Algorithm NCL for constrained optimization

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CME 510 Linear Algebra and Optimization Seminar
ICME, Stanford University, April 5, 2018

Abstract

Standard optimization solvers have difficulty if too many constraints are essentially active at a solution. For example, problems of the form

$$\min f(x) \quad \text{st} \quad c(x) \geq 0 \qquad (m \text{ constraints and } n \text{ variables})$$

may have more than n constraints active at a solution. Such problems arise in the modeling of tax policy (with perhaps millions of constraints and thousands of variables).

Algorithm NCL solves a sequence of about 10 subproblems with regularized constraints $c(x) + r \geq 0$ and an augmented Lagrangian objective $f(x) - y_k^T r + \frac{1}{2}\rho_k r^T r$ that drives the extra variables r to zero. Interior methods are able to warm start each subproblem. Assuming second derivatives are available, NCL expands the use of interior methods for large-scale optimization.

Partially supported by the
National Institute of General Medical Sciences
of the National Institutes of Health (NIH)
Award U01GM102098

Optimization

UCO

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \phi(x)$$

- $\phi(x)$ is a smooth nonlinear function
- $\nabla\phi(x)$ known: Quasi-Newton method
- $\nabla^2\phi(x)$ known: Newton's method

NCO

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \phi(x) \\ \text{subject to} \quad & c(x) = 0 \end{aligned}$$

- $c(x) \in \mathbb{R}^m$ is a vector of smooth nonlinear functions

Constrained Optimization

NCO

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & \phi(x) \\ \text{subject to} & c(x) = 0 \end{array}$$

Penalty function

$$P(x, \rho_k) = \phi(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2$$

- Penalty parameter $\rho_k \to \infty$

Augmented Lagrangian

$$L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2$$

- Lagrange multiplier estimate y_k ρ_k can remain finite

A general optimization problem

A general optimization problem

NLP
$$\min_{x\in\mathbb{R}^n} \ \phi(x) \quad \text{subject to} \quad c(x)=0, \quad \ell \le x \le u$$

- $c(x)=0$ includes linear constraints $Ax=b$
- Bounds on the variables x

LANCELOT (1992) solves large-scale optimization problems of this form

- Inequalities $\bar{\ell} \le c(x) \le \bar{u}$ are equalities with more variables and bounds:

NLP
$$\min_{x,s} \ \phi(x) \quad \text{subject to} \quad c(x)-s=0, \quad \begin{pmatrix}\ell\\ \bar{\ell}\end{pmatrix} \le \begin{pmatrix}x\\ s\end{pmatrix} \le \begin{pmatrix}u\\ \bar{u}\end{pmatrix}$$

LANCELOT's BCL algorithm for general NLP

Conn, Gould & Toint (1992)

LANCELOT

$$\min \phi(x) \quad \text{st} \quad c(x) = 0, \quad \ell \le x \le u$$

BCL subproblems (Bound-Constrained augmented Lagrangian):

$$\text{BC}_k \qquad \underset{x}{\text{minimize}} \quad \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2 \qquad \text{subject to} \quad \ell \le x \le u$$

Loop:

solve BC_k to get x_k^*	decreasing opttol ω_k
if $\|c(x_k^*)\| \le \eta_k$, $y_{k+1} \leftarrow y_k - \rho_k c(x_k^*)$	decreasing featol η_k
else $\rho_{k+1} \leftarrow 10\rho_k$	

Our optimization problem

Our NLP problem

NLP	$$\min_{x\in\mathbb{R}^n} \ \phi(x) \quad \text{subject to} \quad c(x) \ge 0, \quad \ell \le x \le u$$

- $\phi(x)$ is a smooth nonlinear function
- $c(x) \in \mathbb{R}^m$ is a vector of smooth nonlinear functions
- General bounds
- Many inequalities $c(x) \ge 0$ might not satisfy LICQ at x^*

Example: $m = 571,000, \quad n = 1500$

10,000 constraints essentially active: $c_i(x^*) \le 10^{-6}$

BCL

LCL

NCL

Sequence of subproblems minimizing X-constrained (augmented) Lagrangian

BCL	LANCELOT	Conn, Gould & Toint (1992)
LCL	linearized constraints	Robinson (1972)
	MINOS	Murtagh and S (1982)
sLCL	KNOSSOS	Friedlander (2002)
NCL	New form of **BCL**!	Today's talk
	AMPL loop + IPOPT	

