Analytical Value-At-Risk with Jumps and Credit Risk

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Preliminary

Abstract

This paper provides an analytical method for computing value at risk, and other risk measures, for portfolios that may include options and other derivatives, with defaultable counterparties or borrowers. The risk setting is that of a classical multi-factor jump-diffusion for default intensities and asset returns, under which between-jump returns are correlated Brownian motions, with return jumps at Poisson arrivals that are jointly normally distributed. This allows for fat-tailed and skewed return distributions.

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1 Introduction

This paper provides an analytical approximation for the value at risk, and other risk measures, of over-the-counter (OTC) portfolios. The portfolios may include options and other derivatives, with defaultable counterparties or borrowers. The risk setting is that of a classical multi-factor jump-diffusion for default intensities and asset returns, under which between-jump returns are correlated Brownian motions, with return jumps at Poisson arrivals that are jointly normally distributed. This allows for fat-tailed and skewed return distributions. We find that, at least for the cases we examined, extreme tail losses are dominated by either credit risk (for example in loan portfolios) or by the underlying market price risk (for example with equity option portfolios), but that only rarely do both sources of risk make a large contribution to tail risk.

In our previous work in this direction, Duffie and Pan [1997], we used a relatively standard Monte-Carlo procedure, under which returns are simulated, and delta-gamma approximations to derivative prices are used to estimate marks to market for each scenario. Repeated simulation is used to estimate a confidence interval on losses that is typically known as "value at risk," or "VaR." For example, under an accord[1] set through the Bank of International Settlements (BIS), regulated banks are required to report, and to maintain capital based on, an estimate of VaR defined as the loss in portfolio value that is exceeded with 1% probability over a two-week period. Banks and many other financial firms often compute *VaR* for internal purposes based on 1-day changes in portfolio value, and sometimes at other confidence levels. We believe that delta-gamma-based approximation of changes in derivative prices, under which the actual derivative pricing formula is treated up to a second-order Taylor series approximation, is commonly used for this purpose. The use of Monte-Carlo simulation also appears to be popular, although in some cases actual historical returns are used, in the spirit of a "boot-strap," rather than simulation based on theoretical return distributions. For a recent Monte-Carlo-based VaR estimation methodology, see Cardenas, Fruchard, Picron, Reyes, Walters, and Yang [1999], and Jamshidian and Zhu [1997], who also consider default risk.

The BIS accords do not currently address "specific" (default) risk in VaR measurement, although there is some discussion of eventual model-based credit-risk measurement for setting regulatory capital requirements (BIS [1999]). For a pure-diffusion (no-jump) model, Cardenas, Fruchard, Koehler, Michel, and Thomazeau [1997] and Rouvinez [1997] showed how to replace the simulation step with an analytical method based on the characteristic function of the delta-gamma approximation Y of the change in portfolio value.[2] After deriving an explicit solution for this characteristic function, which is essentially the Fourier transform of the density of Y, they inverted for the distribution (or, in the case of Cardenas, Fruchard, Koehler, Michel, and Thomazeau [1997], the density) of Y, and then obtained the VaR. We take essentially the same approach here, *after conditioning on the number of jumps*, exploiting a single and efficient numerical integration step, designed by Davies [1980], for inverting the characteristic function so as to directly obtain the cumulative distribution function of Y, from which VaR is immediate. The jump-conditional VaR is then easily

[1] See BIS [1995].

[2] Another approach, also based on characteristic functions, is due to Jahel, Perraudin, and Sellin [1998].

weighted by the probability of a given number of jumps, and combined to obtained the unconditional VaR including the effects of jumps. For short time horizons, such as one day or two weeks, ignoring more than a small number of jumps achieves a given accuracy, for which we explicitly control. A similar approach can be used to compute other risk measures, such as the expected loss in excess of a given critical value. See Artzner, Delbaen, Eber, and Heath [1998] for a critique of VaR and an axiomatic basis for "coherent" risk measures, such as the expected excess loss.

For a given level of accuracy, and for the option portfolios and return processes that we have examined to this point, we find a substantial reduction in computational effort over the alternative of Monte-Carlo simulation.

We next fold into the *VaR* calculation the risk of changes in credit quality, including default. Based on a correlated default-intensity model, we obtain an approximation of the probability distribution of the total loss of a portfolio that is exposed to a number of correlated default risks, applying the same analytical Fourier-transform approach. Fixing a given joint distribution of counterparty default intensity processes, we show a significant impact of common credit events, at which more than one counterparty may default.

While we do not pursue it here, our method would make it easy to analytically calculate *conditional* measures of VaR, as suggested by Kupiec [1998], allowing one to plot VaR conditional on a given market return or a given scenario for various markets, as one varies that scenario parametrically.

