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A simple isentropic model of electron transport in Hall thrusters

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Abstract
A simple model is presented for the time-averaged electron mobility within a Hall thruster. The model is predicated on the experimental evidence for isentropic electron flow and, when used in a one-dimensional simulation, captures plasma properties that are in reasonable agreement with experiment.

1. Introduction

Understanding the mechanism that determines the mobility of electrons across the magnetic field has been one of the greatest challenges in Hall thruster research. The cross-field electron transport measured in Hall thrusters is anomalously high when compared with classical predictions. Explanations have emerged that attempt to quantify this phenomenon. The presence of strong fluctuations over a wide frequency range suggest that this anomalous transport is due to plasma turbulence and may follow a Bohm-type scaling with magnetic field [1], However, such a model is unsuccessful at capturing the spatial variation in important plasma properties throughout the discharge channel when used in hybrid simulations [2]. Early theories pointed to the possible role played by electron-wall scattering in enhancing the so-called near-wall conductivity [3], although it appears that the loss of the high energy electrons to the wall may not be replenished in the bulk plasma at a sufficient rate to account for the anomalous electron current [4]. Researchers have used, with some success, models with ad hoc coefficients that combine these two mechanisms, i.e. apply a Bohm transport description in the near-field beyond the exit plane, and wall-scattering models within the discharge channel [5]. However, the coefficients used to predict the discharge current at nominal operating conditions do not seem to capture the behaviour of the discharge over the broader operating envelope. Clearly, the absence of a physical model for electron transport has precluded the development of robust simulations of Hall thrusters. In this paper, we discuss a model based on observed isentropic behaviour in the electron fluid. We use this model in a 1D simulation of Hall thrusters and find that it captures the plasma properties reasonably well. We propose that this model may be applicable to a broad range of operating conditions and thruster designs, and should be relatively easy to implement in multi-dimensional hybrid Hall thruster simulations.

The question as to why this isentropic flow condition seems to apply throughout the discharge channel is not yet understood. As discussed by Robertson *et al* [6], a minimum entropy production is expected in plasmas that are electrically isolated by non-conducting channel walls. In a Hall thruster, collisionless electron migration in the absence of random scattering from the wall (i.e. most of the electrons bounce off the wall sheath) would not be expected to lead to entropy production. Whatever the explanation may be, the finding that the electron entropy remains reasonably constant leads to a convenient way of predicting the net electron transport.

2. Spatial variation of electron entropy

The spatial variation of specific electron entropy (entropy per electron mass) can be expressed in terms of the second law of thermodynamics in differential form:

$$\dot{P}_s = \rho_e \vec{u}_e \cdot \nabla s_e + \nabla \cdot \left(\frac{\kappa \nabla T_e}{T_e} \right). \quad (1)$$

In deriving this equation, we have made use of the electron conservation equation, and so $\dot{P}_s$ represents the net rate of electron entropy production through both elastic and non-elastic collision processes, s_e is the electron entropy per unit mass, ρ_e is the electron mass density, $\vec{u}_e$ is the electron velocity, κ is the thermal conductivity of the electron fluid and T_e is the electron temperature. The first term on the right-hand side represents the mass-associated transport of electron entropy, while the second term is the heat-associated transport. In the presence of a magnetic field, the thermal conductivity is a tensor. If we consider the flow to be uniform along directions other than the direction transverse to the magnetic field (x-direction), and if we relate the transverse thermal conductivity, $\kappa_\perp$, to the transverse electron mobility, $\mu_\perp$, through

$$\kappa_\perp = \frac{5}{2}\left(\frac{k}{e}\right) n_e k T_e \mu_\perp \tag{2}$$

(here, k is the Boltzmann constant, n_e is the electron number density and e is the elementary charge) then the entropy balance reduces to

$$\dot{P}_s = \rho_e \mu_\perp E_x \frac{\mathrm{d}s_e}{\mathrm{d}x} + \frac{5}{2}\frac{k^2}{e}\frac{\mathrm{d}}{\mathrm{d}x}\left(n_e \mu_\perp \frac{\mathrm{d}T_e}{\mathrm{d}x}\right), \tag{3}$$

where we have expressed the transverse electron drift velocity in terms of the axial electric field, E_x, and the transverse electron mobility, $\mu_\perp$, with $u_\perp = \mu_\perp E_x$. The non-dimensionalization of this equation introduces the smallness parameter