Algorithm NCL
for general NLP

NCL subproblems

NLP

$$\begin{aligned} \underset{x,s}{\text{minimize}} \quad & \phi(x) \\ \text{subject to} \quad & c(x) = 0, \quad \ell \le x \le u \end{aligned}$$

NCL subproblems

NLP

$$\underset{x,s}{\text{minimize}} \quad \phi(x) \qquad \text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$$

LANCELOT-type subproblems:

BC_k

$$\underset{x,s}{\text{minimize}} \quad L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2 \qquad \text{subject to} \quad \ell \le x \le u$$

NCL subproblems

NLP

$$\underset{x,s}{\text{minimize}} \quad \phi(x)$$
$$\text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$$

LANCELOT-type subproblems:

BC_k

$$\underset{x,s}{\text{minimize}} \quad L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k \|c(x)\|^2$$
$$\text{subject to} \quad \ell \le x \le u$$

Introduce $r = -c(x)$:

NC_k

$$\underset{x,r}{\text{minimize}} \quad \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k \|r\|^2$$
$$\text{subject to} \quad c(x) + r = 0, \quad \ell \le x \le u$$

Free vars r make the nonlinear constraints independent and feasible IPOPT is happy!

NCL subproblems

NLP

$$\underset{x,s}{\text{minimize}} \quad \phi(x)$$
$$\text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$$

NC$_k$

$$\underset{x,r}{\text{minimize}} \quad \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k \|r\|^2$$
$$\text{subject to} \quad c(x) + r = 0, \quad \ell \le x \le u$$

Free vars r make the nonlinear constraints independent and feasible IPOPT is happy!

NCL subproblems for our problem

NLP

$$\begin{array}{ll} \underset{x,\,s}{\text{minimize}} & \phi(x) \\ \text{subject to} & c(x) \geq 0, \quad \ell \leq x \leq u \end{array}$$

NC_k

$$\begin{array}{ll} \underset{x,\,r}{\text{minimize}} & \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k \|r\|^2 \\ \text{subject to} & c(x) + r \geq 0, \quad \ell \leq x \leq u \end{array}$$

Free vars r make the nonlinear constraints independent and feasible IPOPT is happy!

Optimal Taxation Policy

Optimal income taxation with multidimensional taxpayer types

Kenneth L. Judd and Che-Lin Su

Manuscript, Dec 2011


Optimal tax policy

TAX	$\text{maximize}_{c,\,y}$	$\sum_i \lambda_i U^i(c_i, y_i)$	
	subject to	$U^i(c_i, y_i) - U^i(c_j, y_j) \geq 0$	for all i, j
		$\lambda^T(y - c) \geq 0$	
		$c,\ y \geq 0,$	

where c_i and y_i are the consumption and income of taxpayer i, and λ is a vector of positive weights. The utility functions $U^i(c_i, y_i)$ are each of the form

$$U(c, y) = \frac{(c - \alpha)^{1-1/\gamma}}{1 - 1/\gamma} - \psi \frac{(y/w)^{1/\eta+1}}{1/\eta + 1},$$

where w is the wage rate and α, γ, ψ, η are taxpayer heterogeneities.

Optimal tax policy

More precisely,

$$U^{i,j,k,g,h}(c_{p,q,r,s,t}, y_{p,q,r,s,t}) = \frac{(c_{p,q,r,s,t} - \alpha_k)^{1-1/\gamma_h}}{1 - 1/\gamma_h} - \psi_g \frac{(y_{p,q,r,s,t}/w_i)^{1/\eta_j+1}}{1/\eta_j + 1},$$

where (i, j, k, g, h) and (p, q, r, s, t) run over 5 dimensions:

na	wage types w	$= 5$
nb	elasticities of labor supply η	$= 3$
nc	basic need types α	$= 3$
nd	levels of distaste for work ψ	$= 2$
ne	elasticities of demand for consumption γ	$= 2$

$T =$	$na \times nb \times nc \times nd \times ne$	$= 180$
$m =$	$T(T-1)$ nonlinear constraints	$= 32220$
$n =$	$2T$ variables	$= 360$

