Algorithm NCL for constrained optimization

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Abstract

Standard optimization solvers have difficulty if the active-constraint gradients are not independent at a solution. For example, problems of the form

$$\min f(x) \quad \text{st} \quad c(x) \geq 0 \qquad (m \text{ constraints and } n \text{ variables})$$

may have more than n constraints active at a solution. Such problems arise in the modeling of tax policy (with perhaps millions of constraints and thousands of variables).

Algorithm NCL solves a sequence of about 10 augmented Lagrangian subproblems with constraints $c(x) + r \geq 0$. The extra variables r make the constraints linearly independent, and the subproblem solutions converge to the required solution as r is driven to zero. Assuming second derivatives are available, NCL expands the use of interior methods for large-scale optimization.

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Optimization

UCO
$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \ \phi(x)$$

- $\phi(x)$ is a smooth nonlinear function
- $\nabla\phi(x)$ known: Quasi-Newton method
- $\nabla^2\phi(x)$ known: Newton's method

NCO
$$\begin{aligned}\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \ & \phi(x)\\ \text{subject to}\ \ & c(x)=0\end{aligned}$$

- $c(x)\in\mathbb{R}^m$ is a vector of smooth nonlinear functions

Constrained Optimization

NCO

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & \phi(x) \\ \text{subject to} & c(x) = 0 \end{array}$$

Penalty function

$$P(x, \rho_k) = \phi(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2$$

- Penalty parameter $\rho_k \to \infty$

Augmented Lagrangian

$$L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2$$

- Lagrange multiplier estimate y_k $\qquad$ ρ_k can remain finite

A general optimization problem

A general optimization problem

NLP
$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \phi(x) \quad \text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$$

- $c(x) = 0$ includes linear constraints $Ax = b$
- Bounds on the variables x

LANCELOT (1992) solves large-scale optimization problems of this form

- Inequalities $\bar{\ell} \le c(x) \le \bar{u}$ are equalities with more variables and bounds:

NLP
$$\underset{x,s}{\text{minimize}} \quad \phi(x) \quad \text{subject to} \quad c(x) - s = 0, \quad \begin{pmatrix} \ell \\ \bar{\ell} \end{pmatrix} \le \begin{pmatrix} x \\ s \end{pmatrix} \le \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$$

LANCELOT's BCL algorithm for general NLP

Conn, Gould & Toint (1992)

LANCELOT

$$\boxed{\min \phi(x) \quad \text{st} \quad c(x) = 0, \quad \ell \le x \le u}$$

BCL subproblems (Bound-Constrained augmented Lagrangian):

$$\boxed{\begin{array}{lll} \text{BC}_k & \underset{x}{\text{minimize}} & \phi(x) - y_k^T c(x) + \frac{1}{2}\rho_k \|c(x)\|^2 \\ & \text{subject to} & \ell \le x \le u \end{array}}$$

Loop:

solve BC_k to get x_k^*	decreasing opttol ω_k
if $\|c(x_k^*)\| \le \eta_k$, $y_{k+1} \leftarrow y_k - \rho_k c(x_k^*)$	decreasing featol η_k
else $\rho_{k+1} \leftarrow 10\rho_k$	

Our optimization problem

Our NLP problem

$$\text{NLP} \qquad \begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & \phi(x) \\ \text{subject to} & c(x) \ge 0, \quad \ell \le x \le u \end{array}$$

- $\phi(x)$ is a smooth nonlinear function
- $c(x) \in \mathbb{R}^m$ is a vector of smooth nonlinear functions
- General bounds
- Many inequalities $c(x) \ge 0$ might not satisfy LICQ at x^*

Example: $m = 571,000, \quad n = 1500$
10,000 constraints essentially active: $c_i(x^*) \le 10^{-6}$

BCL

LCL

NCL

Sequence of subproblems minimizing
X-constrained (augmented) Lagrangian

BCL	LANCELOT	Conn, Gould & Toint (1992)
LCL	linearized constraints	Robinson (1972)
	MINOS	Murtagh and S (1982)
sLCL	KNOSSOS	Friedlander (2002)
NCL	New form of **BCL**!	Today's talk
	AMPL loop + IPOPT or KNITRO	