$$\epsilon = \frac{5}{2}\frac{k^2 T_0}{m_e e s_0 \phi_0}, \tag{4}$$

in terms of T_0, s_0 and ϕ_o, characteristic electron temperature, characteristic electron entropy and discharge potential respectively, which determines the importance of the thermal conduction term. Here m_e is the electron mass. If we assume that the net production of entropy is zero (reversible thermodynamic process), then the spatial variation in entropy is determined by the strength of this smallness parameter:

$$\frac{\mathrm{d}\tilde{s}_e}{\mathrm{d}\tilde{x}} = -\frac{\epsilon}{\tilde{n}_e \tilde{\mu}_\perp \tilde{E}_x}\frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\left(\tilde{n}_e \tilde{\mu}_\perp \frac{\mathrm{d}\tilde{T}_e}{\mathrm{d}\tilde{x}}\right), \tag{5}$$

where the tilde indicates non-dimensional parameters. For most cases in Hall discharges, $\epsilon \ll 1$, and so the reversible process is also isentropic (electron entropy should be uniform within the discharge). Of course, this isentropic condition rests on the assumption of reversibility and requires justification. Experimental data gathered on a laboratory Hall thruster in previous studies by Meezan *et al* [7] allow us to test this condition. The experimental data include the axial variation in electron temperature and number density in the acceleration channel of the discharge. The spatial variation of electron entropy is estimated from the Sackur–Tetrode equation [8],

$$s_e = \frac{5k}{2m_e}\ln(T_e) - \frac{k}{m_e}\ln(p_e) + \frac{k}{m_e}\left\{\ln\left[\left(\frac{2\pi m_e}{h^2}\right)^{3/2} k^{5/2}\right] + \frac{5}{2}\right\} + \frac{k}{m_e}\ln(2), \tag{6}$$

where p_e is the electron pressure and h is Planck's constant. It should be noted that local thermodynamic equilibrium is assumed in the derivation of the Sackur–Tetrode equation. While this assumption holds well for the bulk of the electrons, certain departures from a Maxwellian velocity distribution do occur. Notably, some of the high energy electrons are removed from the plasma as a result of wall interactions, which acts to deplete the high energy tail of the velocity distribution. Additionally, the strong azimuthal electron drift increases the probability of finding electrons in the vicinity of the $E \times B$ drift velocity.

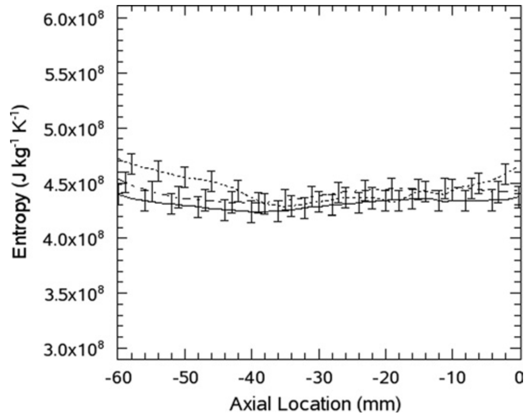

Figure 1. The axial variation of electron entropy for 100 V (——), 160 V (— · —) and 200 V (----) discharge conditions.

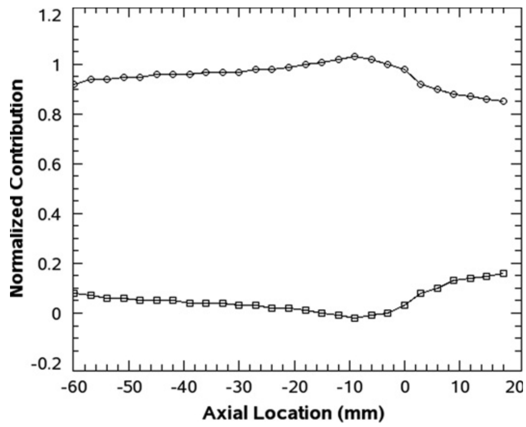

Figure 2. Fractional temperature contributions (○) and pressure contributions (□) to the electron entropy for the 160 V discharge condition.

