Algorithm NCL to the rescue when LICQ fails

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Algorithm NCL to the rescue when LICQ fails

For general constrained optimization problems, LANCELOT is not troubled by LICQ because it solves a short sequence of bound-constrained subproblems. We call it a BCL method (bound-constrained augmented Lagrangian). Algorithm NCL solves an equivalent sequence of nonlinearly constrained subproblems that are suitable for interior methods such as IPOPT and KNITRO.

The AMPL implementation of NCL solved a specific (taxation policy) model with many nonlinear inequality constraints. The Julia implementation can solve the same model and more general problems from CUTEst.

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Constrained Optimization

NCO	$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & \phi(x) \\ \text{subject to} \quad & c(x) = 0 \qquad (c \in \mathbb{R}^m,\ \ m < n) \end{aligned}$$

Penalty function

$$P(x, \rho_k) = \phi(x) + \tfrac{1}{2}\rho_k c(x)^T c(x)$$

Penalty parameter $\rho_k \to \infty$

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Penalty parameter $\rho_k \to \infty$

Augmented Lagrangian

$$L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k c(x)^T c(x)$$

If Lagrange multiplier estimate $y_k \to y^*$, ρ_k can remain finite

LANCELOT's BCL algorithm for general NLP

Conn, Gould & Toint (1992)

LANCELOT

$$\min \phi(x) \quad \text{st} \quad c(x) = 0, \quad \ell \le x \le u$$

BCL subproblems (Bound-Constrained augmented Lagrangian):

$$\text{BC}_k \qquad \underset{x}{\text{minimize}} \quad \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k c(x)^T c(x)$$
$$\text{subject to} \quad \ell \le x \le u$$

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Loop:

solve BC_k to get x_k^*	decreasing opttol ω_k
if $\lVert c(x_k^*) \rVert \le \eta_k$, $y_{k+1} \leftarrow y_k - \rho_k c(x_k^*)$	decreasing featol η_k
else $\rho_{k+1} \leftarrow 10\rho_k$	

LANCELOT

$$\min \phi(x) \quad \text{st} \quad c(x) = 0, \quad \ell \le x \le u$$

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Only about 10 subproblems, no LICQ worries

Our optimization problem

Our NLP problem

NLP	$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \phi(x)$$ $$\text{subject to} \quad c(x) \ge 0, \quad \ell \le x \le u$$

Many inequalities $c(x) \ge 0$ might not satisfy LICQ at x^*

Example: $m = 571,000, \quad n = 1500$

10,000 constraints essentially active: $c_i(x^*) \le 10^{-6}$

BCL

LCL

NCL

Sequence of subproblems minimizing
X-constrained (augmented) Lagrangian

BCL	LANCELOT	Conn, Gould & Toint (1992)
LCL	linearized constraints	Robinson (1972)
	MINOS	Murtagh and S (1982)
sLCL	KNOSSOS	Friedlander (2002)
NCL	New form of **BCL**	
	AMPL or Julia loop + IPOPT or KNITRO	

Algorithm NCL
for general NLP

NCL subproblems

NLP

$$\underset{x}{\text{minimize}} \quad \phi(x)$$
$$\text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$$

LANCELOT-type subproblems:

BC_k

$$\underset{x}{\text{minimize}} \quad L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k c(x)^T c(x)^2$$
$$\text{subject to} \quad \ell \le x \le u$$

NCL subproblems

NLP

$$\underset{x}{\text{minimize}} \quad \phi(x)$$
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LANCELOT-type subproblems:

BC_k

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$$\text{subject to} \quad \ell \le x \le u$$

Introduce $r = -c(x)$:

NC_k

$$\underset{x,\, r}{\text{minimize}} \quad \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k r^T r$$
$$\text{subject to} \quad c(x) + r = 0, \quad \ell \le x \le u$$

Free vars r make the nonlinear constraints independent and feasible   Interior solvers happy!

NCL subproblems

NLP

$$\underset{x}{\text{minimize}} \quad \phi(x)$$
$$\text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$$

NC_k

$$\underset{x,\,r}{\text{minimize}} \quad \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k r^T r$$
$$\text{subject to} \quad c(x) + r = 0, \quad \ell \le x \le u$$

Free vars r make the nonlinear constraints independent and feasible Interior solvers happy!