2 Portfolio Delta-Gamma Approach

We suppose that the portfolio under consideration consists of cash-market positions and derivatives on n underlying assets, whose price processes, $S_1, \ldots, S_n$, form a jump-diffusion. Specifically, we let $R = (\ln S_1, \ldots, \ln S_n)$ denote the log-price process, and suppose that

$$R_t = R_0 + \left(\overline{\mathcal{R}} - \lambda\mu\right) t + \sqrt{t}\,\Sigma^{1/2} X_0 + \sum_{j=1}^{\mathcal{N}(t)} (\mu + V^{1/2} X_j)\,, \tag{2.1}$$

where $\overline{\mathcal{R}}$ in $\mathbb{R}^n$ is a mean-return vector;[3] for $j = \{0, 1, 2, \ldots.\}$, X_j is standard normal in $\mathbb{R}^n$, and X_j and X_k are independent for $j \neq k$, and where Σ and V are symmetric and positive semi-definite matrices of dimension $n \times n$; and $\mathcal{N}$ is a jump-counting Poisson process, independent of $\{X_1, X_2, \ldots\}$, with constant arrival intensity λ. For now, we suppose only one type of normally distributed joint jump distribution. We later consider multiple jump types. We defer to Section 5 the consideration of credit risk.

To summarize, the jump-diffusion model specified in (2.1) has jumps at Poisson arrivals with mean arrival rate λ, and conditional on j jump events before time t, R_t is jointly normally distributed with covariance $\Sigma\, t + jV$, and with mean $R_0 + \left(\overline{\mathcal{R}} - \lambda\mu\right)\, t + j\mu$.

Given the dynamics of the underlying assets, we are now interested in assessing the risk of a portfolio of cash-market positions and derivatives, such as options. The total portfolio

[3] If a risk-neutral distribution were required, which is not generally the case for VaR calculations, the k-th element of $\overline{\mathcal{R}}$ would be defined by $\overline{\mathcal{R}}_k = r - \frac{1}{2}\Sigma_{kk} - \lambda\left(\exp(\mu_k + \frac{1}{2}V_{kk}) - 1\right) + \lambda\mu_k$, where r is the risk-free short rate.

value may be expressed by $C(R) = \sum_{k=1}^{m} C_k(R)$, where $C_1, \ldots, C_m$ are m derivatives written on the n underlying assets. For example, a long cash-market position of K units of asset i is defined by a derivative function $C_k(\cdot)$ given by $C_k(R) = K \exp(R_i)$. A classical European option could be defined by a linear combination of Black-Scholes option pricing formulas, by conditioning on the number of jumps and evaluating the Black-Scholes formula associated with the risk-neutral jump-conditional return distribution.[4]

For the purpose of VaR and other risk measures, we are interested in the tail distribution of $C(R_T)$ over a fixed time horizon T. We adopt the well known[5] delta-gamma second-order approximation

$$\begin{aligned} C(R_T) \approx C^{\Delta,\Gamma}(R_T) &= C(R_0 + \overline{\mathcal{R}}T) + \Delta \cdot (R_T - R_0 - \overline{\mathcal{R}}T) \\ &\quad + \frac{1}{2}(R_T - R_0 - \overline{\mathcal{R}}T)^\top \,\Gamma\, (R_T - R_0 - \overline{\mathcal{R}}T)\,, \end{aligned} \tag{2.2}$$

where Δ and Γ are defined by

$$\Delta_i = \sum_{k=1}^{m} \left.\frac{\partial C_k(R)}{\partial R_i}\right|_{R_0+\overline{\mathcal{R}}T} \quad ; \quad \Gamma_{i,j} = \sum_{k=1}^{m} \left.\frac{\partial^2 C_k(R)}{\partial R_i\, \partial R_j}\right|_{R_0+\overline{\mathcal{R}}T} . \tag{2.3}$$

In practice, banks often take $\overline{\mathcal{R}} = 0$ in (2.2)-(2.3). For the purpose of a short-horizon VaR estimate, the distinction is small in typical cases, and one could adopt either convention for the following calculations.

For European options, Δ and Γ may be computed explicitly, by conditioning on the number of jumps, as a linear combination of the corresponding explicit Black-Scholes deltas and gammas. (For more on computing deltas and gammas, see Glasserman and Zhao [1998].) In any case, it is somewhat common in practice to maintain a database of delta and gamma exposures, which may be available for this VaR application.

3 Calculation of Tail Probabilities

We propose now an approach based on characteristic functions to calculate $P\left(C^{\Delta,\Gamma}(R_T) < \bar{c}\right)$, for any fixed critical value $\bar{c}$. For notational simplicity, we re-express (2.2) as

$$C(R_T) \simeq C^{\Delta,\Gamma}(R_T) = A + B \cdot R_T + \frac{1}{2} R_T^\top \,\Gamma\, R_T\,, \tag{3.1}$$

where $A = C(R_0+\overline{\mathcal{R}}T) - \Delta\cdot(R_0+\overline{\mathcal{R}}T) + \frac{1}{2}(R_0+\overline{\mathcal{R}}T)^\top\Gamma(R_0+\overline{\mathcal{R}}T)$, and $B = \Delta - \Gamma\,(R_0+\overline{\mathcal{R}}T)$.

3.1 Characteristic Function

We first derive the characteristic function $\phi : \mathbb{R} \to \mathbb{C}$ of $C^{\Delta,\Gamma}(R_T)$, where $\mathbb{C}$ is the set of complex numbers, defined by

$$\phi(u) = E\left[\exp\left(iuC^{\Delta,\Gamma}(R_T)\right)\right] . \tag{3.2}$$

[4] See Duffie and Pan [1997] for details.