Figure 1 shows the spatial variation in electron entropy for three operating conditions of the Hall thruster: 100, 160 and 200 V. Indeed the electron entropy is very nearly constant for all three cases, however, on this scale, variations in entropy change are masked by the fact that changes in entropy depend on the logarithm of the electron temperature and pressure. A better examination of this is seen in figure 2, for the 160 V discharge case, where we plot the fractional contribution of the temperature (open circles) and pressure (open squares) to the total entropy. We see a clear correlation between the two, following a trend that would result in the cancellation of their contributions. This correlation is seen despite the uncertainty in the experimental measurements. This result, the reason for which merits further study, leads us to conclude that it may be reasonable to assume isentropic transverse flow conditions in the electron fluid and, through the Sackur–Tetrode equation,

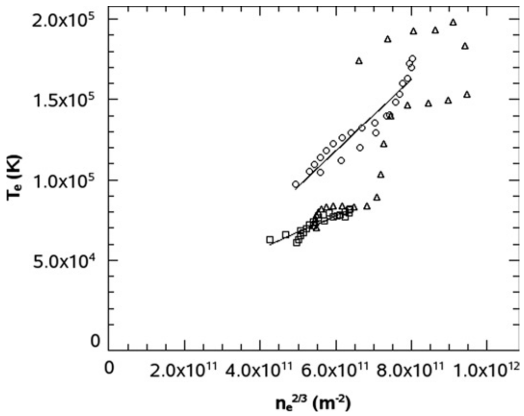

Figure 3. Electron temperature versus number density raised to the 2/3 power for 100 V (□), 160 V (○) and 200 V (△) discharge conditions.

provides a simple polytropic constraint on the axial variation in the plasma density:

$$\frac{dn_e}{dx} = \frac{3}{2}\frac{n_e}{T_e}\frac{dT_e}{dx}. \tag{7}$$

It follows from equation (7) that there should exist a linear relationship, that intercepts the origin, between the electron temperature (T_e) and the number density raised to the 2/3 power ($n_e^{2/3}$) in an isentropic electron fluid. This relationship is investigated using the available experimental data in figure 3. For the 100 and 160 V conditions this linear relationship is apparent. However, it can be observed that the 200 V discharge condition departs from this trend, particularly as we move outwards into the plasma plume. It is speculated that this departure may be due in a large part to the challenge of measuring the electron temperature and number density accurately under conditions of high directional ion flux. These measurements were conducted using a combination of Langmuir and emissive type probes. The accuracy of these probes under conditions of high ion flux have not been well established.

It is noteworthy that other data have surfaced suggesting a polytropic relation between plasma density and plasma potential in Hall thruster discharges [9]. Such a polytropic condition greatly simplifies the flow equations, as illustrated below, which can be used in both 1D and multi-dimensional fluid and/or hybrid models.

3. Second law electron transport model in an $E \times B$ field

The isentropic condition presented in section 2 allows us to derive a relatively simple expression for the transverse electron fluid velocity in an orthogonal $E \times B$ field. For this analysis we assume that the electric field is aligned with the x-axis, which is parallel to the axis of the thruster. The magnetic field is aligned with the z-axis along the radius of the thruster leading to an $E \times B$ drift in the negative y-axis direction. We limit our analysis to spatial variations in the plasma properties transverse to the B-field direction.

The differential form of the steady electron energy equation is given by

$$\frac{d(n_e m_e u_x e_e)}{dx} = -\frac{d(u_x n_e k T_e)}{dx} - e n_e u_x E_x - \beta_e \varphi \dot{W} \epsilon_i. \tag{8}$$

Here u_x is the transverse component of the electron velocity, e_e is the electron energy per unit mass, $\dot{W}$ is the production rate of electrons due to ionization and ϵ_i is the ionization energy of the neutral particles. The term φ is the ionization cost, which depends on the neutral species being ionized and the electron temperature. This study uses the method introduced by Dugan *et al* [10] to model the ionization cost. The β_e parameter modifies the ionization rate to account for energy losses to the wall. This term will be discussed in more detail in section 4.