Algorithm NCL
for general NLP

NCL subproblems

NLP

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \phi(x) \\ \text{subject to} & c(x) = 0, \quad \ell \le x \le u \end{array}$$

NCL subproblems

NLP	$$\begin{array}{ll} \underset{x}{\text{minimize}} & \phi(x) \\ \text{subject to} & c(x) = 0, \quad \ell \le x \le u \end{array}$$

LANCELOT-type subproblems:

BC_k	$$\begin{array}{ll} \underset{x}{\text{minimize}} & L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k \left\| c(x) \right\|^2 \\ \text{subject to} & \ell \le x \le u \end{array}$$

NCL subproblems

NLP	$\underset{x}{\text{minimize}} \quad \phi(x)$ $\text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$

LANCELOT-type subproblems:

BC_k	$\underset{x}{\text{minimize}} \quad L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \frac{1}{2}\rho_k \left\|c(x)\right\|^2$ $\text{subject to} \quad \ell \le x \le u$

Introduce $r = -c(x)$:

NC_k	$\underset{x,\, r}{\text{minimize}} \quad \phi(x) + y_k^T r + \frac{1}{2}\rho_k \left\|r\right\|^2$ $\text{subject to} \quad c(x) + r = 0, \quad \ell \le x \le u$

Free vars r make the nonlinear constraints independent and feasible   Solvers happy!

NCL subproblems

NLP

$$\underset{x}{\text{minimize}} \quad \phi(x)$$
$$\text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$$

NC$_k$

$$\underset{x,\, r}{\text{minimize}} \quad \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k \|r\|^2$$
$$\text{subject to} \quad c(x) + r = 0, \quad \ell \le x \le u$$

Free vars r make the nonlinear constraints independent and feasible   Solvers happy!

NCL subproblems for our problem

NLP

$$\begin{array}{ll} \displaystyle\min_{x} & \phi(x) \\ \text{subject to} & c(x) \geq 0, \quad \ell \leq x \leq u \end{array}$$

NC$_k$

$$\begin{array}{ll} \displaystyle\min_{x,\,r} & \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k \|r\|^2 \\ \text{subject to} & c(x) + r \geq 0, \quad \ell \leq x \leq u \end{array}$$

Free vars r make the nonlinear constraints independent and feasible Solvers happy!

Subproblems for convex QP

QP
$$\begin{array}{ll} \underset{x}{\text{minimize}} & \phi(x) \\ \text{subject to} & Ax = b, \quad \ell \le x \le u \end{array}$$

Chris Maes, ICME PhD thesis (2010)   QPBLUR

QP_k
$$\begin{array}{ll} \underset{x,\,r}{\text{minimize}} & \phi(x) + y_k^T r + \frac{1}{2}\rho_k \|r\|^2 \\ \text{subject to} & Ax + r = b, \quad \ell \le x \le u \end{array}$$

Free vars r make the constraints independent and feasible   Solvers happy!

Optimal Tax Policy

Kenneth Judd and Che-Lin Su 2011

Optimal tax policy

$$
\begin{array}{llll}
\text{TAX} & \text{maximize}_{c,\,y} & \sum_i \lambda_i U^i(c_i, y_i) & \\
 & \text{subject to} & U^i(c_i, y_i) - U^i(c_j, y_j) \ge 0 & \text{for all } i, j \ (*) \\
 & & \lambda^T (y - c) \ge 0 & \\
 & & c,\ y \ge 0 &
\end{array}
$$

where c_i and y_i are the consumption and income of taxpayer i, and λ is a vector of positive weights. The utility functions $U^i(c_i, y_i)$ are each of the form

$$U(c, y) = \frac{(c - \alpha)^{1-1/\gamma}}{1 - 1/\gamma} - \psi \frac{(y/w)^{1/\eta+1}}{1/\eta + 1}$$

where w is the wage rate and α, γ, ψ and η are taxpayer heterogeneities

$(*)$ = incentive-compatibility or self-selection constraints (zillions of them)