[5] See, for example, Page and Feng [1995].

For this calculation, we use the fact that, conditional on the event of j jumps before T, R_T is jointly normally distributed with mean $M_j = R_0 + \left(\overline{\mathcal{R}} - \lambda\mu\right) T + j\mu$, and covariance matrix $\Omega_j = \Sigma T + jV$. We therefore have

$$\phi(u) = \sum_{j=0}^{\infty} p_j \phi_j(u)\,, \tag{3.3}$$

where the probability p_j of the event of j jumps by T is

$$p_j = P(\mathcal{N}_T = j) = e^{-\lambda T}\frac{(\lambda T)^j}{j!}\,,$$

and where the characteristic function ϕ_j of $C^{\Delta,\Gamma}(R_T)$, conditional on the event of $\mathcal{N}_T = j$ jumps, is given, based on calculations whose history can be traced back from sources cited in Davies [1973], by

$$\begin{aligned}\phi_j(u) &= E\left[\exp\left(i\,u\,C^{\Delta,\Gamma}(R_T)\right)\middle|\mathcal{N}_T = j\right]\\ &= \frac{\exp\left(iu\sum_{k=1}^{n}\eta_j^k(\delta_j^k)^2/(1-2iu\eta_j^k) - \frac{1}{2}u^2\beta_j^2 + iu\gamma_j\right)}{\prod_{k=1}^{n}(1-2iu\eta_j^k)^{1/2}}\,,\end{aligned} \tag{3.4}$$

where the parameters δ, η, β, and γ are defined as follows.

For fixed j, $\eta_j = (\eta_j^1, \dots, \eta_j^n)$ is defined by

$$D_j^\top \begin{pmatrix} \eta_j^1 & & \\ & \ddots & \\ & & \eta_j^n \end{pmatrix} D_j = \frac{1}{2}\Omega_j^{1/2}\,\Gamma\,\Omega_j^{1/2}\,, \tag{3.5}$$

where D_j is an $(n \times n)$ orthogonal matrix with $D_j^\top = D_j^{-1}$. Denoting the k-th row of D_j by D_j^k, we recognize that η_j^k is the j-th eigenvalue of the matrix on the right-hand side, while the associated eigenvector is D_j^k. Next, for fixed j, we define $\delta_j = (\delta_j^1, \dots, \delta_j^n)$ by

$$\delta_j^k = \frac{D_j^k \cdot a_j}{2\eta_j^k}\,, \quad \text{if } \eta_j^k \neq 0\,; \qquad \delta_j^k = 0\,, \quad \text{otherwise}\,, \tag{3.6}$$

where $a_j = \Omega_j^{1/2}(B + \Gamma M_j)$. Finally, we have

$$\beta_j = \sqrt{a_j \cdot a_j - 4\sum_k (\eta_j^k \delta_j^k)^2}\,, \tag{3.7}$$

$$\gamma_j = A + B \cdot M_j + \frac{1}{2}M_j^\top \Gamma M_j - \sum_k \eta_j^k (\delta_j^k)^2\,. \tag{3.8}$$

3.2 Fourier Inversion

Given the characteristic function ϕ of $C^{\Delta,\Gamma}(R_T)$ from (3.3), a standard Fourier-inversion formula[6] implies that

$$P(C^{\Delta,\Gamma}(R_T) \leq \bar{c}) = \frac{1}{2} - \frac{1}{\pi} \sum_{j=0}^{\infty} p_j \int_0^{\infty} \frac{I_j(u)}{u}\, du\,, \tag{3.9}$$

where, for any $u \in \mathbb{R}$,

$$\begin{aligned} I_j(u) &= \operatorname{Im}\left[\phi_j(u) \exp(-iu\bar{c})\right] \\ &= N_j(u) \sin\left[\frac{1}{2}\sum_k \arctan(2u\eta_j^k) + \sum_k \frac{u\eta_j^k(\delta_j^k)^2}{1+4u^2(\eta_j^k)^2} + u(\gamma_j - \bar{c})\right], \end{aligned}$$

where $\operatorname{Im}(z)$ denotes the imaginary part of a complex number z, and where

$$N_j(u) = \frac{\exp\left(-2u^2\sum_k (\eta_j^k\delta_j^k)^2/(1+4u^2(\eta_j^k)^2) - u^2\beta_j^2/2\right)}{\prod_k (1+4u^2(\eta_j^k)^2)^{1/4}}.$$

3.3 Numerical Inversion with Error Analysis

We evaluate (3.9) by numerical integration, based on equally spaced abscissas with steps:

$$P(C^{\Delta,\Gamma}(R_T) \leq \bar{c}) \simeq \sum_{j=0}^{J} p_j \left(\frac{1}{2} - \frac{1}{\pi}\sum_{k=0}^{K_j} \frac{I_j\left((k+1/2)h_j\right)}{k+1/2}\right), \tag{3.10}$$

where the summation over jump events is cut off at J, and where the integration, for the event with j jumps, is cut off at $(K_j + 1/2)h_j$, and where h_j is the corresponding step size. Three types of errors are introduced by this:

1. The truncation error introduced by $J < \infty$.
2. The discretization error introduced by $h_j > 0$, for each j.
3. The truncation error introduced by $K_j < \infty$, for each j.