The energy equation (equation (8)) neglects the thermal conduction of the electron fluid and the collisional energy loss due to non-ionizing collisions. The electron energy includes contributions from thermal and bulk fluid motion, i.e.

$$e_e = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \frac{3}{2}\frac{kT_e}{m_e}. \tag{9}$$

Substituting the conservation equation,

$$\frac{d(n_e m_e u_x)}{dx} = m_e \beta_c \dot{W} \tag{10}$$

(where β_c modifies the ionization rate to account for electrons lost to the wall), into the energy equation leads to the following form:

$$\frac{1}{2}u_x^2 m_e \beta_c \dot{W} + n_e m_e u_x^2 \frac{du_x}{dx} + \frac{1}{2}u_y^2 m_e \beta_c \dot{W} + n_e m_e u_x u_y \frac{du_y}{dx} + \frac{3}{2}kT_e\beta_c\dot{W} + \frac{3}{2}k n_e u_x \frac{dT_e}{dx} = -n_e k T_e \frac{du_x}{dx} - u_x k \frac{d(n_e T_e)}{dx} - e n_e u_x E_x - \beta_e \varphi \dot{W} \epsilon_i. \tag{11}$$

We now incorporate the isentropic condition into the energy equation. When we substitute the isentropic condition (equation (7)) into the electron conservation equation (equation (10)), we arrive at the following relation:

$$\frac{3}{2}k n_e u_x \frac{dT_e}{dx} = -n_e k T_e \frac{du_x}{dx} + kT_e \beta_c \dot{W}, \tag{12}$$

which can be combined with the modified energy equation (equation (11)) leading to a quadratic equation for the transverse drift velocity. This expression can be further simplified by assuming that the terms involving the square of the transverse velocity, u_x^2, can be neglected. An expression for the ionization rate can be substituted into the above equations. This is given by

$$\dot{W} = n_N n_e \left(\frac{8kT_e}{\pi m_e}\right)^{1/2} \bar{\sigma}_i. \tag{13}$$

Here n_N is the neutral species number density and $\bar{\sigma}_i$ is the ionization cross section. We then arrive at the following description for the transverse electron drift velocity:

$$u_x\left(m_e u_y\frac{\mathrm{d}u_y}{\mathrm{d}x}+\frac{k}{n_e}\frac{\mathrm{d}(n_eT_e)}{\mathrm{d}x}+eE_x\right)$$
$$=n_N\bar{\sigma}_i\left(-\frac{1}{2}\beta_c m_e u_y^2-\frac{5}{2}\beta_c kT_e-\beta_e\varphi\epsilon_i\right)\left(\frac{8kT_e}{\pi m_e}\right)^{1/2}. \quad (14)$$

On the basis of experimental results collected by Meezan *et al* [7], we can identify two terms that can be neglected from the above expression in the context of Hall thrusters. The first is the electron shear term, $m_e u_y(\mathrm{d}u_y/\mathrm{d}x)$, and the second is the electron momentum in the $E\times B$ direction, $\frac{1}{2}m_e u_y^2$. With these two simplifications, we can solve for the transverse electron fluid velocity

$$u_x=-n_N\bar{\sigma}_i\left(\frac{\frac{5}{2}\beta_c kT_e+\beta_e\varphi\epsilon_i}{(k/n_e)(\mathrm{d}(n_eT_e)/\mathrm{d}x)+eE_x}\right)\left(\frac{8kT_e}{\pi m_e}\right)^{1/2}. \quad (15)$$

Although simple, this electron transport model captures the spatial variation in the electron mobility. In the next section, we will test the transportability of this model to a 1D simulation of a Hall thruster and compare the predicted variations in plasma properties to those measured experimentally.

4. One-dimensional Hall thruster simulation

This 1D quasi-neutral simulation is based on the geometry of the Stanford Hall Thruster; a custom-built low power Hall thruster with a channel diameter of 90 mm, width of 11 mm and length of 80 mm. The unknowns in this simulation include the electron temperature T_e, electron number density n_e, electron velocity u_x, and the plasma potential ϕ which is related to the electric field by $E_x=-\mathrm{d}\phi/\mathrm{d}x$. The four equations used to solve for this system are provided below.

Electron transport equation:

$$u_x\left(\frac{k}{n_e}\frac{\mathrm{d}(n_eT_e)}{\mathrm{d}x}+eE_x\right)=-n_N\bar{\sigma}_i\left(\frac{5}{2}\beta_c kT_e+\beta_e\varphi\epsilon_i\right)\times\left(\frac{8kT_e}{\pi m_e}\right)^{1/2}. \quad (16)$$

Electron entropy equation:

$$\frac{\mathrm{d}n_e}{\mathrm{d}x}=\frac{3}{2}\frac{n_e}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}. \quad (17)$$

