Computing approximate PageRank vectors by Basis Pursuit Denoising

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Joint work with
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Outline

Abstract

The PageRank eigenvector problem involves a square system $Ax = b$ in which x is naturally nonnegative and somewhat sparse (depending on b). We seek an approximate x that is nonnegative and extremely sparse. We experiment with an active-set optimization method designed for the dual of the BPDN problem, and find that it tends to extract the important elements of x in a greedy fashion.

Supported by the Office of Naval Research

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$$(I - \alpha H^T)x = v \qquad H, v \text{ sparse}$$

where $\quad 0 < \alpha < 1 \qquad He \leq e \qquad x \geq 0$, sparse

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Perhaps BPDN can find a very sparse approximate x

PageRank

PageRank

Langville and Meyer 2006

$$H = \begin{pmatrix} 1/3 & & 1/3 & 1/3 \\ & 1/2 & 1/2 & \\ 0 & 0 & 0 & 0 \\ & 1/2 & & 1/2 \end{pmatrix}$$ Hyperlink matrix $He \le e$

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eigenvector	$\equiv$	linear system
$G^T x = x$	$\equiv$	$(I - \alpha H^T)x = v$

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eigenvector	$\equiv$	linear system
$G^T x = x$	$\equiv$	$(I - \alpha H^T) x = v$

In both cases, $x \leftarrow x / e^T x$ (and $x \ge 0$)

Web matrices H

Name	n (pages)	nnz (links)	i	$(v = e_i)$
csStanford	9914	36854	4	cs.stanford.edu
Stanford	275689	1623817	23036	cs.stanford.edu
			6753	calendus.stanford.edu
Stanford-Berkeley	683446	7583376	6753	unknown

All data collected around 2001

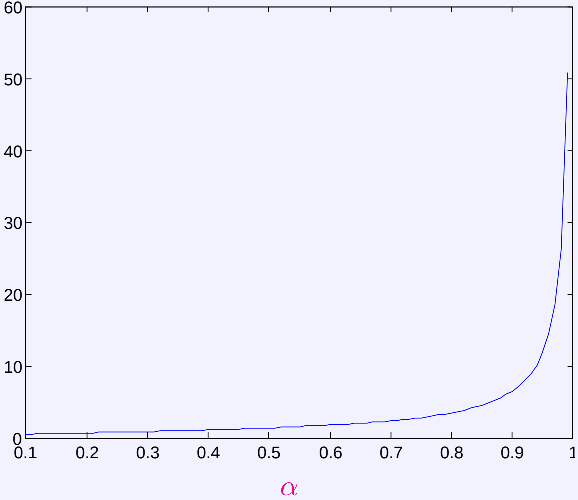
Cleve Moler
Sep Kamvar
David Gleich

$$(I - \alpha H^T)x = v$$

Power method on $G^T x = x$

Stanford, $n = 275000$, $v = e_{6753}$ (calendus.stanford.edu)

Time (secs) vs $\alpha = 0.1 : 0.01 : 0.99$



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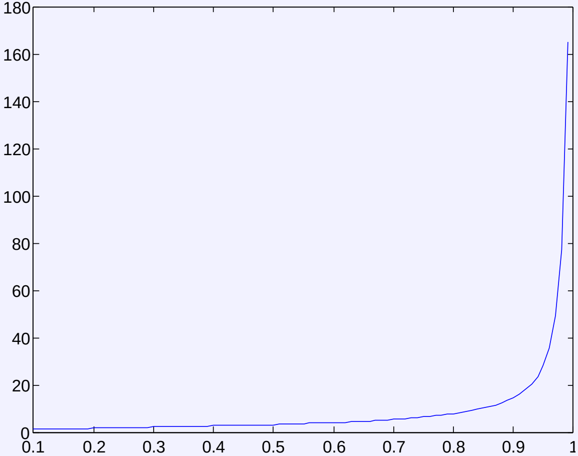
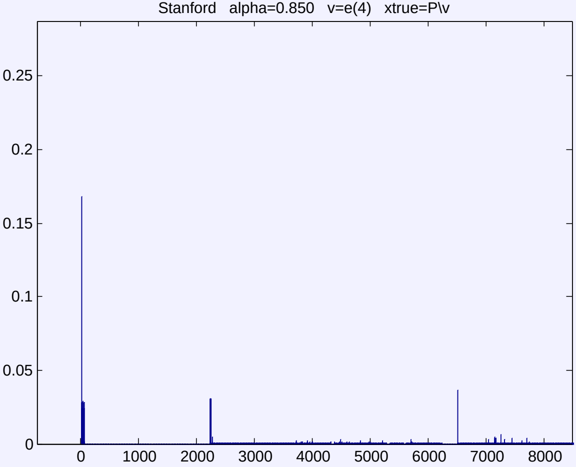