Optimal tax policy

More precisely,

$$U^{i,j,k,g,h}(c_{p,q,r,s,t}, y_{p,q,r,s,t}) = \frac{(c_{p,q,r,s,t} - \alpha_k)^{1-1/\gamma_h}}{1 - 1/\gamma_h} - \psi_g \frac{(y_{p,q,r,s,t}/w_i)^{1/\eta_j+1}}{1/\eta_j + 1}$$

where (i, j, k, g, h) and (p, q, r, s, t) run over 5 dimensions:

na	wage types	$= 5$	21
nb	elasticities of labor supply	$= 3$	3
nc	basic need types	$= 3$	3
nd	levels of distaste for work	$= 2$	2
ne	elasticities of demand for consumption	$= 2$	2
$T =$	$na \times nb \times nc \times nd \times ne$	$= 180$	756
$m =$	$T(T-1)$ nonlinear constraints	$= 32220$	570780
$n =$	$2T$ variables	$= 360$	1512

AMPL model

TAX	maximize$_{c,\,y}$	$\sum_i \lambda_i U^i(c_i, y_i)$	
	subject to	$U^i(c_i, y_i) - U^i(c_j, y_j) \geq 0$	for all i, j
		$\lambda^T(y - c) \geq 0$	
		$c,\ y \geq 0$	

```
Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
          !(i=p and j=q and k=r and g=s and h=t)}:
      (c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
    - (c[p,q,r,s,t] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       + psi[g]*(y[p,q,r,s,t]/w[i])^mu1[j] / mu1[j]
      >= 0;

Technology:
   sum{(i,j,k,g,h) in T} lambda[i,j,k,g,h]*(y[i,j,k,g,h] - c[i,j,k,g,h]) >= 0;
```

Piecewise-smooth extension

```
Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
          !(i=p and j=q and k=r and g=s and h=t)}:
   (if c[i,j,k,g,h] - alpha[k] >= epsilon then
      (c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
    else
       -  0.5/gamma[h] *epsilon^(-1/gamma[h]-1)*(c[i,j,k,g,h] - alpha[k])^2
       + (1+1/gamma[h])*epsilon^(-1/gamma[h]  )*(c[i,j,k,g,h] - alpha[k])
       + (1/(1-1/gamma[h]) - 1 - 0.5/gamma[h])*epsilon^(1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
   )
 - (if c[p,q,r,s,t] - alpha[k] >= epsilon then
      ...
   ) >= 0;
```

SNOPT on problem TAX (1st derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

```
Major Minors    Step   nCon Feasible  Optimal  MeritFunction   nS  condHz Penalty
    0    866              1 (3.7E-15) 4.9E-04  4.1745522E+02    4 4.1E+08 1.0E+04 _   r  t
    1    503 2.7E-02      6 (3.6E-15) 6.5E-02  4.1746922E+02   24 3.2E+05 1.0E+04 _n r  t
    2    134 1.0E-01     11 (1.4E-07) 2.7E-05  4.1755749E+02    8 2.6E+09 1.8E+06 _s
    3    313 9.8E-02     16 (1.4E-07) 8.9E-05  4.1764438E+02   43 1.0E+07 1.8E+06 _
    4    153 2.8E-02     21 (5.5E-08) 1.8E-04  4.1767129E+02   35 2.2E+04 1.8E+06 _
    5    103 2.2E-02     26 (5.4E-08) 9.5E-04  4.1769616E+02   34 6.7E+07 1.8E+06 _

  194  30811 1.0E+00    795   8.6E-01  9.7E-01  2.8330244E+21    2 1.8E+01 3.5E+13 _n   it
  195   1819 1.1E-04    800   8.6E-01  1.0E+00  2.6326936E+22    3 1.4E+02 1.1E+15 _n R it
  195   3314            800   8.6E-01  1.0E+00  2.8661156E+22              1.0E+04 _n r it
  195   4439            800   8.6E-01  9.9E-01  2.8661156E+22              1.0E+04 _n r it

  SNOPTB EXIT  40 -- terminated after numerical difficulties
  SNOPTB INFO  41 -- current point cannot be improved
```

