0438

---- 122868 VARIABLE  X1.L          amount product j1

j1m06n   0.698,    j1y05nc3 0.302


---- 122868 VARIABLE  X2.L          amount product j2

           j2m03n      j2m06n      j2y03n    j2y03nc1      j2y05n    j2y05nc3

w20001                  0.046                               0.418       0.052
w20003       0.201                   0.292
w20004       0.698
w20005       0.119                               0.145
w20006       0.698
w20007       0.603
w20008       0.333
w20009       0.302
w20010       0.698

     +       j2y10n    j2y10nc1    j2y10nc3      j2y30n

w20001       0.129                               0.053
w20002                  0.698
w20003       0.039      0.039        0.127
w20005       0.221      0.214
w20007                  0.095
w20008       0.280      0.085
w20009       0.095      0.301
```

Table 15: Model Flat, Multi-Stage, Duration and Convexity Constrained

```
**** OBJECTIVE VALUE           -8383.6563

---- 127082 VARIABLE  X1.L          amount product j1

j1m06n   0.448,    j1y05n   0.225,    j1y05nc1 0.201,    j1y07n   0.125


---- 127082 VARIABLE  X2.L          amount product j2
```

	j2m03n	j2m06n	j2y02n	j2y03n	j2y03nc1	j2y10n
w20001	0.567					0.082
w20002	0.102					
w20003	0.370			0.078		
w20004		0.251	0.033			0.164
w20005	0.018				0.631	
w20006		0.397				0.051
w20007						0.036
w20008	0.634					0.016
w20009	0.448					
w20010	0.277	0.372				

+	j2y10nc1
w20002	0.346
w20007	0.412

```
---- 127082 VARIABLE  CX12.L        call amount product j1
```

	j1y05nc1
w20010	0.201

9.9. Model Steep, Multi-Stage, Duration and Convexity Hedged

Table 16: Model Steep, Multi-Stage, Minimum Downside Risk

```
**** OBJECTIVE VALUE            1590.0012

---- 125459 VARIABLE  X1.L          amount product j1

j1m03n 1.000


---- 125459 VARIABLE  X2.L          amount product j2

           j2m03n      j2y01n      j2y03n    j2y03nc1      j2y05n      j2y10n

w20001                  0.299                               0.389       0.215
w20002       0.644                              0.356
w20003                                          1.000
w20004       1.000
w20005                                          0.734                   0.088
w20006       1.000
w20007       1.000
w20008       0.600                  0.005                               0.246
w20009       0.958                                                      0.042
w20010       1.000

     +    j2y10nc1      j2y30n

w20001                  0.097
w20005       0.178
w20008       0.149 5.126743E-4
w20009 8.234504E-4
```

Table 17: Model Steep, Multi-Stage, Duration and Convexity Constrained

```
**** OBJECTIVE VALUE          -14977.1521

---- 130337 VARIABLE  X1.L          amount product j1

j1y01n 0.682,    j1y03n 0.305,    j1y10n 0.013


---- 130337 VARIABLE  X2.L          amount product j2

            j2m03n      j2y01n      j2y02n      j2y03n    j2y03nc1      j2y10n

w20001       0.598
w20002                   0.053                   0.629
w20003       0.049       0.633
w20004                                           0.531                   0.151
w20005                                                       0.682
w20006                   0.158                   0.524
w20007                   0.500                                           0.182
w20008       0.682
w20009       0.682
w20010                   0.392       0.289

     +      j2y30n

w20001       0.084
```

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