AMPL model

$$
\begin{array}{llll}
\text{TAX} & \text{maximize}_{c,\,y} & \sum_i \lambda_i U^i(c_i, y_i) & \\
& \text{subject to} & U^i(c_i, y_i) - U^i(c_j, y_j) \geq 0 & \text{for all } i, j \\
& & \lambda^T(y - c) \geq 0 & \\
& & c,\ y \geq 0, &
\end{array}
$$

```
Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
          !(i=p and j=q and k=r and g=s and h=t)}:
      (c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
    - (c[p,q,r,s,t] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       + psi[g]*(y[p,q,r,s,t]/w[i])^mu1[j] / mu1[j]
      >= 0;

Technology:
   sum{(i,j,k,g,h) in T} lambda[i,j,k,g,h]*(y[i,j,k,g,h] - c[i,j,k,g,h]) >= 0;
```

Piecewise-smooth extension

```
Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
          !(i=p and j=q and k=r and g=s and h=t)}:
   (if c[i,j,k,g,h] - alpha[k] >= epsilon then
      (c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
    else
       -  0.5/gamma[h] *epsilon^(-1/gamma[h]-1)*(c[i,j,k,g,h] - alpha[k])^2
       + (1+1/gamma[h])*epsilon^(-1/gamma[h]  )*(c[i,j,k,g,h] - alpha[k])
       + (1/(1-1/gamma[h]) - 1 - 0.5/gamma[h])*epsilon^(1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
   )
 - (if c[p,q,r,s,t] - alpha[k] >= epsilon then
      ...
   ) >= 0;
```

SNOPT on problem TAX

$$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$$

	Total	Normal	Free	Fixed	Bounded
Rows	32222	32221	1	0	0
Columns	360	0	0	0	360

```
No. of matrix elements            129600     Density       1.117
Biggest                       1.0000E+00  (excluding fixed columns,
Smallest                      0.0000E+00   free rows, and RHS)

Nonlinear constraints   32220     Linear constraints       2
Nonlinear variables       360     Linear variables         0
Jacobian  variables       360     Objective variables    360
Total constraints       32222     Total variables        360
```

SNOPT on problem TAX

Major	Minors	Step	nCon	Feasible	Optimal	MeritFunction	nS	condHz	Penalty	
0	866		1	(3.7E-15)	4.9E-04	4.1745522E+02	4	4.1E+08	1.0E+04	_ r t
1	503	2.7E-02	6	(3.6E-15)	6.5E-02	4.1746922E+02	24	3.2E+05	1.0E+04	_n r t
2	134	1.0E-01	11	(1.4E-07)	2.7E-05	4.1755749E+02	8	2.6E+09	1.8E+06	_s
3	313	9.8E-02	16	(1.4E-07)	8.9E-05	4.1764438E+02	43	1.0E+07	1.8E+06	_
4	153	2.8E-02	21	(5.5E-08)	1.8E-04	4.1767129E+02	35	2.2E+04	1.8E+06	_
5	103	2.2E-02	26	(5.4E-08)	9.5E-04	4.1769616E+02	34	6.7E+07	1.8E+06	_
6	87	2.1E-02	31	(5.3E-08)	1.2E-03	4.1772079E+02	32	1.6E+08	1.8E+06	_
7	161	7.3E-03	35	(5.2E-08)	9.7E-04	4.1772930E+02	31	2.4E+05	1.8E+06	_
8	64	7.3E-03	39	(5.2E-08)	1.1E-03	4.1773783E+02	33	9.7E+05	1.8E+06	_
9	83	7.2E-03	43	(5.2E-08)	1.2E-03	4.1774637E+02	37	1.9E+07	1.8E+06	_
10	72	7.2E-03	47	(5.1E-08)	9.9E-04	4.1775484E+02	31	1.3E+08	1.8E+06	_
99	154	7.2E-03	459	(1.0E-08)	2.7E-05	4.1839776E+02	32	7.9E+06	1.8E+06	_
100	343	7.2E-03	463	(1.0E-08)	4.9E-03	4.1839867E+02	50	4.3E+07	1.8E+06	_ R
106	413	7.3E-03	496	(1.0E-08)	1.8E-05	4.1840397E+02	17	5.0E+07	1.8E+06	_
107	630	7.2E-03	500	(1.0E-08)	1.4E-01	1.5496435E+06	6	1.0E+12	1.8E+06	_ it

