QPBLUR: An active-set convex QP solver

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QPBLUR: An active-set convex QP solver based on regularized KKT systems

SNOPT obtains search directions from convex QP subproblems, currently solved by SQOPT. For problems with many degrees of freedom, the nullspace active-set method of SQOPT becomes inefficient.

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QPBLUR is effectively a penalty method with bounds. QPBCL (bound-constrained Lagrangian) includes a Lagrangian term to satisfy $Ax = b$ more accurately.

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Motivation

Why another QP solver?

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- can handle a large number of free variables
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- can handle a large number of free variables
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Such a solver would be useful

- inside SQP methods (like SNOPT)
- for related QPs (e.g. model predictive control)

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$$G = D(I + s_1 v_1^T)(I + s_2 v_2^T)\dots$$

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- Currently use SQOPT (active-set null-space method)

SQOPT: Large-scale convex QP

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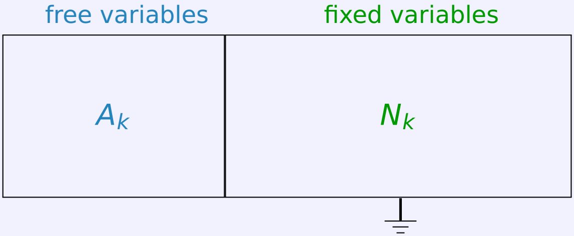
$AP_k =$



OK if A_k is nearly square:

$$10000 \times 12000 \quad \text{or} \quad 100000 \times 102000$$

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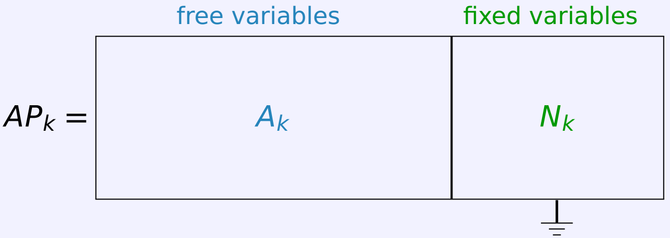
Active-set null-space method



For A_k 100000×400000, we need a
QP solver based on KKT systems like QPA in GALAHAD

KKT system for current active set

Solve $K_0 \begin{pmatrix} \Delta x_k \\ \Delta y \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$

$$K_0 = \left[\begin{array}{c|c} H_k & A_k^T \\ \hline A_k & \end{array}\right]$$

QPBLU

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- Uses black-box factorizer on K_0
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- Active-set method keeps K_0 nonsingular in theory

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$$\left(\begin{array}{c|c} K_0 & V \\ \hline V^T & E \end{array}\right) = \begin{bmatrix} L_0 & \\ Z^T & I \end{bmatrix} \begin{bmatrix} U_0 & Y \\ & C \end{bmatrix}$$

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- Y, Z sparse, C small
- Quasi-Newton updates to H_k handled same way
- Singular K_0 is a difficulty in practice

QPBLUR: Large-scale QP with Regularization

QPBLUR (Thesis of Chris Maes 2010)

$$\min_{x,y} \quad c^T x + \tfrac{1}{2} x^T H x + \tfrac{1}{2}\delta \|x\|_2^2 + \tfrac{1}{2}\mu \|y\|_2^2$$

$$Ax + \mu y = b, \qquad l \le x \le u$$

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- Can use Hanh's block-LU updates without change

QPBLUR strategy

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Matlab implementation

- Scale problem

QPBLUR strategy

$$\min_{x,y} \quad c^Tx + \tfrac{1}{2}x^THx + \tfrac{1}{2}\delta\|x\|_2^2 + \tfrac{1}{2}\mu\|y\|_2^2$$
$$Ax + \mu y = b, \qquad l \le x \le u$$

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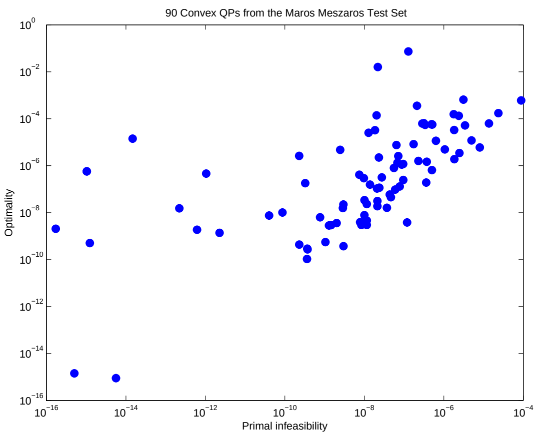
- Exit if small relative change in obj