IPOPT on problem TAX (2nd derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

```
This is Ipopt version 3.12.4, running with linear solver mumps.

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 -4.1745522e+02 0.00e+00 2.52e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -4.1734473e+02 6.18e-03 7.36e+00  -1.0 1.34e+00    -  7.69e-01 2.05e-01f  1
   2 -4.1682694e+02 4.93e-03 1.78e+01  -1.0 5.48e+00    -  2.23e-01 1.34e-01f  1

  10 -4.1428766e+02 1.22e-03 1.50e+04  -1.0 3.01e-01   0.6 4.75e-01 5.39e-01h  1

 160 -4.1641067e+02 0.00e+00 1.50e-03  -3.8 1.25e-01    -  1.00e+00 1.00e+00f  1

 449r-4.1630403e+02 1.13e-05 2.79e-05  -8.1 2.92e-01    -  1.00e+00 9.77e-01h  1

                                   (scaled)                 (unscaled)
Dual infeasibility......:   1.1130803588695777e+00    1.1130803588695777e+00
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   1.3412941119075164e-08    1.3412941119075164e-08
```

LANCELOT on problem TAX (2nd derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	rhok	omegak	etak	Obj	itns	CGit	TRradius	active
1	1.0e+1	1.0e-1	1.0e-1	-417.455	18	12000	4.1e-01	2831
2	1.0e+1	1.0e-2	1.2e-2	-421.606	39	9000	1.6e-01	2568
3	1.0e+2	1.0e-2	7.9e-2	-421.011	23	11000	2.4e-01	1662
4	1.0e+2	1.0e-4	1.3e-3	-420.188	282	104000	8.6e-02	1444
5	1.0e+3	1.0e-3	6.3e-2	-419.967	134	64000	5.7e-02	1004
6	1.0e+3	1.0e-6	1.3e-4	-419.819	198	156000	3.1e-02	901
7	1.0e+4	1.0e-4	5.0e-2	-419.741	300	308000	3.1e-12	710
8	1.0e+4	1.0e-6	1.3e-5	-419.698	327	623000	5.5e-04	709
9	1.0e+5	1.0e-5	4.0e-2	-419.682	253	724000	4.7e-03	653
10	1.0e+5	1.0e-6	1.3e-6	-419.676	154	1031000	4.2e-11	663
11	1.0e+6	1.0e-6	3.2e-2	...				

1970 iterations, 8 hours CPU on NEOS

AMPL implementation of NCL

pTax5Dnclipopt.run

```
reset;   model pTax5Dinitial.run;

reset;   model pTax5Dncl.mod;
         data  pTax5Dncl.dat;
         data; var include p5Dinitial.dat;

model;   option solver ipopt;
         option show_stats 1;
         option ipopt_options  'dual_inf_tol=1e-6   max_iter=5000';
```

pTax5Dnclipopt.run

```
option opt2 $ipopt_options ' warm_start_init_point=yes';

for {K in 1..kmax}
{  if K == 2 then {option ipopt_options $opt2 ' mu_init=1e-4'};
   if K == 4 then {option ipopt_options $opt2 ' mu_init=1e-5'};
   if K == 6 then {option ipopt_options $opt2 ' mu_init=1e-6'};
   if K == 8 then {option ipopt_options $opt2 ' mu_init=1e-7'};
   if K ==10 then {option ipopt_options $opt2 ' mu_init=1e-8'};

   solve;

   let rmax := max({(i,j,k,g,h) in T, (p,q,r,s,t) in T:
      !(i=p and j=q and k=r and g=s and h=t)} R[i,j,k,g,h,p,q,r,s,t]);
   let rmin := ...
   let rnorm := max(abs(rmax), abs(rmin));   # ||r||_inf
   if rnorm <= rtol then { printf "Stopping: rnorm is small\n"; break; }
```