NCL subproblems for our problem

NLP

$$\underset{x}{\text{minimize}} \quad \phi(x)$$
$$\text{subject to} \quad c(x) \ge 0, \quad \ell \le x \le u$$

NC$_k$

$$\underset{x,\, r}{\text{minimize}} \quad \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k r^T r$$
$$\text{subject to} \quad c(x) + r \ge 0, \quad \ell \le x \le u$$

Free vars r make the nonlinear constraints independent and feasible   Interior solvers happy!

Interior Methods (IPMs)

For

$$\underset{x}{\text{minimize}}\ \phi(x) \quad \text{st} \quad c(x) = 0, \quad x \geq 0,$$

each search direction $(\Delta x, \Delta y)$ comes from solving

$$\begin{pmatrix} -(H + X^{-1}Z) & J^T \\ J & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} r_2 \\ r_1 \end{pmatrix}$$

Interior Methods (IPMs)

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$$\underset{x}{\text{minimize}} \;\; \phi(x) \;\; \text{st} \;\; c(x) = 0, \quad x \ge 0,$$

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For the NCL problem

$$\underset{x,\, r}{\text{minimize}} \;\; \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k r^T r \;\; \text{st} \;\; c(x) + r = 0, \quad x \ge 0$$

the linear system is

$$\begin{pmatrix} -(H + X^{-1}Z) & & J^T \\ & -\rho_k I & I \\ J & I & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta r \\ \Delta y \end{pmatrix} = \begin{pmatrix} r_2 \\ r_3 \\ r_1 \end{pmatrix}$$

Optimal Tax Policy

Kenneth Judd and Che-Lin Su 2011

Optimal tax policy

TAX

$$\begin{array}{lrl} \text{maximize}_{c,\,y} & \sum_i \lambda_i U^i(c_i, y_i) & \\ \text{subject to} & U^i(c_i, y_i) - U^i(c_j, y_j) \geq 0 & \text{for all } i, j \ (*) \\ & \lambda^T(y - c) \geq 0 & \\ & c,\ y \geq 0 & \end{array}$$

Optimal tax policy

TAX	$\text{maximize}_{c,\,y}$	$\sum_i \lambda_i U^i(c_i, y_i)$	
	subject to	$U^i(c_i, y_i) - U^i(c_j, y_j) \geq 0$	for all i, j $(*)$
		$\lambda^T(y - c) \geq 0$	
		$c,\ y \geq 0$	

where c_i and y_i are the consumption and income of taxpayer i, and λ is a vector of positive weights. Each utility function $U^i(c_i, y_i)$ has the form

$$U(c, y) = \frac{(c - \alpha)^{1 - 1/\gamma}}{1 - 1/\gamma} - \psi \frac{(y/w)^{1/\eta + 1}}{1/\eta + 1}$$

where w is the wage rate and α, γ, ψ, η are taxpayer heterogeneities

$(*)$ = zillions of incentive-compatibility constraints

Optimal tax policy

More precisely,

$$U^{i,j,k,g,h}(c_{p,q,r,s,t}, y_{p,q,r,s,t}) = \frac{(c_{p,q,r,s,t} - \alpha_k)^{1-1/\gamma_h}}{1 - 1/\gamma_h} - \psi_g \frac{(y_{p,q,r,s,t}/w_i)^{1/\eta_j+1}}{1/\eta_j + 1}$$

where (i, j, k, g, h) and (p, q, r, s, t) run over 5 dimensions:

na	wage types	$= 5$	21
nb	elasticities of labor supply	$= 3$	3
nc	basic need types	$= 3$	3
nd	levels of distaste for work	$= 2$	2
ne	elasticities of demand for consumption	$= 2$	2
$T =$	$na \times nb \times nc \times nd \times ne$	$= 180$	756
$m =$	$T(T-1)$ nonlinear constraints	$= 32220$	570780
$n =$	$2T$ variables	$= 360$	1512