The first type of error is associated with the truncation of the number of jumps before time T, and can be easily managed by choosing J so that $\sum_{j=J+1}^{\infty} p_j$ is less than the desired accuracy. For example, even for an expected time between jumps of as little as one month, an accuracy of 0.00001 in the probability calculation over a time horizon of two weeks corresponds to a cut-off level of $J = 6$ jumps.

For an error tolerance of α, and given the choice of J, we assign an error tolerance of α/J to each of the J terms in (3.10). For the j-th term, the error analysis then comes down to choosing a step size h_j and a truncation level K_j so that the error introduced in the summation over k is less than the desired accuracy α/J for that term, divided by p_j.

[6] See, for example, Gil-Pelaez [1951].

For the rest of the section, we sketch out a mechanism to control the discretization and truncation errors listed in (2) and (3) above. The treatment is the same for each of the J terms. Thus, we will discuss the case of a generic j and, for notational simplicity, we will drop the subscript j from our notation. Our treatment of these error bounds follows closely that of Davies [1973] and Davies [1980], where more details can be found.

We first address the issue of truncation error. Given that the numerical integration is cut off at $U = (K + 1/2)h$, the truncation error can be managed by finding bounds on $I(\,\cdot\,)$. That is,

$$\text{Truncation Error} = \frac{1}{\pi} \sum_{k=K+1}^{\infty} \frac{I\left((k+1/2)h\right)}{k+1/2} \leq \frac{1}{\pi} \int_{u=U}^{\infty} \frac{\bar{I}(u)}{u}\, du\,, \tag{3.11}$$

where $|I(u)| \leq \bar{I}(u)$, and $\bar{I}(u)$ is some monotonically decreasing function of u, for $u \geq U$.

Three different bounds on I are considered in Davies [1980], which result in three bounds on the truncation error, for a given cut-off value U, given by

$$B^{(1)}(U) = \frac{2}{\pi J} N(U) \prod_{4U^2(\eta^k)^2 > 1} \left(\frac{1 + 4U^2(\eta^k)^2}{4U^2(\eta^k)^2} \right)^{1/4},$$

$$B^{(2)}(U) = \frac{N(U)}{\pi U^2 \beta^2}, \quad B^{(3)}(U) = \frac{2.5}{\pi} N(U)\,,$$

where $B^{(3)}(U)$ applies if $\sum_k \ln(1 + 4U^2(\eta^k)^2) + 2\beta^2 U^2 \geq 1$. The truncation error is then bounded by the minimum of $B^{(1)}(U)$, $B^{(2)}(U)$, and $B^{(3)}(U)$.

We next address the issue of discretization error introduced by $h > 0$, which is derived in Davies [1973] to be

$$\text{Discretization Error} = \left| \sum_{n=1}^{\infty} (-1)^n \Big(P(C < \bar{c} - 2\pi n/h) - P(C > \bar{c} + 2\pi n/h) \Big) \right|, \tag{3.12}$$

where for notational simplicity, we denote $C^{\Delta,\Gamma}(R_T)$ by C. The discretization error can be managed by choosing h so that

$$\max \Big\{ P(C < \bar{c} - 2\pi/h), P(C > \bar{c} + 2\pi/h) \Big\} \tag{3.13}$$

is less than the desired accuracy.

For the purpose of calculating the tail probabilities in (3.13), Davies [1973] adopts the following approach. For $j \in \{1, \ldots, J\}$, let $\Psi_j(u)$ denote the logarithm of the moment-generating function of $C^{\Delta,\gamma}(R_T)$, conditioning on the event of j jumps, which can be derived to be

$$\Psi_j(u) = \ln E\left[\exp\left(u C^{\Delta,\Gamma}(R_T)\right) \,\middle|\, \mathcal{N}_T = j \right] \tag{3.14}$$

$$= \sum_k \frac{u \eta_j^k (\delta_j^k)^2}{1 - 2u\eta_j^k} + \frac{1}{2} u^2 \beta_j^2 + u\gamma_j - \frac{1}{2} \sum_k \ln(1 - 2u\eta_j^k)\,. \tag{3.15}$$

Then, after conditioning on a given number of jumps, we have

$$P\left(C > \Psi_j'(u) \,|\, \mathcal{N}_T = j\right) \leq \exp\left(\Psi_j(u) - u\Psi_j'(u)\right). \tag{3.16}$$

Finally, the unconditional probability that C exceeds a given number is obtained by conditioning, applying the given probability p_j of j jumps and the above expression for the conditional probability of exceedence given j jumps, and summing over j in the obvious way.

3.4 Multiple Jump Types

We have so far supposed that there is only one type of joint normally distributed return jump distribution. One could allow several joint normal jump types, for a mixture-of-normals jump size, although computational efficiency may suggest an attempt to limit the number of types.