α

Exact $x(\alpha)$

csStanford, $n = 9914$, $v = e_4$ (cs.stanford.edu)

$x_j(\alpha), \;\; j = 1:n$



$\alpha = 0.85$

Exact $x(\alpha)$

Stanford, $n = 275000$, $v = e_{23036}$ (cs.stanford.edu)

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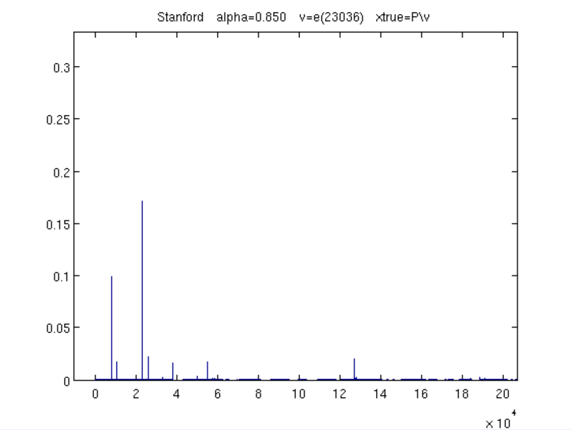
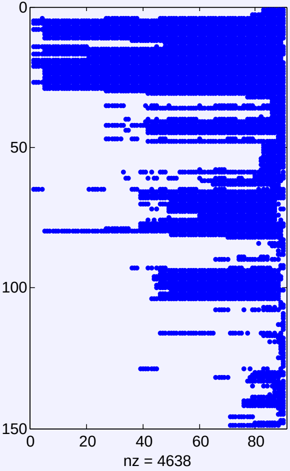


$\alpha = 0.85$

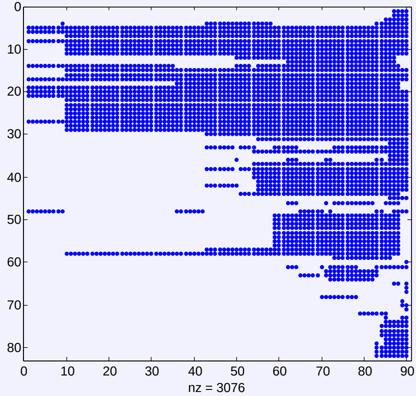
Sparsity of $x(\alpha)$

Stanford, $n = 275000$, $v = e_{23036}$ (cs.stanford.edu)

$x > 250/n$



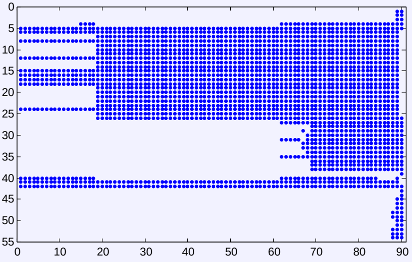
$x > 500/n$



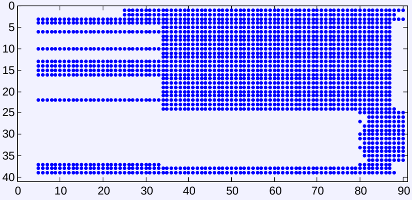
$\alpha = 0.1 : 0.01 : 0.99$ (90 values)