AMPL model

TAX	$\text{maximize}_{c,\,y}$	$\sum_i \lambda_i U^i(c_i, y_i)$	
	subject to	$U^i(c_i, y_i) - U^i(c_j, y_j) \geq 0$	for all $i \neq j$
		$\lambda^T(y - c) \geq 0$	
		$c,\ y \geq 0$	

```
Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
          !(i=p and j=q and k=r and g=s and h=t)}:
      (c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
    - (c[p,q,r,s,t] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       + psi[g]*(y[p,q,r,s,t]/w[i])^mu1[j] / mu1[j]
      >= 0;

Technology:
   sum{(i,j,k,g,h) in T} lambda[i,j,k,g,h]*(y[i,j,k,g,h] - c[i,j,k,g,h]) >= 0;
```

Piecewise-smooth extension

```
Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
          !(i=p and j=q and k=r and g=s and h=t)}:
   (if c[i,j,k,g,h] - alpha[k] >= epsilon then
      (c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
    else
       -  0.5/gamma[h] *epsilon^(-1/gamma[h]-1)*(c[i,j,k,g,h] - alpha[k])^2
       + (1+1/gamma[h])*epsilon^(-1/gamma[h]  )*(c[i,j,k,g,h] - alpha[k])
       + (1/(1-1/gamma[h]) - 1 - 0.5/gamma[h])*epsilon^(1-1/gamma[h])
       - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
   )
 - (if c[p,q,r,s,t] - alpha[k] >= epsilon then
      ...
   ) >= 0;
```

SNOPT on problem TAX (SQP method, 1st derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

```
Major Minors    Step   nCon Feasible  Optimal  MeritFunction   nS  condHz Penalty
   0    866               1 (3.7E-15) 4.9E-04  4.1745522E+02    4 4.1E+08 1.0E+04 _  r  t
   1    503 2.7E-02       6 (3.6E-15) 6.5E-02  4.1746922E+02   24 3.2E+05 1.0E+04 _n r  t
   2    134 1.0E-01      11 (1.4E-07) 2.7E-05  4.1755749E+02    8 2.6E+09 1.8E+06 _s
   3    313 9.8E-02      16 (1.4E-07) 8.9E-05  4.1764438E+02   43 1.0E+07 1.8E+06 _
   4    153 2.8E-02      21 (5.5E-08) 1.8E-04  4.1767129E+02   35 2.2E+04 1.8E+06 _
   5    103 2.2E-02      26 (5.4E-08) 9.5E-04  4.1769616E+02   34 6.7E+07 1.8E+06 _

 194  30811 1.0E+00     795   8.6E-01  9.7E-01  2.8330244E+21    2 1.8E+01 3.5E+13 _n   it
 195   1819 1.1E-04     800   8.6E-01  1.0E+00  2.6326936E+22    3 1.4E+02 1.1E+15 _n R it
 195   3314             800   8.6E-01  1.0E+00  2.8661156E+22              1.0E+04 _n r it
 195   4439             800   8.6E-01  9.9E-01  2.8661156E+22              1.0E+04 _n r it

 SNOPTB EXIT  40 -- terminated after numerical difficulties
 SNOPTB INFO  41 -- current point cannot be improved
```

IPOPT on problem TAX (IPM, 2nd derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

```
This is Ipopt version 3.12.4, running with linear solver mumps.

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 -4.1745522e+02 0.00e+00 2.52e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -4.1734473e+02 6.18e-03 7.36e+00  -1.0 1.34e+00    -  7.69e-01 2.05e-01f  1
   2 -4.1682694e+02 4.93e-03 1.78e+01  -1.0 5.48e+00    -  2.23e-01 1.34e-01f  1

  10 -4.1428766e+02 1.22e-03 1.50e+04  -1.0 3.01e-01   0.6 4.75e-01 5.39e-01h  1

 160 -4.1641067e+02 0.00e+00 1.50e-03  -3.8 1.25e-01    -  1.00e+00 1.00e+00f  1

 449r-4.1630403e+02 1.13e-05 2.79e-05  -8.1 2.92e-01    -  1.00e+00 9.77e-01h  1

                               (scaled)                 (unscaled)
Dual infeasibility......:   1.1130803588695777e+00    1.1130803588695777e+00
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   1.3412941119075164e-08    1.3412941119075164e-08
```