pTax5Dnclipopt.run

if $\|r_k^*\| \le \eta_k$, $y_{k+1} \leftarrow y_k + \rho_k r_k^*$

```
if rnorm <= etak then   # update dual estimate dk; save new solution
{let {(i,j,k,g,h) in T, (p,q,r,s,t) in T:
      !(i=p and j=q and k=r and g=s and h=t)}
         dk[i,j,k,g,h,p,q,r,s,t] :=
         dk[i,j,k,g,h,p,q,r,s,t] + rhok*R[i,j,k,g,h,p,q,r,s,t];
 let {(i,j,k,g,h) in T} ck[i,j,k,g,h] := c[i,j,k,g,h];
 let {(i,j,k,g,h) in T} yk[i,j,k,g,h] := y[i,j,k,g,h];

 if  etak == etamin then { printf "Stopping: etak = etamin\n"; break; }
 let etak := max(etak*etafac, etamin);
}
```

pTax5Dnclipopt.run

else $\rho_{k+1} \leftarrow 10\rho_k$

```
else   # keep previous solution; increase rhok
{ let {(i,j,k,g,h) in T} c[i,j,k,g,h] := ck[i,j,k,g,h];
  let {(i,j,k,g,h) in T} y[i,j,k,g,h] := yk[i,j,k,g,h];

  if  rhok == rhomax then { printf "Stopping: rhok = rhomax\n"; break; }
  let rhok := min(rhok*rhofac, rhomax);
}
}  # end main loop
```

Numerical results

Interior Methods (IPMs)

FOLKLORE: We don't know how to warm-start IPMs

Interior Methods (IPMs)

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NCL:

- Sequence of related problems

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- Sequence of related problems
- Only the objective changes

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- Many extra variables r

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FOLKLORE: We don't know how to warm-start IPMs

NCL:

- Sequence of related problems
- Only the objective changes
- Many extra variables r
- r stabilizes iterations, doesn't affect sparsity of factorizations

Interior Methods (IPMs)

FOLKLORE: We don't know how to warm-start IPMs

NCL:

- Sequence of related problems
- Only the objective changes
- Many extra variables r
- r stabilizes iterations, doesn't affect sparsity of factorizations

Maybe warm starts are practical after all!

Warm-start options for Nonlinear Interior Methods

`IPOPT`	`warm_start_init_point=yes`	
	`mu_init=1e-4`	`(1e-5, ..., 1e-8)`

NCL/IPOPT on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	95	41.1
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	7.2
3	10^3	10^{-3}	1.3e-03	-4.1986069e+02	10^{-4}	20	8.1
4	10^4	10^{-3}	4.4e-04	-4.1972958e+02	10^{-4}	48	25.0
5	10^4	10^{-4}	2.2e-04	-4.1968646e+02	10^{-4}	43	20.5
6	10^5	10^{-4}	9.8e-05	-4.1967560e+02	10^{-4}	64	32.9
7	10^5	10^{-5}	6.6e-05	-4.1967177e+02	10^{-4}	57	26.8
8	10^6	10^{-5}	4.2e-06	-4.1967150e+02	10^{-4}	87	46.2
9	10^6	10^{-6}	9.4e-07	-4.1967138e+02	10^{-4}	96	53.6

527 iterations, 5 mins CPU

NCL/IPOPT on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	95	40.8
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	7.0
3	10^3	10^{-3}	1.3e-03	-4.1986069e+02	10^{-4}	20	8.5
4	10^4	10^{-3}	4.4e-04	-4.1972958e+02	10^{-5}	57	32.6
5	10^4	10^{-4}	2.2e-04	-4.1968646e+02	10^{-5}	29	14.6
6	10^5	10^{-4}	9.8e-05	-4.1967560e+02	10^{-6}	36	18.7
7	10^5	10^{-5}	3.9e-05	-4.1967205e+02	10^{-6}	35	19.7
8	10^6	10^{-5}	4.2e-06	-4.1967150e+02	10^{-7}	18	7.7
9	10^6	10^{-6}	9.4e-07	-4.1967138e+02	10^{-7}	15	6.8