SNOPT on problem TAX

```
Major Minors    Step   nCon Feasible  Optimal   MeritFunction   nS  condHz Penalty
  108     503 5.1E-02   503 (1.0E-08) 3.7E-02  1.2973689E+06   12 2.2E+04 1.8E+06 _ m
  109    1805 4.0E-01   506  1.3E-05  7.5E-01  2.3393279E+05    9 4.8E+04 1.8E+06 _ m  it
  110   16517 2.0E-01   509  1.1E-05  1.0E+00  2.6854996E+05    2 7.5E+03 1.8E+06 _    it

  117 175544 1.0E+00    526  1.0E-04  1.0E+00  1.0282166E+09   13 7.6E+07 1.8E+06 _ mr it
  118    1681 1.0E+00   528  3.0E-04  1.0E+00  1.7677788E+10   23 1.7E+09 2.9E+06 _smr it
  119     919 1.0E+00   530  2.0E-05  2.4E-02  8.1274014E+07   39 1.1E+06 3.0E+07 _sm   t

  194   30811 1.0E+00   795  8.6E-01  9.7E-01  2.8330244E+21    2 1.8E+01 3.5E+13 _n   it
  195    1819 1.1E-04   800  8.6E-01  1.0E+00  2.6326936E+22    3 1.4E+02 1.1E+15 _n R it
  195    3314           800  8.6E-01  1.0E+00  2.8661156E+22              1.0E+04 _n r it
  195    4439           800  8.6E-01  9.9E-01  2.8661156E+22              1.0E+04 _n r it

  SNOPTB EXIT  40 -- terminated after numerical difficulties
  SNOPTB INFO  41 -- current point cannot be improved
```

IPOPT on problem TAX

$$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$$

```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in inequality constraint Jacobian.:   129240
Number of nonzeros in Lagrangian Hessian.............:      360

Total number of variables............................:      360
                     variables with only lower bounds:      360
Total number of inequality constraints...............:    32221
        inequality constraints with only lower bounds:    32221
```

IPOPT on problem TAX

iter	objective	inf_pr	inf_du	lg(mu)	||d||	lg(rg)	alpha_du	alpha_pr	ls
0	-4.1745522e+02	0.00e+00	2.52e+00	-1.0	0.00e+00	-	0.00e+00	0.00e+00	0
1	-4.1734473e+02	6.18e-03	7.36e+00	-1.0	1.34e+00	-	7.69e-01	2.05e-01f	1
2	-4.1682694e+02	4.93e-03	1.78e+01	-1.0	5.48e+00	-	2.23e-01	1.34e-01f	1
3	-4.1656532e+02	4.60e-03	1.66e+02	-1.0	6.95e+00	-	3.06e-01	5.29e-02f	1
4	-4.1553526e+02	3.69e-03	2.13e+02	-1.0	9.40e+00	-	2.38e-01	1.60e-01f	1
5	-4.1470856e+02	3.45e-03	2.96e+02	-1.0	1.14e+01	-	1.65e-01	1.17e-01f	1
6	-4.1433731e+02	3.72e-03	9.79e+02	-1.0	1.30e+01	-	1.73e-01	5.83e-02f	1
7	-4.1433774e+02	3.53e-03	5.22e+03	-1.0	2.47e-02	2.0	5.60e-01	5.64e-02h	1
8	-4.1433988e+02	3.09e-03	5.10e+04	-1.0	3.46e-01	1.5	1.00e+00	1.24e-01h	1
9	-4.1434074e+02	2.15e-03	2.87e+04	-1.0	3.40e-01	1.0	5.03e-01	3.01e-01h	1
10	-4.1428766e+02	1.22e-03	1.50e+04	-1.0	3.01e-01	0.6	4.75e-01	5.39e-01h	1
11	-4.1429091e+02	1.05e-03	1.27e+04	-1.0	2.21e-01	1.0	3.87e-01	1.24e-01h	1
77	-4.0990702e+02	0.00e+00	4.57e-02	-1.0	6.32e-03	-	1.00e+00	1.00e+00h	1
78	-4.0995159e+02	0.00e+00	5.00e+04	-2.5	4.68e+00	-	9.27e-01	1.00e+00f	1