LANCELOT on problem TAX (BCL method, 2nd derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	rhok	omegak	etak	Obj	itns	CGit	TRradius	active
1	1.0e+1	1.0e-1	1.0e-1	-417.455	18	12000	4.1e-01	2831
2	1.0e+1	1.0e-2	1.2e-2	-421.606	39	9000	1.6e-01	2568
3	1.0e+2	1.0e-2	7.9e-2	-421.011	23	11000	2.4e-01	1662
4	1.0e+2	1.0e-4	1.3e-3	-420.188	282	104000	8.6e-02	1444
5	1.0e+3	1.0e-3	6.3e-2	-419.967	134	64000	5.7e-02	1004
6	1.0e+3	1.0e-6	1.3e-4	-419.819	198	156000	3.1e-02	901
7	1.0e+4	1.0e-4	5.0e-2	-419.741	300	308000	3.1e-12	710
8	1.0e+4	1.0e-6	1.3e-5	-419.698	327	623000	5.5e-04	709
9	1.0e+5	1.0e-5	4.0e-2	-419.682	253	724000	4.7e-03	653
10	1.0e+5	1.0e-6	1.3e-6	-419.676	154	1031000	4.2e-11	663
11	1.0e+6	1.0e-6	3.2e-2	...				

1970 iterations, 8 hours CPU on NEOS

AMPL implementation of NCL

pTax5Dnclipopt.run

```
reset;   model pTax5Dinitial.run;    # Get initial values

reset;   model pTax5Dncl.mod;
         data  pTax5Dncl.dat;
         data; var include p5Dinitial.dat;

model;   option solver ipopt;
         option ipopt_options  'dual_inf_tol=1e-6   max_iter=5000';
```

pTax5Dnclipopt.run

```
option opt2 $ipopt_options ' warm_start_init_point=yes';

for {K in 1..kmax}
{  if K == 2 then {option ipopt_options $opt2 ' mu_init=1e-4'};
   if K == 4 then {option ipopt_options $opt2 ' mu_init=1e-5'};
   if K == 6 then {option ipopt_options $opt2 ' mu_init=1e-6'};
   if K == 8 then {option ipopt_options $opt2 ' mu_init=1e-7'};
   if K ==10 then {option ipopt_options $opt2 ' mu_init=1e-8'};

   solve;

   let rmax := max({(i,j,k,g,h) in T, (p,q,r,s,t) in T:
      !(i=p and j=q and k=r and g=s and h=t)} R[i,j,k,g,h,p,q,r,s,t]);
   let rmin := ...
   let rnorm := max(abs(rmax), abs(rmin));
   if rnorm <= rtol then { printf "Stopping: rnorm is small\n"; break; }
```

Numerical results

Warm Starts for IPMs

Sequence of related subproblems

- The whole world knows we can't warm-start IPMs

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NCL:

- Only the objective changes: $\phi(x) + y_k^T r + \frac{1}{2}\rho_k r^T r$
- Many extra variables r
- r stabilizes iterations, doesn't affect sparsity of factorizations

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- Many extra variables r
- r stabilizes iterations, doesn't affect sparsity of factorizations

In this context, IPM warm starts are practical

Warm-start options for Nonlinear Interior Methods

```
IPOPT     warm_start_init_point=yes
          mu_init=1e-4                 (1e-5, ..., 1e-8)
```