Rather than treating each jump type separately, it may be efficient to view the model as one with jumps of any type at total Poisson arrival rate $\lambda = \lambda_1 + \cdots + \lambda_K$, where K is the number of types of joint normally distributed jumps, and λ_k is the arrival rate of jumps of type k. Any jump is of type k with constant conditional probability $\pi_k = \lambda_k/\lambda$. The characteristic function ϕ_j associated with j jumps (in total) may be viewed as a linear combination of the characteristic functions associated with the various combinations of numbers of jumps of each type that add up to j. That is, we would replace ϕ_j in (3.4) with

$$\phi_j(u) = \sum_{m \in I(j)} \pi^*(m)\phi_m^*(u), \tag{3.17}$$

where:

- $m = (m_1, \ldots, m_K)$ indicates the number of jumps of each of the K types, with $I(j) = \{m : m_1 + \cdots + m_K = j\}$.
- $\pi^*(m)$ is the probability that $(m_1, \ldots, m_K)$ is the outcome of the number of jumps of each type, for the multinomial distribution associated with $m_1 + \cdots + m_K$ independent trials, with probability π_k for an outcome of type k at a given trial. That is,

$$\pi^*(m) = \frac{(m_1 + \cdots + m_K)!}{m_1! m_2! \cdots m_K!}\pi_1^{m_1}\pi_2^{m_2}\cdots\pi_K^{m_K}. \tag{3.18}$$

- ϕ_m^* is the explicit characteristic function (for the appropriate joint-normal total returns) conditional on the outcome m for the numbers of jumps of each type.

One can then proceed as above to obtain the unconditional cumulative distribution function of portfolio loss, up to the delta-gamma approximation, with analogous error control. There are perhaps useful efficiencies to be obtained by "pruning" combinations of jump types whose total probability is sufficiently small relative to the allowable error.

4 An Example: Options Portfolio on Equity Indices

As an example, we consider options portfolios on equity indices from 32 countries. For the information on volatility and pair-wise correlations among the 32 markets, we use the Risk-Metrics database on November 20, 1998. The 32 countries and their respective volatilities are characterized in Table 1. The associated correlations are shown in Table 6.

Table 1: Equity Indices of 32 Countries with Respective Volatilities.

ARS	58%	EMB	37%	ITL	43%	PHP	50%
ATS	34%	ESP	43%	JPY	36%	PTE	36%
AUD	16%	FIM	47%	KRW	49%	SEK	44%
BEF	27%	FRF	36%	MXN	47%	SGD	41%
CAD	27%	GBP	29%	MYR	66%	THB	59%
CHF	38%	HKD	45%	NLG	39%	TWD	29%
DEM	40%	IDR	56%	NOK	38%	USD	25%
DKK	29%	IEP	31%	NZD	22%	ZAR	31%

Source: RiskMetrics, November 20, 1998. The three-letter codes are the SWIFT currency code, except for EMB, which represents J.P. Morgan's Emerging Markets Bond Index Plus.

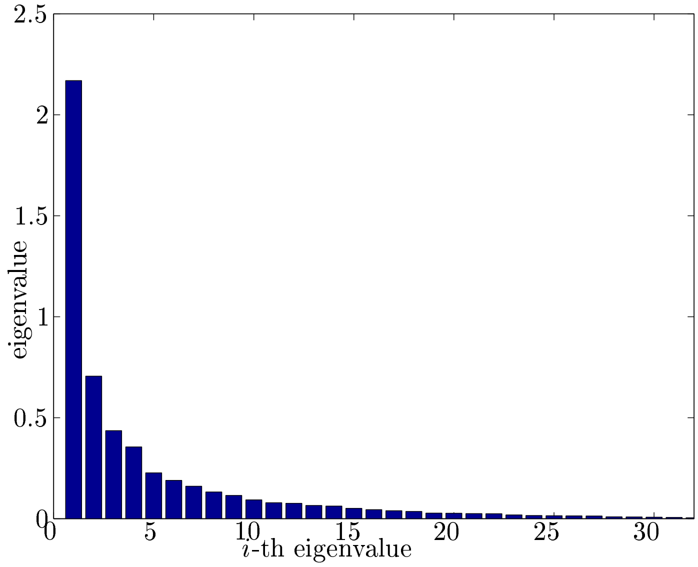

Figure 1: Eigenvalues of the variance-covariance matrix of 32 equity indices. Data Source: RiskMetrics, November, 20, 1998.

We first construct the return covariance matrix, Cov, of the 32 equity indices using RiskMetrics data. Figure 1 shows the eigenvalues of Cov, which partially demonstrates the limited degree of diversification offered by our option portfolio. Two types of returns are considered:

- *pure diffusion*, which has no jump component, $(\lambda = 0, \Sigma = \text{Cov})$.
- *jump diffusion*. The jump arrival intensity, λ, is set for our illustrative purposes at 4 per year. We let $\Sigma = \lambda V = \text{Cov}/2$, so that half of the observed variance is explained by the jump component, while the other half by the diffusion component. The expected jump amplitude in return is set at zero. That is, μ in (2.1) is set so that $\exp(\mu_j + V_{jj}/2) = 1$, for $j = 1, \ldots, 32$.

The jump-diffusion example represents an extremely high level of jump risk, not based on any empirical model, which we have chosen merely to illustrate the accuracy of our *VaR* estimates in the presence of severe jumps.

