SSAI and SSAI_LS:
Sparse approximate inverse preconditioners
for CG and MINRES

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Abstract

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SSAI       Jacobi’s method on $Am = ej$
SSAI_LS     1D least squares on $\min ||Am - ej||^2_2$
SPD $Ax = b$

- $A \in R^{n \times n}$, explicit, sparse
- Diagonal scaling $DADy = Db$, $x = Dy$ can make $\text{diag}(DAD) = I$.
  Assume $A_{ii} = 1$.
- SSAI $\equiv$ Symmetric sparse approximate inverse
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Two methods

- SSAI_LS: SPD version of Chow and Saad (1998), $\min ||AM - I||_F^2$
SSAI and SSAI_LS

Exact $M = [m_1 \ m_2 \ldots \ m_n]$ satisfies $Am_j = e_j$.

For each column $m = m_j$, apply a few iterations of coordinate descent ($m \leftarrow m + \delta e_i$) on either $Am = e_j$ (Jacobi's method) or $\|Am - e_j\|_2^2$ (least squares):

$m = 0, \quad r = e_j$
for $k = 1, 2, \ldots, k_{\text{max}}$
  $i = \arg \max \ |r_i|$
  $\delta = r_i \quad \text{or} \quad \delta = a_i^T r / \|a_i\|^2$
  $m \leftarrow m + \delta e_i$
  $r \leftarrow r - \delta a_i$
end

Limit $\text{nnz}(M)$ to average nonzeros in columns $A$ $\Rightarrow$ $M$ is about as sparse as $A$.

$M \leftarrow (M + M^H)/2$ is initial preconditioner for CG or MINRES.
SSAI and SSAI_LS

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- Limit $\text{nnz}(m)$ to average nonzeros in cols of $A \Rightarrow M$ is about as sparse as $A$
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### Test problems from SuiteSparse collection

<table>
<thead>
<tr>
<th>Name</th>
<th>$n$</th>
<th>$\text{nnz}(A)$</th>
<th>Kind</th>
</tr>
</thead>
<tbody>
<tr>
<td>olafu</td>
<td>16K</td>
<td>1M</td>
<td>Structural</td>
</tr>
<tr>
<td>oilpan</td>
<td>74K</td>
<td>2M</td>
<td>Structural</td>
</tr>
<tr>
<td>cfd2</td>
<td>123K</td>
<td>3M</td>
<td>CFD</td>
</tr>
<tr>
<td>cant</td>
<td>62K</td>
<td>4M</td>
<td>2D/3D</td>
</tr>
<tr>
<td>tmt_sym</td>
<td>727K</td>
<td>5M</td>
<td>Electromag</td>
</tr>
<tr>
<td>consph</td>
<td>83K</td>
<td>6M</td>
<td>2D/3D</td>
</tr>
<tr>
<td>bmw7st_1</td>
<td>141K</td>
<td>7M</td>
<td>Structural</td>
</tr>
<tr>
<td>thermal2</td>
<td>1228K</td>
<td>8M</td>
<td>Thermal</td>
</tr>
<tr>
<td>m_t1</td>
<td>98K</td>
<td>9M</td>
<td>Structural</td>
</tr>
<tr>
<td>crankseg_1</td>
<td>53K</td>
<td>10M</td>
<td>Structural</td>
</tr>
</tbody>
</table>
Time to compute $M$

![Graph comparing SSAI and SSAI_LS times](image)
$\text{nnz}(M)$
Restarting CG or MINRES with $M \leftarrow M + \gamma I$

$M$ is symmetric but may not be SPD

- Monitor certain $p^T M p$ in CG and MINRES that should be positive
- If necessary, set $M \leftarrow M + \gamma I$ to make $M$ more positive definite
- Restart CG or MINRES

Modifications to $M$: typically 0, 1, or 2
MINRES restarts [6]

CG and MINRES can detect if $\beta = p^T M p < 0$ for some $p$, then restart with $M \leftarrow M + \gamma I$ (where $\gamma$ depends on $|\beta|$)

<table>
<thead>
<tr>
<th></th>
<th>SSAI</th>
<th>SSAI_LS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
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</tr>
<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>0</td>
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<td>3</td>
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</tr>
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SSAI_LS restarts a bit more often
Final $\gamma$ in $M \leftarrow M + \gamma I$

but needs smaller $\gamma$ to get SPD $M + \gamma I$
**MINRES iterations**

SSAI always does fewer MINRES iterations
65 other SuiteSparse problems [5]

- SSAI and SSAI_LS succeeded on all problems
- ichol failed on 26 problems
- Backslash failed on 17 problems
- SSAI was better than SSAI_LS (as for the above 10 problems)
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- Both methods are a few iterations of coordinate descent
- Each iteration adds 1 or 0 nonzeros to \( m_j \)
- Embarrassingly parallel
- General-purpose (no assumptions on sparsity pattern of \( A \))