IPOPT on problem TAX

```
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  88 -4.1227845e+02 0.00e+00 4.45e-03  -2.5 2.55e+00    -  1.00e+00 1.00e+00h  1
  89 -4.1320008e+02 2.97e-03 8.22e+02  -3.8 6.90e+01    -  2.57e-01 4.28e-01f  1
 160 -4.1641067e+02 0.00e+00 1.50e-03  -3.8 1.25e-01    -  1.00e+00 1.00e+00f  1
 161 -4.1646520e+02 8.99e-06 9.47e+00  -5.7 4.36e+00    -  2.62e-01 2.63e-01f  1
 384 -4.1702347e+02 4.14e-07 3.89e+09  -5.7 4.11e-03   0.6 2.59e-01 1.15e-01f  1
 385r-4.1702347e+02 4.14e-07 1.00e+03  -5.4 0.00e+00   4.6 0.00e+00 2.07e-07R  2
 419r-4.1580769e+02 0.00e+00 9.69e-06  -5.4 2.10e-02    -  1.00e+00 1.00e+00h  1
 420r-4.1589722e+02 9.33e-07 2.86e+02  -8.1 1.29e+01    -  1.62e-01 3.97e-01f  1
 449r-4.1630403e+02 1.13e-05 2.79e-05  -8.1 2.92e-01    -  1.00e+00 9.77e-01h  1

                                   (scaled)                 (unscaled)
Objective...............:  -4.1630466137535933e+02   -4.1630466137535933e+02
Dual infeasibility......:   1.1130803588695777e+00    1.1130803588695777e+00
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   1.3412941119075164e-08    1.3412941119075164e-08
Total CPU secs in IPOPT (w/o function evaluations)   =    151.4
Total CPU secs in NLP function evaluations           =     65.6
```

LANCELOT on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	rhok	etak	omegak	Obj	itns	CGit	TRradius	active
1	1.0e+1	1.0e-1	1.0e-1	-417.455	18	12000	4.1e-01	2831
2	1.0e+1	1.2e-2	1.0e-2	-421.606	39	9000	1.6e-01	2568
3	1.0e+2	7.9e-2	1.0e-2	-421.011	23	11000	2.4e-01	1662
4	1.0e+2	1.3e-3	1.0e-4	-420.188	282	104000	8.6e-02	1444
5	1.0e+3	6.3e-2	1.0e-3	-419.967	134	64000	5.7e-02	1004
6	1.0e+3	1.3e-4	1.0e-6	-419.819	198	156000	3.1e-02	901
7	1.0e+4	5.0e-2	1.0e-4	-419.741	300	308000	3.1e-12	710
8	1.0e+4	1.3e-5	1.0e-6	-419.698	327	623000	5.5e-04	709
9	1.0e+5	4.0e-2	1.0e-5	-419.682	253	724000	4.7e-03	653
10	1.0e+5	1.3e-6	1.0e-6	-419.676	154	1031000	4.2e-11	663
11	1.0e+6	3.2e-2	1.0e-6	...				

1970 iterations, 8 hours CPU on NEOS

AMPL implementation of NCL

pTax5Dnclipopt.run

```
reset;   model pTax5Dinitial.run;

reset;   model pTax5Dncl.mod;
         data  pTax5Dncl.dat;
         data; var include p5Dinitial.dat;

model;   option solver ipopt;
         option show_stats 1;
         option ipopt_options  'dual_inf_tol=1e-6   max_iter=5000';
```

pTax5Dnclipopt.run

```
option opt2 $ipopt_options ' warm_start_init_point=yes';

for {K in 1..kmax}
{  if K == 2 then {option ipopt_options $opt2 ' mu_init=1e-4'};
   if K == 4 then {option ipopt_options $opt2 ' mu_init=1e-5'};
   if K == 6 then {option ipopt_options $opt2 ' mu_init=1e-6'};
   if K == 8 then {option ipopt_options $opt2 ' mu_init=1e-7'};
   if K ==10 then {option ipopt_options $opt2 ' mu_init=1e-8'};

   solve;

   let rmax := max({(i,j,k,g,h) in T, (p,q,r,s,t) in T:
      !(i=p and j=q and k=r and g=s and h=t)} R[i,j,k,g,h,p,q,r,s,t]);
   let rmin := ...
   let rnorm := max(abs(rmax), abs(rmin));   # ||r||_inf
   if rnorm <= rtol then { printf "Stopping: rnorm is small\n"; break; }
```