322 iterations, 3 mins CPU

NCL/IPOPT on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

Constraints within tol of being active: $c_i(x) \le tol$

	tol	$count$	$count/n$		
	10^{-10}	548	1.5		
	10^{-9}	550	1.5		
	10^{-8}	591	1.6		
	10^{-7}	890	2.5		
$\rightarrow$	10^{-6}	1104	3.1	$\leftarrow$	About $3n$ active constraints
	10^{-5}	1225	3.4		
	10^{-4}	1301	3.6		
	10^{-3}	1655	4.6		
	10^{-2}	3483	9.7		
	10^{-1}	10280	28.6		

NCL/IPOPT bigger example

$na, nb, nc, nd, ne = 21, 3, 3, 2, 2 \quad m = 570780 \quad n = 1512$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	5.1e-03	-1.7656816e+03	10^{-1}	825	7763
2	10^2	10^{-3}	2.4e-03	-1.7648480e+03	10^{-4}	66	473
3	10^3	10^{-3}	1.3e-03	-1.7644006e+03	10^{-4}	106	771
4	10^4	10^{-3}	3.8e-04	-1.7639491e+03	10^{-5}	132	1347
5	10^4	10^{-4}	3.2e-04	-1.7637742e+03	10^{-5}	229	2451
6	10^5	10^{-4}	8.6e-05	-1.7636804e+03	10^{-6}	104	1097
7	10^5	10^{-5}	4.9e-05	-1.7636469e+03	10^{-6}	143	1633
8	10^6	10^{-5}	1.5e-05	-1.7636252e+03	10^{-7}	71	786
9	10^7	10^{-5}	2.8e-06	-1.7636196e+03	10^{-7}	67	726
10	10^7	10^{-6}	5.1e-07	-1.7636187e+03	10^{-8}	18	171

1761 iterations, 5 hours CPU

NCL/IPOPT bigger example

$na, nb, nc, nd, ne = 21, 3, 3, 2, 2 \quad m = 570780 \quad n = 1512$

Constraints within tol of being active: $c_i(x) \le tol$

	tol	*count*	*count*/*n*		
	10^{-10}	3888	2.6		
	10^{-9}	3941	2.6		
	10^{-8}	4430	2.9		
	10^{-7}	7158	4.7		
$\rightarrow$	10^{-6}	10074	6.6	$\leftarrow$	$\approx 6.6n$ active constraints
	10^{-5}	11451	7.6		
	10^{-4}	13109	8.7		
	10^{-3}	23099	15.3		
	10^{-2}	66361	43.9		
	10^{-1}	202664	134.0		

Warm-start options for Nonlinear Interior Methods

IPOPT	warm_start_init_point=yes	
	mu_init=1e-4	(1e-5, ..., 1e-8)
KNITRO	algorithm=1	Thanks, Richard Waltz!
	bar_directinterval=0	
	bar_initpt=2	
	bar_murule=1	
	bar_initmu=1e-4	(1e-5, ..., 1e-8)
	bar_slackboundpush=1e-4	(1e-5, ..., 1e-8)

Comparison of IPOPT, KNITRO, NCL (2nd derivs)

na = increasing $\quad nb = 3 \quad nc = 3 \quad nd = 2 \quad ne = 2$

			IPOPT		KNITRO		NCL/IPOPT		NCL/KNITRO	
na	m	n	itns	time	itns	time	itns	time	itns	time
5	32220	360	449	217	168	53	322	146	2320	8.0mins
9	104652	648	> 98*	> 360*	928	825	655	1023	9697	1.9hrs
11	156420	792	> 87*	∞!	2769	4117	727	1679	26397	7.0hrs
17	373933	1224			2598	11447	1021	6347		
21	570780	1512					1761	17218	45039	1.9 days

*duals diverge

MUMPS needs more mem

!Loop

Warm starts (NCL/IPOPT) Cold starts (NCL/KNITRO)