Electron conservation equation:

$$\frac{\mathrm{d}(n_eu_x)}{\mathrm{d}x}=\beta_c n_e n_N\left(\frac{8kT_e}{\pi m_e}\right)^{1/2}\bar{\sigma}_i. \quad (18)$$

Ion conservation equation:

$$\frac{\mathrm{d}(n_eu_{ix})}{\mathrm{d}x}=\beta_c n_e n_N\left(\frac{8kT_e}{\pi m_e}\right)^{1/2}\bar{\sigma}_i. \quad (19)$$

Ordinarily, one would solve for the ion velocity, u_{ix}, using the ion momentum equation

$$u_{ix}\frac{\mathrm{d}u_{ix}}{\mathrm{d}x}=\frac{e}{m_i}E_x-u_{ix}\beta_c n_N\left(\frac{8kT_e}{\pi m_e}\right)^{1/2}\bar{\sigma}_i. \quad (20)$$

However, this equation is known to be problematic as the ions have a highly non-Maxwellian velocity distribution and should more appropriately be solved using the Boltzmann equation. In this simple model we will instead use experimentally measured values for the ion velocity.

The temperature dependence of the ionization cross section $\bar{\sigma}_i$ was evaluated using a model derived by Ahedo *et al* [11]. This approximation works well for the electron temperature range of interest for a Hall thruster (between 1 and 100 eV). This cross section is given by

$$\bar{\sigma}_i(T_e)=\sigma_{i0}\left(1+\frac{T_e\epsilon_i}{(T_e+\epsilon_i)^2}\right)\exp\left(-\frac{\epsilon_i}{T_e}\right), \quad (21)$$

with $\sigma_{i0}=5\times10^{-20}\,\mathrm{m}^2$.

The neutral density, n_N, is not solved for explicitly and instead was taken from experimental measurements. The model for the ionization production cost, φ, was taken from the work by Dugan *et al* [10]. The β_e and β_c parameters, which modify the ionization rate to account for wall losses in the energy equation and conservation equation, respectively, were both set to a constant value of 0.4. These values were established by solving the electron energy equation and conservation equation with the available experimental data, and taking the average value of β_e and β_c over all axial locations within the channel. This admittedly *ad hoc* way of dealing with wall interactions would not be needed in a more advanced two-dimensional simulation, which would model the wall interactions explicitly.

The boundary conditions for the simulation were established by imposing a discharge voltage, and allowing the discharge current to be determined self-consistently. The plasma potential at the anode and the cathode of the simulation domain was imposed from the experimental conditions

$$\phi_1=162\,\mathrm{V}, \quad (22)$$

$$\phi_2=36\,\mathrm{V}. \quad (23)$$

Here we use the subscript 1 to indicate the anode boundary and subscript 2 for the cathode boundary. We can see from the electron entropy equation (equation (17)) that the plasma density is related to the electron temperature everywhere by

$$T_e=c_1n_e^{2/3}. \quad (24)$$

The constant of proportionality, c_1, was determined from experimental measurements to be $2.2\times10^{-7}\,\mathrm{m}^2\,\mathrm{K}$. The final boundary condition was established by forcing the peak of the plasma density, and therefore the electron temperature, to be aligned with the peak of the imposed magnetic field. This was accomplished by setting

$$\left[\frac{\mathrm{d}n_e}{\mathrm{d}x}\right]_{x=-9\,\mathrm{mm}}=0 \quad (25)$$

at the peak magnetic field location.

The above equations were solved numerically using the specified boundary conditions. The results of this simulation are shown in figures 4(*a*)–(*e*), and are compared with the