Warm-start options for Nonlinear Interior Methods

```
IPOPT     warm_start_init_point=yes
          mu_init=1e-4                  (1e-5, ..., 1e-8)
```

mu_init is the initial value of μ (the barrier parameter)

$$\mu \to 0$$

NCL/IPOPT on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	95	41.1
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	7.2
3	10^3	10^{-3}	1.3e-03	-4.1986069e+02	10^{-4}	20	8.1
4	10^4	10^{-3}	4.4e-04	-4.1972958e+02	10^{-4}	48	25.0
5	10^4	10^{-4}	2.2e-04	-4.1968646e+02	10^{-4}	43	20.5
6	10^5	10^{-4}	9.8e-05	-4.1967560e+02	10^{-4}	64	32.9
7	10^5	10^{-5}	6.6e-05	-4.1967177e+02	10^{-4}	57	26.8
8	10^6	10^{-5}	4.2e-06	-4.1967150e+02	10^{-4}	87	46.2
9	10^6	10^{-6}	9.4e-07	-4.1967138e+02	10^{-4}	96	53.6

527 iterations, 5 mins CPU

NCL/IPOPT on problem TAX

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	95	40.8
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	7.0
3	10^3	10^{-3}	1.3e-03	-4.1986069e+02	10^{-4}	20	8.5
4	10^4	10^{-3}	4.4e-04	-4.1972958e+02	10^{-5}	57	32.6
5	10^4	10^{-4}	2.2e-04	-4.1968646e+02	10^{-5}	29	14.6
6	10^5	10^{-4}	9.8e-05	-4.1967560e+02	10^{-6}	36	18.7
7	10^5	10^{-5}	3.9e-05	-4.1967205e+02	10^{-6}	35	19.7
8	10^6	10^{-5}	4.2e-06	-4.1967150e+02	10^{-7}	18	7.7
9	10^6	10^{-6}	9.4e-07	-4.1967138e+02	10^{-7}	15	6.8

322 iterations, 3 mins CPU

NCL/IPOPT bigger example

$na, nb, nc, nd, ne = 21, 3, 3, 2, 2 \quad m = 570780 \quad n = 1512$

k	ρ_k	η_k	$\|r_k^*\|_\infty$	$\phi(x_k^*)$	mu_init	Itns	Time
1	10^2	10^{-2}	5.1e-03	-1.7656816e+03	10^{-1}	825	7763
2	10^2	10^{-3}	2.4e-03	-1.7648480e+03	10^{-4}	66	473
3	10^3	10^{-3}	1.3e-03	-1.7644006e+03	10^{-4}	106	771
4	10^4	10^{-3}	3.8e-04	-1.7639491e+03	10^{-5}	132	1347
5	10^4	10^{-4}	3.2e-04	-1.7637742e+03	10^{-5}	229	2451
6	10^5	10^{-4}	8.6e-05	-1.7636804e+03	10^{-6}	104	1097
7	10^5	10^{-5}	4.9e-05	-1.7636469e+03	10^{-6}	143	1633
8	10^6	10^{-5}	1.5e-05	-1.7636252e+03	10^{-7}	71	786
9	10^7	10^{-5}	2.8e-06	-1.7636196e+03	10^{-7}	67	726
10	10^7	10^{-6}	5.1e-07	-1.7636187e+03	10^{-8}	18	171

1761 iterations, 5 hours CPU

NCL/IPOPT bigger example

$na, nb, nc, nd, ne = 21, 3, 3, 2, 2 \quad m = 570780 \quad n = 1512$

Constraints within tol of being active: $c_i(x) \le tol$

tol	*count*	*count/n*	
10^{-10}	3888	2.6	
10^{-9}	3941	2.6	
10^{-8}	4430	2.9	
10^{-7}	7158	4.7	
$\rightarrow 10^{-6}$	10074	6.6 $\leftarrow$	$\approx 6.6n$ active constraints
10^{-5}	11451	7.6	
10^{-4}	13109	8.7	
10^{-3}	23099	15.3	
10^{-2}	66361	43.9	
10^{-1}	202664	134.0	

Warm-start options for Nonlinear Interior Methods

IPOPT	warm_start_init_point=yes	
	mu_init=1e-4	(1e-5, ..., 1e-8)
KNITRO	algorithm=1	Thanks, Richard Waltz!
	bar_directinterval=0	
	bar_initpt=2	
	bar_murule=1	
	bar_initmu=1e-4	(1e-5, ..., 1e-8)
	bar_slackboundpush=1e-4	(1e-5, ..., 1e-8)