$n = 275000$

$x > 1000/n$



$x > 2000/n$



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Basis Pursuit Denoising

min L1 $\Rightarrow$ sparse x

Lasso(ν) Tibshirani 1996

$$\min \ \tfrac{1}{2}\|Ax - b\|^2 \quad \text{st} \quad \|x\|_1 \leq \nu \qquad\qquad A = \boxed{}$$

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$A =$ [tall rectangle]

Basis Pursuit Chen, Donoho & S 1998

$$\min \ \|x\|_1 \quad \text{st} \quad Ax = b$$

$A =$ fast operator

(Ax, A^Ty are efficient)

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(Ax, $A^T y$ are efficient)

BPDN(λ) Same paper

$$\min \lambda\|x\|_1 + \tfrac{1}{2}\|Ax - b\|^2$$

fast operator, any shape

BP and BPDN algorithms

$$\min \lambda\|x\|_1 + \tfrac{1}{2}\|Ax - b\|^2$$

OMP	Davis, Mallat et al 1997	Greedy
BPDN-interior	Chen, Donoho & S, 1998, 2001	Interior, CG
PDSCO, PDCO	Saunders 1997, 2002	Interior, LSQR
BCR	Sardy, Bruce & Tseng 2000	Orthogonal blocks
Homotopy	Osborne et al 2000	Active-set, all λ
LARS	Efron, Hastie, Tibshirani 2004	Active-set, all λ
STOMP	Donoho, Tsaig, et al 2006	Double greedy
l1_ls	Kim, Koh, Lustig, Boyd et al 2007	Primal barrier, PCG
GPSR	Figueiredo, Nowak & Wright 2007	Gradient Projection
BCLS	Friedlander 2006	Projection, LSQR
SPGL1	van den Berg & Friedlander 2007	Spectral GP, all λ
BPdual	Friedlander & S 2007	Active-set on dual
LPdual	Friedlander & S 2007	For $x \geq 0$

$\|x\|_1$ when $x \geq 0$

Suggests regularized LP problems:

$$\text{LPprimal}(\lambda) \quad \min_{x,\,y} \quad e^T x + \tfrac{1}{2}\lambda\|y\|^2 \quad \text{st} \quad Ax + \lambda y = b,\ x \geq 0$$

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Friedlander and S 2007: New active-set MATLAB codes

LPdual solver for $Ax \approx b$, $x \geq 0$

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Initially B is empty, $y = 0$

Select columns of B in almost *greedy* manner

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Form $dy = (g - Bx)/\lambda$

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Sparse PageRank with Neumann Series

Neumann series

$$(I - \alpha H^T)x = v \qquad (*)$$

$$\begin{aligned} x &= (I - \alpha H^T)^{-1} v \\ &= (I + \alpha H^T + (\alpha H^T)^2 + \cdots)v \end{aligned}$$

A sum of nonnegative terms

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See Gleich and Polito (2006) for sparse Power Method

Sparse PageRank with BPDN (LPdual)