pTax5Dnclipopt.run

if $\|r_k^*\| \leq \eta_k$, $y_{k+1} \leftarrow y_k + \rho_k r_k^*$

```
if rnorm <= etak then   # update dual estimate dk; save new solution
{let {(i,j,k,g,h) in T, (p,q,r,s,t) in T:
      !(i=p and j=q and k=r and g=s and h=t)}
         dk[i,j,k,g,h,p,q,r,s,t] :=
         dk[i,j,k,g,h,p,q,r,s,t] + rhok*R[i,j,k,g,h,p,q,r,s,t];
 let {(i,j,k,g,h) in T} ck[i,j,k,g,h] := c[i,j,k,g,h];
 let {(i,j,k,g,h) in T} yk[i,j,k,g,h] := y[i,j,k,g,h];

 if  etak == etamin then { printf "Stopping: etak = etamin\n"; break; }
 let etak := max(etak*etafac, etamin);
}
```

pTax5Dnclipopt.run

else $\rho_{k+1} \leftarrow 10\rho_k$

```
   else   # keep previous solution; increase rhok
   { let {(i,j,k,g,h) in T} c[i,j,k,g,h] := ck[i,j,k,g,h];
     let {(i,j,k,g,h) in T} y[i,j,k,g,h] := yk[i,j,k,g,h];

     if  rhok == rhomax then { printf "Stopping: rhok = rhomax\n"; break; }
     let rhok := min(rhok*rhofac, rhomax);
   }
}  # end main loop
```

Numerical results

NCL/IPOPT on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	95	41.1
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	7.2
3	10^3	10^{-3}	1.3e-03	-4.1986069e+02	10^{-4}	20	8.1
4	10^4	10^{-3}	4.4e-04	-4.1972958e+02	10^{-4}	48	25.0
5	10^4	10^{-4}	2.2e-04	-4.1968646e+02	10^{-4}	43	20.5
6	10^5	10^{-4}	9.8e-05	-4.1967560e+02	10^{-4}	64	32.9
7	10^5	10^{-5}	6.6e-05	-4.1967177e+02	10^{-4}	57	26.8
8	10^6	10^{-5}	4.2e-06	-4.1967150e+02	10^{-4}	87	46.2
9	10^6	10^{-6}	9.4e-07	-4.1967138e+02	10^{-4}	96	53.6

527 iterations, 5 mins CPU

NCL/IPOPT on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	`mu_init`	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	95	40.8
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	7.0
3	10^3	10^{-3}	1.3e-03	-4.1986069e+02	10^{-4}	20	8.5
4	10^4	10^{-3}	4.4e-04	-4.1972958e+02	10^{-5}	57	32.6
5	10^4	10^{-4}	2.2e-04	-4.1968646e+02	10^{-5}	29	14.6
6	10^5	10^{-4}	9.8e-05	-4.1967560e+02	10^{-6}	36	18.7
7	10^5	10^{-5}	3.9e-05	-4.1967205e+02	10^{-6}	35	19.7
8	10^6	10^{-5}	4.2e-06	-4.1967150e+02	10^{-7}	18	7.7
9	10^6	10^{-6}	9.4e-07	-4.1967138e+02	10^{-7}	15	6.8

322 iterations, 3 mins CPU

NCL/IPOPT on problem TAX

$$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$$

Constraints within tol of being active: $c_i(x) \leq tol$

tol	$count$	$count/n$
10^{-10}	548	1.5
10^{-9}	550	1.5
10^{-8}	591	1.6
10^{-7}	890	2.5
$\rightarrow 10^{-6}$	1104	3.1 $\leftarrow$ About $3n$ active constraints
10^{-5}	1225	3.4
10^{-4}	1301	3.6
10^{-3}	1655	4.6
10^{-2}	3483	9.7
10^{-1}	10280	28.6