Comparison of IPOPT, KNITRO, NCL (2nd derivs)

na = increasing $nb = 3$ $nc = 3$ $nd = 2$ $ne = 2$

			IPOPT		KNITRO		NCL/IPOPT		NCL/KNITRO	
na	*m*	*n*	itns	time	itns	time	itns	time	itns	time
5	32220	360	449	217	168	53	322	146	2320	8.0mins
9	104652	648	> 98*	> 360*	928	825	655	1023	9697	1.9hrs
11	156420	792	> 87*	∞!	2769	4117	727	1679	26397	7.0hrs
17	373933	1224			2598	11447	1021	6347		
21	570780	1512					1761	17218	45039	1.9 days

*duals diverge
MUMPS needs more mem
!Loop

Warm starts — Cold starts

NCL/KNITRO with Warm Starts

na = increasing $nb = 3$ $nc = 3$ $nd = 2$ $ne = 2$

			IPOPT		KNITRO		NCL/IPOPT		NCL/KNITRO	
na	m	n	itns	time	itns	time	itns	time	itns	time
5	32220	360	449	217	168	53	322	146	339	63
9	104652	648	> 98*	> 360*	928	825	655	1023	307	239
11	156420	792	> 87*	∞!	2769	4117	727	1679	383	420
17	373933	1224			2598	11447	1021	6347	486	1200
21	570780	1512					1761	17218	712	2880

Warm starts   Warm starts

Julia/NCL

Dominique Orban and Pierre-Élie Personnaz

A Julia Implementation of NCL

Features:

- generic implementation using a full-blown programming language
- rests upon the JuliaSmoothOptimizers[1] infrastructure for optimization
- here, we use the AMPL models for the TAX problems
- can use IPOPT, KNITRO[2] interchangeably

Differences from AMPL/NCL:

- accepts problems modeled with SIF, AMPL, JuMP or plain Julia
- subproblems solved inexactly ($\omega_k \searrow$)
- we are currently experimenting with warm-starting multipliers

[1] https://juliasmoothoptimizers.github.io
[2] Thanks to the authors of IPOPT.jl and to Artelys for supporting KNITRO.jl

Illustration on TAX Problems with Realistic Data

- Use KNITRO 12
- Progressively decrease ω_k
- Stop when $\|r\| \leq$ `feas_tol` and $\|\nabla L\| \leq$ `opt_tol`

```
julia> using NCL

julia> using AmplNLReader  # Julia module to read a nl file

julia> tax1D = AmplModel("data/tax1D")
Maximization problem data/tax1D
nvar = 24, ncon = 133 (1 linear)

julia> NCLSolve(tax1D, outlev=0)
```

outer	inner	NCL obj	$\|r\|$	η	$\|\nabla L\|$	ω	ρ	μ init	$\|y\|$	$\|x\|$	time
1	5	-8.00e+02	9.7e-02	1.0e-02	7.6e-03	1.0e-02	1.0e+02	1.0e-01	1.0e+00	2.0e+02	0.13
2	12	-7.89e+02	4.2e-02	1.0e-02	4.3e-03	1.0e-02	1.0e+03	1.0e-03	1.0e+00	1.9e+02	0.00
3	7	-7.83e+02	5.7e-03	1.0e-02	1.0e-03	1.0e-02	1.0e+04	1.0e-03	1.0e+00	1.9e+02	0.00
4	3	-7.82e+02	1.3e-04	1.0e-03	1.0e-05	1.0e-03	1.0e+04	1.0e-05	5.8e+01	1.9e+02	0.00
5	2	-7.82e+02	2.3e-06	1.0e-04	1.0e-05	1.0e-04	1.0e+04	1.0e-05	5.9e+01	1.9e+02	0.00
6	2	-7.82e+02	9.3e-08	1.0e-05	1.0e-06	1.0e-05	1.0e+04	1.0e-06	5.9e+01	1.9e+02	0.00
7	2	-7.82e+02	7.7e-09	1.0e-06	1.0e-08	1.0e-06	1.0e+04	1.0e-06	5.9e+01	1.9e+02	0.00