Sparse PageRank

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Regard as

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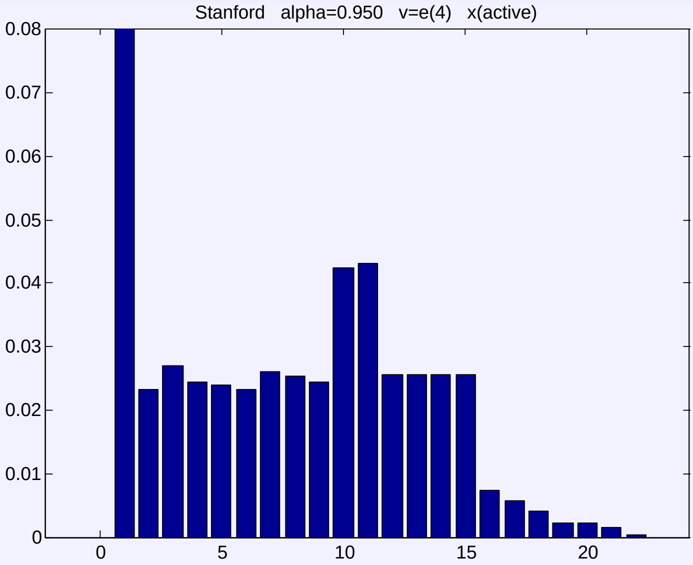
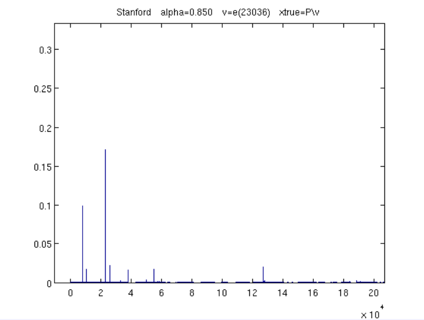
Apply active-set solver LPdual to dual of

$$\min_{x,\,r} \ \lambda e^T x + \tfrac{1}{2}\|r\|^2 \quad \text{st} \quad Ax + r = v, \quad x \geq 0$$

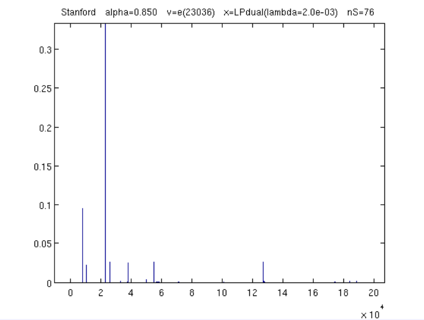

Stanford alpha=0.950 v=e(4) x(active)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
5
10
15
20

Stanford web, $n = 275000$
$\alpha = 0.85$
$v = e_{23036}$ (cs.stanford.edu)

Direct solve:
`xtrue = A\v`

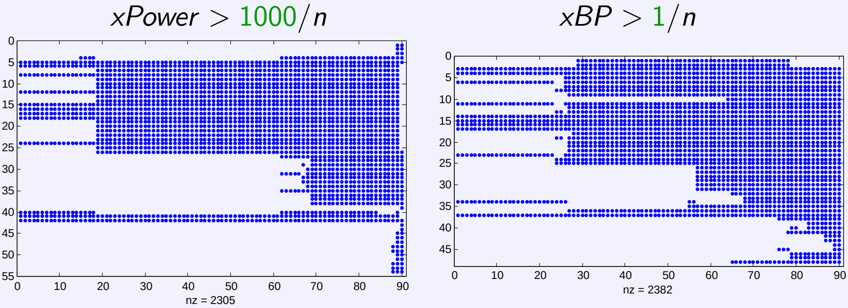


BPDN
$\lambda = 2\text{e-}3$
76 nonzero x_j



Sparsity of $xBP(\alpha)$

Stanford, $n = 275000$, $v = e_{23036}$ (cs.stanford.edu)



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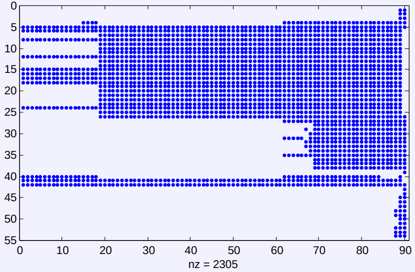
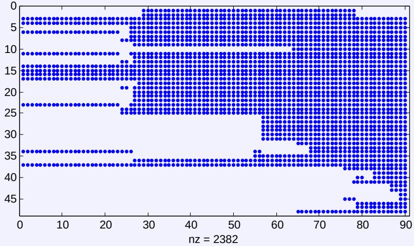


$xBP > 1/n$



$\alpha = 0.1 : 0.01 : 0.99$ (90 values)