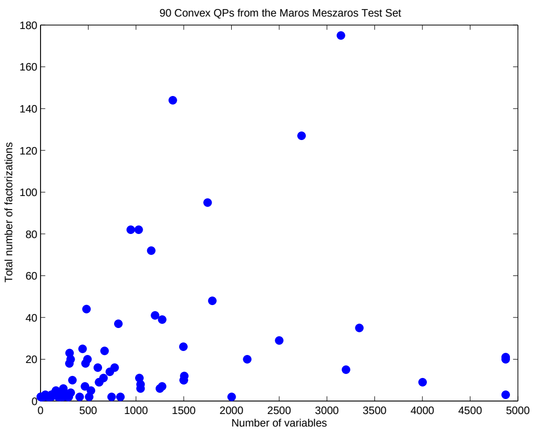
Numerical Results

Accuracy of solutions



Residuals for 90 Meszaros QP test problems

KKT factorizations



No. of factorizations on 90 Meszaros QP test problems

QPBLUR pros and cons

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Advantages

- Warm starts (any x_0, any working set)

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- Tiny regularization: risks ill-conditioned KKT (but so far so good)

Penalty vs Augmented Lagrangian

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Why not Augmented Lagrangian?:

$$\min_{\ell \le x \le u} \; \phi(x) - \hat{y}^T(b - Ax) + \frac{1}{2}\rho\|b - Ax\|_2^2$$

BCL method as in LANCELOT

BCL Method (LANCELOT)

$$\min\ \phi(x)\ \ \text{st}\ \ c(x) = 0,\ \ \ell \le x \le u$$

Bound-constrained augmented Lagrangian method

$$L(x, \hat{y}, \rho) = \phi(x) - \hat{y}^T c(x) + \tfrac{1}{2}\rho\|c(x)\|^2$$

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- Otherwise, increase ρ
- Repeat

BCL Method for QP

$$\min_{x} \ \phi(x) \ \text{ st } \ Ax = b, \ \ \ell \le x \le u$$

BCL subproblem

$$\begin{aligned} \min_{x,r} \ L(x, \hat{y}, \rho) \ &= \ \phi(x) + \hat{y}^T r + \tfrac{1}{2}\rho\|r\|^2 \\ & \ \ \ \ Ax + r = b, \qquad \ell \le x \le u \end{aligned}$$

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$$\min_{x} \ \phi(x) \ \text{ st } \ Ax = b, \ \ \ell \le x \le u$$

BCL subproblem

$$\begin{aligned} \min_{x,r} \ L(x, \hat{y}, \rho) &= \phi(x) + \hat{y}^T r + \tfrac{1}{2}\rho \|r\|^2 \\ & \quad Ax + r = b, \qquad \ell \le x \le u \end{aligned}$$

- Solve subproblem to get $\hat{x}$, $\hat{r}$ (optimality tol $\omega \to 0$)
- If $\|\hat{r}\| < \eta$, update $\hat{y}$ (feasibility tol $\eta \to 0$)
- Otherwise, increase ρ
- Repeat

QPBCL

QPBCL: Active-set method for convex QP

$$AP = \begin{pmatrix} A_k & N_k \end{pmatrix}$$

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$$\begin{pmatrix} -(H_k + \delta I) & A_k^T \\ A_k & \frac{1}{\rho} I \end{pmatrix} \begin{pmatrix} \Delta x_k \\ \Delta y \end{pmatrix} = \begin{pmatrix} g_k - A_k^T y \\ \frac{1}{\rho}(\hat{y} - y) + r \end{pmatrix}$$

$$\Delta r = \frac{1}{\rho}(y + \Delta y - \hat{y}) - r$$

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$$\Delta r = \frac{1}{\rho}(y + \Delta y - \hat{y}) - r$$

- Same system as QPBLUR – just different rhs
- Still single phase
- Still use LUSOL, MA57, UMFPACK, . . .
- Still use Hanh's block-LU updates

Numerical Results

Penalty and Regularization parameter

QPBCL allows for smaller ρ (hence larger $\mu \equiv 1/\rho$) compared to QPBLUR

QPBCL vs SQOPT

Compare QPBCL iterations and SQOPT minor iterations

QPBCL vs SQOPT

Compare QPBCL iterations and SQOPT minor iterations

There are 3 kinds of people: Those who can count, and those who cannot.

— George Carlin

Augmented Lagrangian methods for QP

- Conn, Gould, and Toint (1992)
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- Philip Gill and Elizabeth Wong (2009–$\cdots$)
  Continuing Hanh's f90 QPBLU for use in SNOPT
  BCL approach for indefinite QP (2nd derivatives in SNOPT)

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