NCL/KNITRO with Warm Starts

na = increasing $nb = 3$ $nc = 3$ $nd = 2$ $ne = 2$

			IPOPT		KNITRO		NCL/IPOPT		NCL/KNITRO	
na	*m*	*n*	itns	time	itns	time	itns	time	itns	time
5	32220	360	449	217	168	53	322	146	339	63
9	104652	648	> 98*	> 360*	928	825	655	1023	307	239
11	156420	792	> 87*	∞!	2769	4117	727	1679	383	420
17	373933	1224			2598	11447	1021	6347	486	1200
21	570780	1512					1761	17218	712	2880

Warm starts　　Warm starts

Related work

- C. M. Maes, A Regularized Active-Set Method for Sparse Convex Quadratic Programming. PhD thesis, ICME, Stanford University, 2010.
- M. P. Friedlander and D. Orban, A primal-dual regularized interior-point method for convex quadratic programs. Math. Prog. Comp., 4(1):71–107, 2012.
- S. Arreckx and D. Orban, A regularized factorization-free method for equality-constrained optimization, Technical Report GERAD G-2016-65, GERAD, Montréal, QC, Canada, 2016, doi:10.13140/RG.2.2.20368.00007.

- P. E. Gill, V. Kungurtsev, and D. P. Robinson, A stabilized SQP method: global convergence, IMA J. Numer. Anal., 37 (2017), 407–443.
- P. E. Gill, V. Kungurtsev, and D. P. Robinson, A stabilized SQP method: superlinear convergence, Math. Program., Ser. A, 163 (2017), 369–410.

- O. Hinder and Y. Ye, A one-phase IPM for nonconvex optimization, Oliver's MS&E PhD thesis (2018).

A future possibility

AMPL + IPM + 2nd derivatives

KNOSSOS via AMPL + IPM?

Stablized LCL (Friedlander and S, 2005) is equivalent to a BCL method:

$$\boxed{\begin{array}{lll} \text{ELC}_k & \displaystyle\min_x & L(x, y_k, \rho_k) + \sigma_k \left\|\bar{c}_k(x)\right\|_1 \\ & \text{subject to} & \ell \le x \le u \end{array}}$$

- $\bar{c}_k(x) =$ linear approximation to $c(x)$ at x_k (MINOS has very big σ_k)

KNOSSOS via AMPL + IPM?

Stablized LCL (Friedlander and S, 2005) is equivalent to a BCL method:

ELC_k	$\underset{x}{\text{minimize}} \quad L(x, y_k, \rho_k) + \sigma_k \, \|\bar{c}_k(x)\|_1$ subject to $\ell \le x \le u$

- $\bar{c}_k(x)$ = linear approximation to $c(x)$ at x_k (MINOS has very big σ_k)

ELC''_k	$\underset{x,v,w}{\text{minimize}} \quad M_k(x, v, w) + \sigma_k e^T(v + w)$ subject to $\bar{c}_k(x) + v - w = 0, \quad \ell \le x \le u, \quad v, w \ge 0$

- M_k = modified augmented Lagrangian $d_k(x, v, w) = c(x) - \bar{c}_k(x) - v + w$
- IPOPT, KNITRO, . . . won't mind the extra elastic variables v, w

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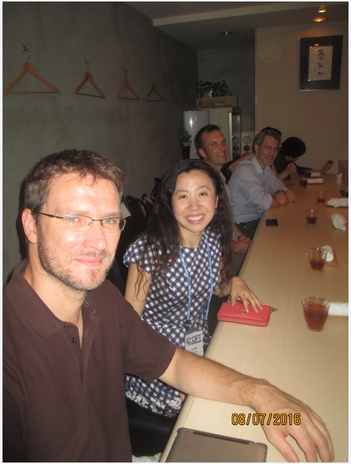
- KNITRO 10.3.0: Richard Waltz, Jorge Nocedal, Todd Plantenga, Richard Byrd
- Coauthors: Ken Judd, Che-Lin Su, Ding Ma, Dominique Orban
- Yuja Wang, YouTube

Coauthors Ken, Che-Lin, Dominique, Ding



YouTube companion Yuja Wang