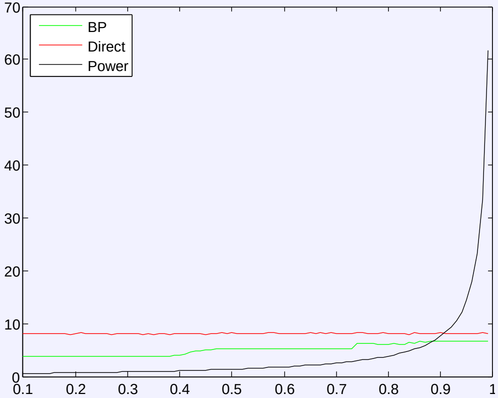
$xPower > 1/n$ has
6000 rows, 155000 nonzeros

$\lambda = 4e\text{-}3$

Power method vs BP

Stanford, $n = 275000$, $v = e_{23036}$ (cs.stanford.edu)

Time (secs) for each α



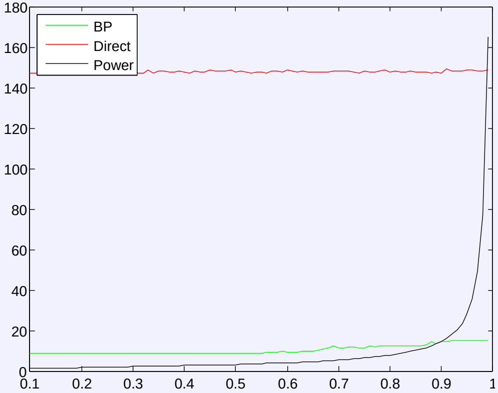
$\alpha = 0.1 : 0.01 : 0.99$ (90 values)

$\lambda = 4\text{e-}3$

Power method vs BP

Stanford-Berkeley, $n = 683000$, $v = e_{6753}$ (unknown)

Time (secs) for each α



$\alpha = 0.1 : 0.01 : 0.99$ (90 values)

$\lambda = 5e\text{-}3$

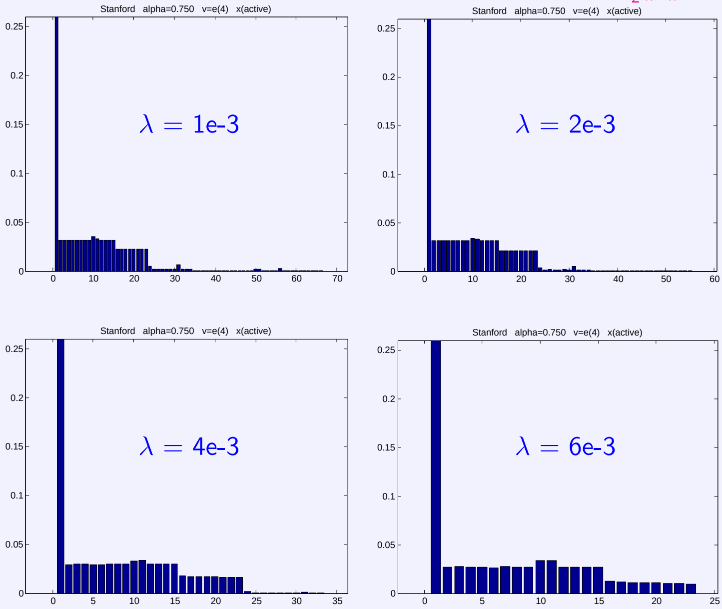
Varying α and λ

$$\min_{x,r} \lambda\|x\|_1 + \tfrac{1}{2}\|r\|^2 \quad \text{st} \quad (I - \alpha H^T)x + r = v$$

$$\begin{aligned} H &= 9914 \times 9914 \text{ csStanford web} \\ v &= e_4 \text{ (cs.stanford.edu)} \\ \alpha &= 0.75,\ 0.85,\ 0.95 \\ \lambda &= \text{1e-3, 2e-3, 4e-3, 6e-3} \end{aligned}$$