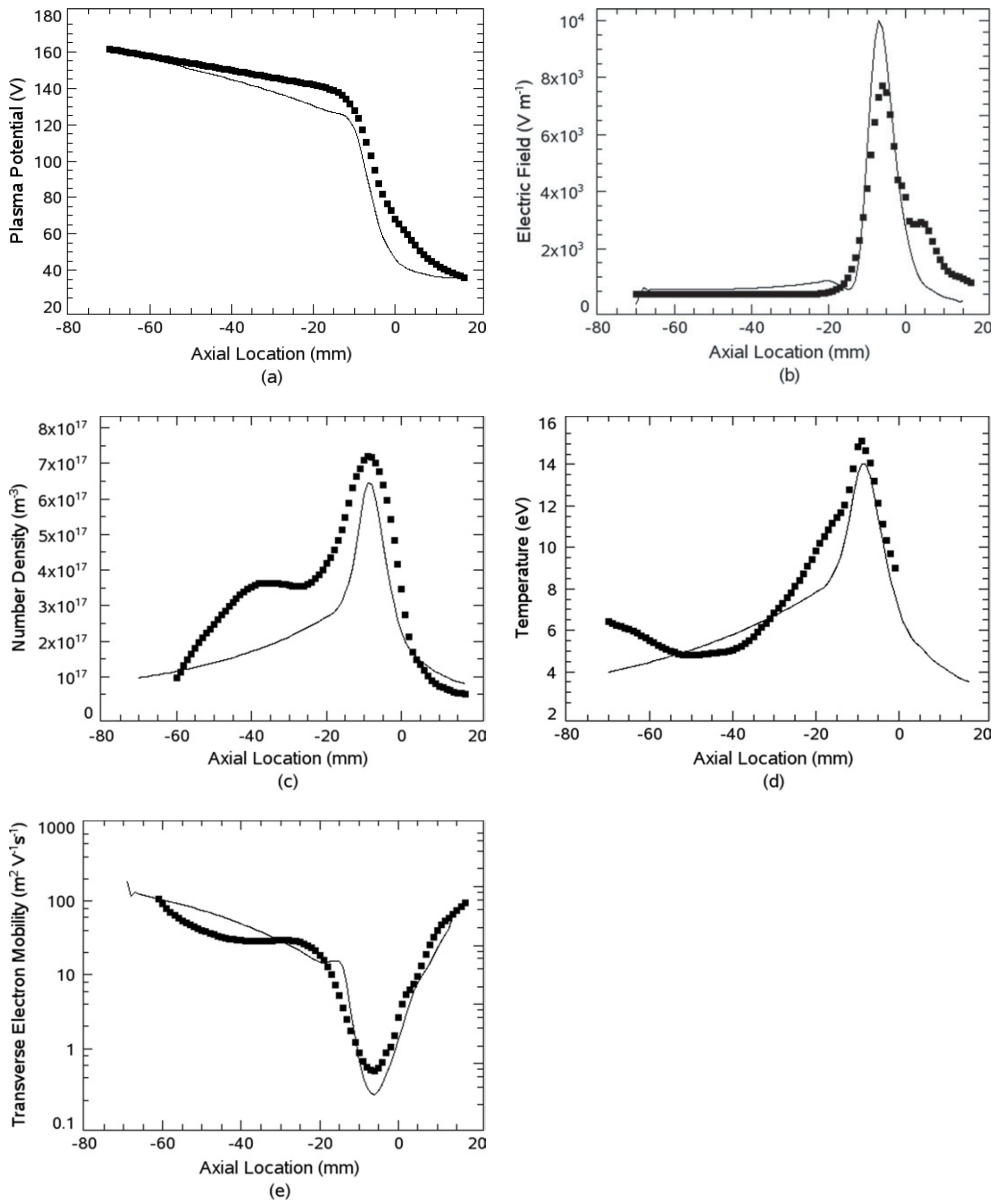

Figure 4. Simulated (——) versus experimental (■) plasma potential, electric field, electron density, electron temperature and transverse electron mobility.

experimental results described by Meezan *et al* [7]. The agreement between the computed and measured plasma properties is reasonable. Recognizing, however, that some parameters used in the model were in fact extracted from these experimental data, this agreement is not entirely unexpected. The success of this model is that it reproduces the structure seen in the transverse mobility (figure 4(*e*)), capturing the presence of a strong transport barrier in the region of strong magnetic fields.

5. Summary

This paper presents a novel approach to modelling the transverse mobility of electrons across a magnetic field in a Hall thruster. The model is based on the empirical observations that the spatial variation of the electron entropy is small over a wide range of conditions studied. This isentropic condition was used to derive an expression for the transverse electron velocity, which replaces the axial electron momentum equation in 1D or 2D Hall thruster simulations.

A 1D simulation was constructed to illustrate the use of this transport model developed in this study. The results of the model are found to be in reasonable agreement with experimental measurements. We note, however, that parameters unique to 1D models that account for plasma interactions with the channel walls are extracted from these experiments, for use in the model. A true test of the usefulness and transportability of this isentropic transport model will be

2D simulations (fluid or hybrid) that do not have to introduce these wall interactions in *ad hoc* ways.

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