SSAI and SSAI_LS: Sparse approximate inverse preconditioners for CG and MINRES

Shaked Regev and Michael Saunders
ICME, Stanford University

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Abstract

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We compare two methods that change 1 element of m at a time: $m \leftarrow m + \delta e_i$.

SSAI	Jacobi's method on $Am = e_j$
SSAI_LS	1D least squares on $\min \|Am - e_j\|_2^2$

SPD $Ax = b$

- $A \in R^{n \times n}$, explicit, sparse
- Diagonal scaling $DADy = Db$, $x = Dy$ can make $\mathrm{diag}(DAD) = I$.
  Assume $A_{ii} = 1$.
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Two methods

- SSAI: inspired by GMRES preconditioner of Salkuyeh and Toutounian (2004)
- SSAI_LS: SPD version of Chow and Saad (1998), $\min \|AM - I\|_F^2$

SSAI and SSAI_LS

Exact $M = [m_1 \ m_2 \ \dots \ m_n]$ satisfies $Am_j = e_j$.

For each col $m = m_j$, apply a few iterations of coordinate descent ($m \leftarrow m + \delta e_i$) on either $Am = e_j$ (Jacobi's method) or $\|Am - e_j\|_2^2$ (least squares):

$$
\begin{aligned}
& m = 0, \quad r = e_j \\
& \text{for } k = 1, 2, \dots, k_{\max} \\
& \quad i = \arg\max |r_i| \\
& \quad \delta = r_i \quad \textbf{or} \quad \delta = a_i^T r / \|a_i\|^2 \\
& \quad m \leftarrow m + \delta e_i \\
& \quad r \leftarrow r - \delta a_i \\
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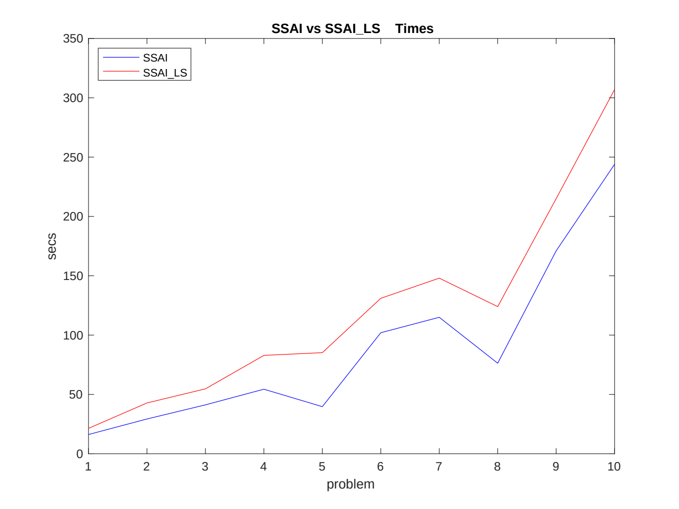
- Limit nnz(m) to average nonzeros in cols of $A \Rightarrow$ M is about as sparse as A
- $M \leftarrow (M + M^H)/2$ is initial preconditioner for CG or MINRES

Test problems from SuiteSparse collection

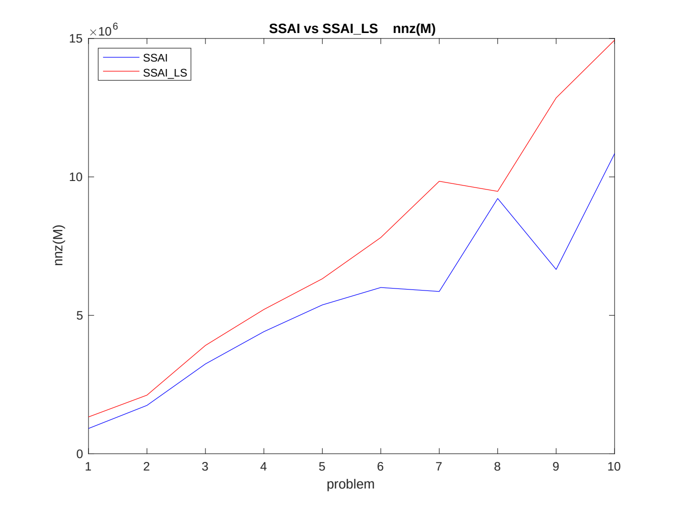
Name	n	nnz(A)	Kind
olafu	16K	1M	Structural
oilpan	74K	2M	Structural
cfd2	123K	3M	CFD
cant	62K	4M	2D/3D
tmt_sym	727K	5M	Electromag
consph	83K	6M	2D/3D
bmw7st_1	141K	7M	Structural
thermal2	1228K	8M	Thermal
m_t1	98K	9M	Structural
crankseg_1	53K	10M	Structural

Note: olafu and oilpan are singular, but the test problems $Ax = b$ are compatible.
PCG's iterations would not be well defined, but preconditioned MINRES converges normally.

Time to compute M



nnz(M)



SSAI vs SSAI_LS nnz(M)
×10^6
15
10
5
0
SSAI
SSAI_LS
nnz(M)
1
2
3
4
5
6
7
8
9
10
problem

Restarting CG or MINRES with $M \leftarrow M + \gamma I$

M is symmetric but may not be SPD

- Monitor certain $p^T M p$ in CG and MINRES that should be positive
- If necessary, set $M \leftarrow M + \gamma I$ to make M more positive definite
- Restart CG or MINRES

Modifications to M: typically 0, 1, or 2

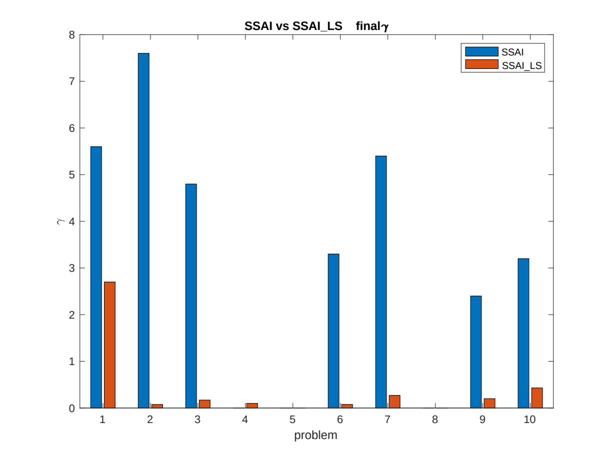
MINRES restarts [6]

CG and MINRES can detect if $\beta = p^T M p < 0$ for some p,
then restart with $M \leftarrow M + \gamma I$ (where γ depends on $|\beta|$)

Name	SSAI	SSAI_LS
olafu	1	1
oilpan	1	3
cfd2	1	2
cant	0	1
tmt_sym	0	0
consph	1	3
bmw7st_1	1	1
thermal2	0	0
m_t1	1	1
crankseg_1	1	1

SSAI_LS restarts a bit more often

Final γ in $M \leftarrow M + \gamma I$



but needs smaller γ to get SPD $M + \gamma I$

MINRES iterations

SSAI always does fewer MINRES iterations

65 other SuiteSparse problems [5]

- SSAI and SSAI_LS succeeded on all problems
- ichol failed on 26 problems
- Backslash failed on 17 problems
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Summary:

SSAI	Jacobi on $Am_j = e_j$
SSAI_LS	$\min \|Am_j - e_j\|_2^2$

- Both methods are a few iterations of coordinate descent
- Each iteration adds 1 or 0 nonzeros to m_j
- Embarrassingly parallel
- General-purpose (no assumptions on sparsity pattern of A)

References

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