Plot nonzero x_j in the order they are chosen

csStanford $v = e_4$ $\alpha = 0.75$ $\min \lambda e^T x + \frac{1}{2}\|r\|^2$



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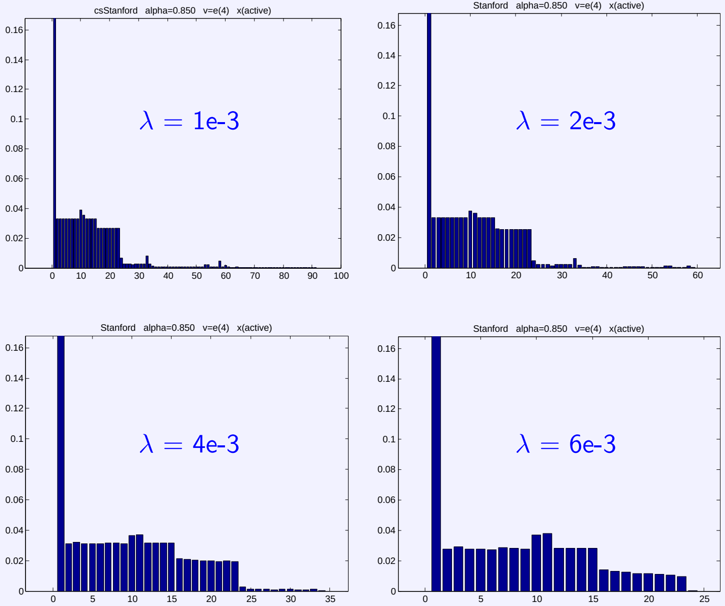
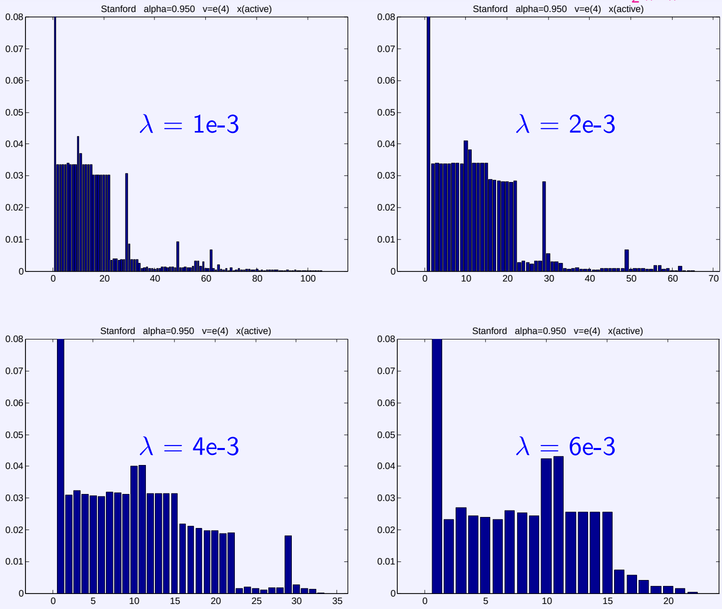


csStanford $v = e_4$ $\alpha = 0.95$ $\min \lambda e^T x + \frac{1}{2}\|r\|^2$



SPAI: Sparse Approximate Inverse

SPAI Preconditioning

A plausible future application

- Find sparse X such that $AX \approx I$

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SPAI Preconditioning

A plausible future application

- Find sparse X such that $AX \approx I$
- Find sparse x_j such that $Ax_j \approx e_j$ for $j = 1 : n$
- Embarrassingly parallel use of sparse BP solutions

Summary

$$(I - \alpha H^T)x \approx v$$

Comments

- Sparse v *sometimes* gives sparse x

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- Sensitive to λ but we can limit iterations

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Questions

- Run much bigger examples
- Which properties of $I - \alpha H^T$ lead to success of "greedy"

Thanks

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