NCL/KNITRO on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	168	34.6
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	3.3
3	10^3	10^{-3}	1.3e-03	-4.1986068e+02	10^{-4}	87	22.8
4	10^4	10^{-3}	4.2e-04	-4.1972698e+02	10^{-5}	4583	2141.9
5	10^4	10^{-4}	1.7e-04	-4.1968557e+02	10^{-5}	63	13.1
6	10^5	10^{-4}	1.1e-04	-4.1967201e+02	10^{-6}	4482	2884.5
7	10^6	10^{-4}	1.1e-05	-4.1964566e+02	10^{-6}	8754	7944.4
8	10^6	10^{-5}	1.2e-06	-4.1964629e+02	10^{-7}	403	194.9
9	10^6	10^{-6}	7.9e-06	-4.1964298e+02	10^{-7}	355	168.7
10	10^7	10^{-6}	4.8e-07	-4.1964798e+02	10^{-8}	7176	10239.7

26088 iterations, 394 mins CPU

NCL/IPOPT bigger example

$na, nb, nc, nd, ne = 21, 3, 3, 2, 2 \quad m = 570780 \quad n = 1512$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	5.1e-03	-1.7656816e+03	10^{-1}	825	7763.3
2	10^2	10^{-3}	2.4e-03	-1.7648480e+03	10^{-4}	66	472.8
3	10^3	10^{-3}	1.3e-03	-1.7644006e+03	10^{-4}	106	771.3
4	10^4	10^{-3}	3.8e-04	-1.7639491e+03	10^{-5}	132	1347.0
5	10^4	10^{-4}	3.2e-04	-1.7637742e+03	10^{-5}	229	2450.9
6	10^5	10^{-4}	8.6e-05	-1.7636804e+03	10^{-6}	104	1096.9
7	10^5	10^{-5}	4.9e-05	-1.7636469e+03	10^{-6}	143	1633.4
8	10^6	10^{-5}	1.5e-05	-1.7636252e+03	10^{-7}	71	786.1
9	10^7	10^{-5}	2.8e-06	-1.7636196e+03	10^{-7}	67	725.7
10	10^7	10^{-6}	5.1e-07	-1.7636187e+03	10^{-8}	18	171.0

1761 iterations, 5 hours CPU

NCL/IPOPT bigger example

$na, nb, nc, nd, ne = 21, 3, 3, 2, 2 \quad m = 570780 \quad n = 1512$

Constraints within tol of being active: $c_i(x) \le tol$

	tol	*count*	*count/n*	
	10^{-10}	3888	2.6	
	10^{-9}	3941	2.6	
	10^{-8}	4430	2.9	
	10^{-7}	7158	4.7	
$\rightarrow$	10^{-6}	10074	6.6	$\leftarrow$ $\approx 6.6n$ active constraints
	10^{-5}	11451	7.6	
	10^{-4}	13109	8.7	
	10^{-3}	23099	15.3	
	10^{-2}	66361	43.9	
	10^{-1}	202664	134.0	

Comparison of IPOPT, KNITRO (via NEOS), NCL

$na = \text{increasing} \quad nb = 3 \quad nc = 3 \quad nd = 2 \quad ne = 2$

			IPOPT		KNITRO		NCL/IPOPT	
na	*m*	*n*	itns	time	itns	time	itns	time
5	32220	360	449	217	168	53	322	146
9	104652	648	> 98*	> 360*	928	825	655	1023
11	156420	792	> 87*	∞!	2769	4117	727	1679
17	373933	1224			2598	11447	1021	6347
21	570780	1512					1761	17218

*duals diverge
 MUMPS needs more mem
!Loop

Comparison of IPOPT, KNITRO (on iMAC), NCL

na = increasing $nb = 3$ $nc = 3$ $nd = 2$ $ne = 2$

			IPOPT		KNITRO		NCL/IPOPT	
na	m	n	itns	time	itns	time	itns	time
5	32220	360	449	217	736	141	322	146
9	104652	648	> 98*	> 360*	862	630	655	1023
11	156420	792	> 87*	∞!	2835	3719	727	1679
17	373933	1224			3856	14746	1021	6347
21	570780	1512					1761	17218

*duals diverge

∞ MUMPS needs more mem

! Loop

Other research

- M. P. Friedlander and D. Orban, A primal-dual regularized interior-point method for convex quadratic programs. Math. Prog. Comp., 4(1):71–107, 2012.
- S. Arreckx and D. Orban, A regularized factorization-free method for equality-constrained optimization, Technical Report GERAD G-2016-65, GERAD, Montréal, QC, Canada, 2016, doi:10.13140/RG.2.2.20368.00007.