Our example options portfolio contains one at-the-money call option on each of the 32 equity indices, in equal amounts of the underlying index (measured in U.S. Dollars). We assume[7] that the mean rate of return on each index is the risk-free rate, $r = 5\%$. Over our one-day or two-week time horizons, the mean rate of return has, in any case, a negligible effect on *VaR* , for conventional parameters such as ours.

As an illustration of the accuracy of the delta-gamma approximation relative to the "true" VaR (that associated with the actual option prices), we show in Figure 2 the one-day and two-week VaRs, at varying confidence levels, for the jump-diffusion model.

Table 2: Values at Risk of Options Portfolio

		One-Day		Two Weeks	
Model	Method	0.4%	1%	0.4%	1%
Long Options Portfolio					
Pure Diffusion	Simulation, Actual	21.912	19.545	65.629	60.491
	Simulation, Delta-Gamma	21.838	19.496	60.816	56.910
	Analytical, Delta-Gamma	21.986	19.547	60.899	57.064
Jump-Diffusion	Simulation, Actual	19.760	15.515	71.948	61.288
	Simulation, Delta-Gamma	18.948	15.391	54.202	50.277
	Analytical, Delta-Gamma	18.713	15.470	54.184	50.406
Short Options Portfolio					
Pure Diffusion	Simulation, Actual	25.959	22.428	117.627	98.842
	Simulation, Delta-Gamma	25.905	22.386	116.689	97.880
	Analytical, Delta-Gamma	25.646	22.232	115.192	97.146
Jump-Diffusion	Simulation, Actual	33.080	17.171	203.903	149.415
	Simulation, Delta-Gamma	34.026	17.224	209.309	151.814
	Analytical, Delta-Gamma	35.339	17.554	207.965	148.690

The 0.4% and 1% critical values of both the long and short positions in the portfolio are calculated using three different methods, 'actual,' 'simulation-delta-gamma,' and 'analytic delta-gamma.' The results are summarized in Table 2. For both the actual and simulation-delta-gamma methods, 100,000 pseudo-independent scenarios are simulated. For the analytic delta-gamma method, we use our inversion method, with a precision set at 0.00001. For this

[7]In effect, we assume "risk-neutrality," which doesn't necessarily hold. In practice, one may wish to assign jump risk premia.

Table 3: Sample Standard Errors of Simulated Values at Risk

		One Day		Two Weeks	
Model	Method	0.4%	1%	0.4%	1%
Long Position on Options Portfolio					
Pure Diffusion	Actual	0.383	0.369	0.832	0.761
	Delta-Gamma	0.377	0.364	0.695	0.600
Jump Diffusion	Actual	2.217	0.250	1.600	0.766
	Delta-Gamma	1.169	0.198	0.668	0.365
Short Position on Options Portfolio					
Pure Diffusion	Actual	0.300	0.439	1.190	2.348
	Delta-Gamma	0.299	0.440	1.207	2.279
Jump Diffusion	Actual	6.797	0.332	6.028	5.045
	Delta-Gamma	6.638	0.328	7.570	4.866

Note: The sample standard errors are from 10 simulated subsamples of 10,000 scenarios each.

example, one calculation of a tail probability by Fourier inversion typically involves 20 to 50 function evaluations for the pure diffusion case, and 80 to 250 function evaluations for the jump-diffusion case. In order to estimate the accuracy of the simulation approach, we split the 100,000 simulations into 10 groups, with 10,000 for each group, and provide the sample standard errors of the critical values. The results are shown in Table 3. One may scale these numbers down by $\sqrt{10}$ to roughly estimate the accuracy of the critical values associated with 100,000 scenarios.

5 Folding in Credit Risk

In this section, we extend our calculations so as to obtain accurate analytical estimates of the value of risk, including the risk of changes in credit quality and of defaults, of a portfolio of securities or loans, including derivatives.

Consider a portfolio of contracts with m credit-risky counterparties. Counterparty (or borrower) i is assumed to have a random default time τ_i whose distribution is governed by an intensity process λ_i, with

$$\lambda_i(t) = \lambda_i^I(t) + p_i \lambda^C(t), \tag{5.1}$$

where p_i is a constant between 0 and 1, and where

$$d\lambda_i^I(t) = \kappa_i^I(\bar{\lambda}_i^I - \lambda_i^I(t))\,dt + \sigma_i^I \sqrt{\lambda_i^I(t)}\,dB_i^I(t)\,, \tag{5.2}$$

$$d\lambda^C(t) = \kappa^C(\bar{\lambda}^C - \lambda^C(t))\,dt + \sigma^C \sqrt{\lambda^C(t)}\,dB^C(t)\,, \tag{5.3}$$

where $(B^C, B_i^I, \ldots, B_m^I)$ is a standard Brownian in $\mathbb{R}^{m+1}$. We assume a Cox-process model for individual default times, in the sense that, conditional on the process λ_i, the default time τ_i of the i-th counterparty is the first jump time of a Poisson process with time-varying intensity $\lambda_i(t)$, for $t \geq 0$. This implies that, for any $T > 0$,

$$P(\tau_i > T) = E\left[\exp\left(-\int_0^T \lambda_i(t)\,dt\right)\right], \tag{5.4}$$

which (as we shall see) is an easy calculation for our choice of λ_i.