TAX Problems with Realistic Data

```
julia> pTax5D = AmplModel("data/pTax5D")
Minimization problem data/pTax5D
nvar = 864, ncon = 186193 (1 linear)

julia> NCLSolve(pTax5D, outlev=0)
outer  inner    NCL obj     ||r||         η     ||∇L||         ω         ρ    μ init      ||y||      ||x||     time
    1     64  -1.76e+05   2.0e-01   1.0e-02   2.3e-03   1.0e-02   1.0e+02   1.0e-01   1.0e+00   1.1e+04    80.43
    2     29  -1.74e+05   4.9e-02   1.0e-02   1.2e-03   1.0e-02   1.0e+03   1.0e-03   1.0e+00   1.1e+04    35.02
    3     23  -1.74e+05   1.6e-02   1.0e-02   1.0e-03   1.0e-02   1.0e+04   1.0e-03   1.0e+00   1.1e+04    28.96
    4     46  -1.74e+05   4.1e-03   1.0e-02   3.6e-05   1.0e-02   1.0e+05   1.0e-05   1.0e+00   1.1e+04    54.50
    5     41  -1.74e+05   2.8e-03   1.0e-03   1.7e-05   1.0e-03   1.0e+05   1.0e-05   4.1e+02   1.1e+04    52.72
    6     28  -1.74e+05   6.1e-04   1.0e-03   1.0e-06   1.0e-03   1.0e+06   1.0e-06   4.1e+02   1.1e+04    34.38
    7     13  -1.74e+05   2.1e-04   1.0e-04   1.4e-06   1.0e-04   1.0e+06   1.0e-06   1.0e+03   1.1e+04    14.81
    8     12  -1.74e+05   5.3e-05   1.0e-04   1.2e-07   1.0e-04   1.0e+07   1.0e-07   1.0e+03   1.1e+04    14.80
    9      7  -1.74e+05   4.5e-06   1.0e-05   1.0e-07   1.0e-05   1.0e+07   1.0e-07   1.0e+03   1.1e+04     9.49
   10      5  -1.74e+05   8.0e-07   1.0e-06   1.2e-08   1.0e-06   1.0e+07   1.0e-08   1.0e+03   1.1e+04     7.02
```

Summary of Algorithm NCL

NLP	
	$\underset{x}{\text{minimize}} \quad \phi(x)$
	subject to $c(x) = 0, \quad \ell \le x \le u$

LANCELOT subproblems:

BC_k	
	$\underset{x}{\text{minimize}} \quad L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \frac{1}{2}\rho_k c(x)^T c(x)$
	subject to $\ell \le x \le u$

Summary of Algorithm NCL

NLP	
	$$\underset{x}{\text{minimize}} \quad \phi(x)$$ $$\text{subject to} \quad c(x) = 0, \quad \ell \le x \le u$$

LANCELOT subproblems:

BC_k	
	$$\underset{x}{\text{minimize}} \quad L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \tfrac{1}{2}\rho_k c(x)^T c(x)$$ $$\text{subject to} \quad \ell \le x \le u$$

NCL subproblems:

NC_k	
	$$\underset{x,\, r}{\text{minimize}} \quad \phi(x) + y_k^T r + \tfrac{1}{2}\rho_k r^T r$$ $$\text{subject to} \quad c(x) + r = 0, \quad \ell \le x \le u$$

Free vars r make the nonlinear constraints independent and feasible IPM solvers happy!

Related work

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- S. Arreckx and D. Orban, A regularized factorization-free method for equality-constrained optimization, Technical Report GERAD G-2016-65, GERAD, Montréal, QC, Canada, 2016, doi:10.13140/RG.2.2.20368.00007.
- D. Ma, K. L. Judd, D. Orban and M. A. Saunders, Stabilized optimization via an NCL algorithm, pp 173–191 in M. Al-Baali et al. (eds.), Numerical Analysis and Optimization, NAO-IV, Muscat, Oman, January 2017, Springer Proceedings in Mathematics & Statistics, Volume 235, 2018.

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- Julia implementation: Pierre-Élie Personnaz
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- Yuja Wang, YouTube (and YouKu!)

Eunae, Courtney, Simge

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