- P. E. Gill, V. Kungurtsev, and D. P. Robinson, A stabilized SQP method: global convergence, IMA J. Numer. Anal., 37 (2017), 407–443.
- P. E. Gill, V. Kungurtsev, and D. P. Robinson, A stabilized SQP method: superlinear convergence, Math. Program., Ser. A, 163 (2017), 369–410.

- O. Hinder and Y. Ye, A one-phase IPM for nonconvex optimization, Oliver's MS&E PhD thesis (2018).

A future possibility
AMPL + IPM + 2nd derivatives

KNOSSOS via AMPL + IPM?

Stablized LCL (Friedlander and S, 2005) is equivalent to a BCL method:

ELC$_k$	$\underset{x}{\text{minimize}}$	$L(x, y_k, \rho_k) + \sigma_k \left\lVert \bar{c}_k(x) \right\rVert_1$
	subject to	$\ell \leq x \leq u$

- $\bar{c}_k(x)$ = linear approximation to $c(x)$ at x_k (MINOS has very big σ_k)

KNOSSOS via AMPL + IPM?

Stablized LCL (Friedlander and S, 2005) is equivalent to a BCL method:

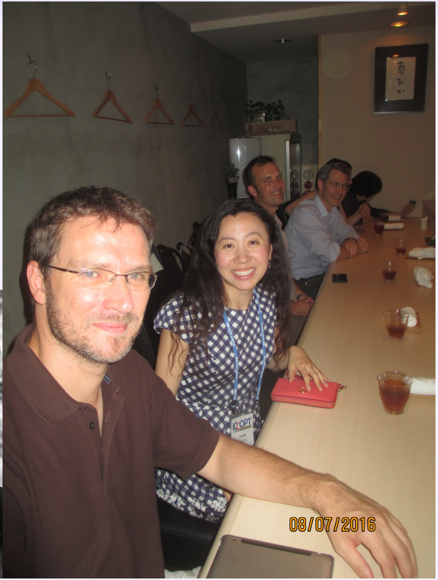
$$\text{ELC}_k \qquad \underset{x}{\text{minimize}} \quad L(x, y_k, \rho_k) + \sigma_k \|\bar{c}_k(x)\|_1 \qquad \text{subject to} \quad \ell \le x \le u$$

- $\bar{c}_k(x)$ = linear approximation to $c(x)$ at x_k (MINOS has very big σ_k)

$$\text{ELC}_k'' \qquad \underset{x,v,w}{\text{minimize}} \quad M_k(x, v, w) + \sigma_k e^T(v + w) \qquad \text{subject to} \quad \bar{c}_k(x) + v - w = 0, \quad \ell \le x \le u, \quad v, w \ge 0$$

- M_k = modified augmented Lagrangian $\quad d_k(x, v, w) = c(x) - \bar{c}_k(x) - v + w$
- IPOPT, KNITRO, ... won't mind the extra elastic variables v, w

Special thanks



- LANCELOT: Andy, Nick, Philippe
- AMPL: Bob Fourer, Dave Gay
- IPOPT: Larry Biegler, Andreas Wächter
- Yuja Wang, youtube
- Coauthors: Ken Judd, Dominique Orban
  Ding Ma, Michael Friedlander

Merida Jan 2016

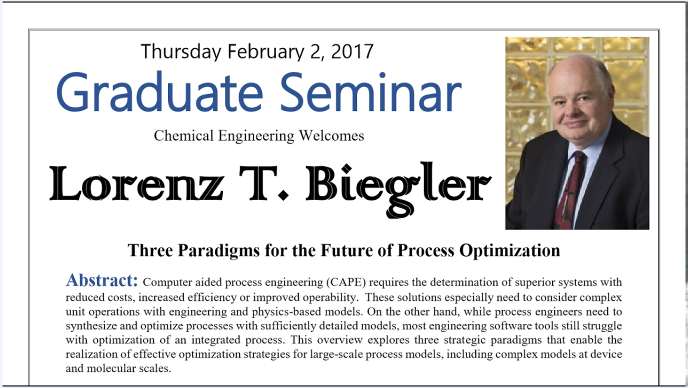
AMPL: Dave Gay and Bob Fourer

Oman Jan 2017

Oman Jan 2017

LANCELOT: Andy Conn, Nick Gould, Philippe Toint

IPOPT: Larry Biegler and Andreas Wächter



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