Correlation is induced both through the common factor λ^C in intensities and through common credit events, as follows. Conditional on *all* of the independent processes $\lambda_1^I, \ldots, \lambda_m^I, \lambda^C$, there are independent Poisson processes $N_1^I, \ldots, N_m^I, N^C$ with these time-varying deterministic intensities. Whenever N^C jumps, any counterparty i defaults with probability p_i, and the events of default of the various counterparties, at any such common event time, are conditionally independent. This means that there is the potential for more than one counterparty to default simultaneously. Intuitively, λ_t^I is the intensity of arrival of a default specific to firm i, while λ^C is the intensity of arrival of common credit events which, with some conditional probabilities, causes the default of a subset of the m firms.

For illustrative simplicity, we assume symmetry in that $p_i = p$, that $\kappa_i^I = \kappa^C = \kappa$, and that $\sigma_i^I = \sqrt{p}\sigma^C = \sigma$. A consequence of this symmetry is that, for each i, λ_i is itself a "Cox-Ingersoll-Ross" process with

$$d\lambda_i(t) = \kappa(\bar{\lambda}_i^I + p\bar{\lambda}^C - \lambda_i(t))\,dt + \sigma\sqrt{\lambda_i(t)}\,dB_i(t), \tag{5.5}$$

where B_i is a Brownian motion.[8] From (5.4) we have

$$P(\tau_i > T) = \exp\left(a_i + b_i\lambda_i(0)\right), \tag{5.6}$$

which is simply the CIR discount formula, where, letting $\bar{\lambda}_i = \bar{\lambda}_i^I + p\bar{\lambda}^C$, we have $a_i = a(T, \kappa, \bar{\lambda}_i, \sigma)$ and $b_i = b(T, \kappa, \bar{\lambda}_i, \sigma)$, with

$$\begin{aligned} b(t, \kappa, \bar{\lambda}_i, \sigma) &= -\frac{2\,(1 - \exp(-\gamma t))}{2\gamma - (\gamma - \kappa)\,(1 - \exp(-\gamma t))} \\ a(t, \kappa, \bar{\lambda}_i, \sigma) &= -\frac{\kappa\bar{\lambda}_i}{\sigma^2}\left((\gamma - \kappa)t + 2\ln\left(1 - \frac{\gamma - \kappa}{2\gamma}(1 - \exp(-\gamma t))\right)\right), \end{aligned} \tag{5.7}$$

and where $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$.

We further assume that if i defaults, then its portfolio loses some fraction L_i of what its market value would have been given survival to T, this market value being denoted $V_i(R_T, \lambda_T)$. This random fractional loss L_i (assumed independent of all else) is assumed to have a characteristic function θ_i, defined by $\theta_i(u) = E(\exp(-iuL_i))$. The time-$T$ value W_T of the entire portfolio is

$$W_T = \sum_{i=1}^m V_i(R_T, \lambda_T) - \sum_{i=1}^m L_i D_{iT}\, V_i(R_T, \lambda_T), \tag{5.8}$$

where D_{iT} is the indicator (1 or 0) of the default event $\{\tau_i < T\}$.

[8] For a proof, and an extension of this idea to include jumps in intensities, see Duffie and Garleanu [1999].

For the purpose of analytical VaR calculations, we approximate by

$$W_T \approx \sum_{i=1}^{m} V_i(R_T, \lambda_T) - \sum_{i=1}^{m} L_i D_{iT} V_i(R_0, \lambda_0), \tag{5.9}$$

neglecting the change in default-free value between 0 and T when computing the default loss. This approximation allows us to disentangle the two components — the total market value assuming survival to the VaR horizon,

$$V(R_T, \lambda_T) = \sum_{i=1}^{m} V_i(R_T, \lambda_T),$$

and the total loss from defaults,

$$H(T) = \sum_{i=1}^{m} L_i D_{iT} \, V_i(R_0, \lambda_0).$$

We are interested in the tail distribution of W_T, which can be obtained using the Fourier-inversion method laid out in Section 3, once the characteristic function ψ of W_T, defined by $\psi(u) = E[\exp(iuW_T)]$, is known. Under the approximation (5.9), we further ignore the dependence between $D_i(T)$ and $\lambda(T)$, which is in any case of small order over small time horizons,[9] and write $\psi(u) \approx \psi^V(u)\, \psi^D(u)$, as a product of Fourier transforms ψ^V of the "value" component $V(R_T, \lambda_T)$, and, ψ^D of the "default" component $H(T)$. We now treat these two components separately.

Value Component

For this component, we make the usual "delta-gamma" approximation by Taylor expanding $V(R_T, \lambda_T)$ in terms of R_T and λ_T, around initial (time-0) values, up to second order (that is, by the "delta-gamma" approximation). For this purpose, we let $Y_T = [R_T, \lambda_T]^\top$, and write

$$V(Y_T) \approx V^{\Delta,\Gamma}(Y_T) = A + B \cdot Y_T + \frac{1}{2} Y_T^\top \Gamma Y_T, \tag{5.10}$$

where

$$A = V(Y_0) - \Delta \cdot Y_0 + \frac{1}{2} Y_0^\top \Gamma Y_0\,, \quad B = \Delta - \Gamma\, Y_0,$$

and where Δ and Γ are the first (gradient) and second (Hessian) derivatives of $V(\,\cdot\,)$ evaluated at Y_0.

[9] For one counterparty, the covariance between D_T and λ_T is

$$\mathrm{cov}(D_T, \lambda_T) = E\left(\lambda_T\right)\, E\left(\exp\left(-\int_0^T \lambda_s\, ds\right)\right) - E\left(\lambda_T \exp\left(-\int_0^T \lambda_s\, ds\right)\right).$$

For our base-case parameters, $\mathrm{corr}(D_T, \lambda_T)$ is 0.00024 over a one-day horizon, and 0.0033 over a two-week horizon.

Under a Gaussian approximation of Y_T, with mean vector M and variance-covariance matrix Ω, the Fourier transform of the value component becomes a straightforward application of (3.4). For simplicity, we assume that λ and R are independent, so that

$$M = \begin{pmatrix} M^R \\ M^\lambda \end{pmatrix} \ ; \quad \Omega = \begin{pmatrix} \Omega^R & 0_{n\times m} \\ 0_{m\times n} & \Omega^\lambda \end{pmatrix}, \tag{5.11}$$

where M^R and Ω^R are the mean vector and variance-covariance matrix, respectively, of the n market risk factors, and where

$$(M^\lambda)_i \equiv E_0(\lambda_i(T)) = \lambda_i(0) + (1-\exp(-\kappa T))(\bar{\lambda}_i^I + p\bar{\lambda}^C - \lambda_i(0))$$

$$(\Omega^\lambda)_{i,i} \equiv \mathrm{var}_0(\lambda_i(T)) = \exp(-\kappa T)\frac{1-\exp(-\kappa T)}{\kappa}\sigma^2\lambda_i(0) + \frac{(1-\exp(-\kappa T))^2}{2\kappa}\sigma^2(\bar{\lambda}_i^I + p\bar{\lambda}^C)$$

$$(\Omega^\lambda)_{i,j} \equiv \mathrm{cov}_0(\lambda_i(T), \lambda_j(T)) = p\sigma^2\left(\exp^{-\kappa T}\frac{1-\exp(-\kappa T)}{\kappa}\lambda^C(0) + \frac{(1-\exp(-\kappa T))^2}{2\kappa}\bar{\lambda}^C\right).$$

We note that only the case in which R and λ are diffusions is considered here. An extension to include jumps in both R and λ can be readily incorporated, as in Section 3, by conditioning on the number of jumps. Specifically, we may allow the same structure of jumps in the market risk factors R as before, and introduce exponential jumps in the credit risk factors λ. Assuming the independence of the jump arrivals and amplitudes for R and λ, the construction of $M_{j,k}$ and $\Omega_{j,k}$, given j jumps in R and k jumps in λ is a straightforward exercise. Allowing for correlation between R and λ is also straightforward.

Default Component

We next focus on the default component $H(T)$, taking the approximation:

$$D_T^i \approx D_T^{iI} + \xi^i D_T^C, \tag{5.12}$$

where D_T^{iI} is the indicator of the event that the i-specific event-counting process N_i^I has jumped by T, D_T^C is the indicator for the event that the common credit-event counting process N^C has jumped by T, and ξ^i is the indicator of the event that i defaults at the first common credit event. Here, we ignore the double-counting of defaults that occurs from both common and idiosyncratic credit events. (The loss from a given counterparty default can in fact occur only once.) The probability of the double-counting event is on the order of $\lambda_i(0)^2T^2$, and we find by Monte Carlo tests that it has a negligible impact for our examples. The use of (5.12) also "under-counts" defaults associated with multiple common credit events before the *VaR* horizon T; this error is of order $\lambda_c(0)^2T^2$. These under and over counting errors are to some extent offsetting.

We have already defined $P(\xi^i = 1) = p$, and we have

$$\begin{aligned} p_i^I(T) &= P(D_T^{iI} = 1) = 1 - E\left[\exp\left(-\int_0^T \lambda_i^I(t)\,dt\right)\right] \\ p^C(T) &= P(D_T^C = 1) = 1 - E\left[\exp\left(-\int_0^T \lambda^C(t)\,dt\right)\right]. \end{aligned}$$

As with the calculation of $P(\tau_i > T)$ in (5.6), $p_i^I(T)$ and $p^C(T)$ can be calculated from the CIR discount formula.

The Fourier transform of the default component is thus approximated as

$$\begin{aligned}\psi^D(u) &= E\left(e^{-iuH(T)}\right)\\ &\approx E\left(e^{-iuH(T)} \mid D_T^C = 1\right) p^C(T) + E\left(e^{-iuH(T)} \mid D_T^C = 0\right)\left(1 - p^C(T)\right)\\ &\approx \left(1 - p^C(T) + p^C(T)\prod_{i=1}^{m}\left(1 - p + p\,\theta_i(uV_i(\lambda_0, R_0))\right)\right)\prod_{i=1}^{m} k_i,\end{aligned}$$

where

$$k_i = 1 - p_i^I(T) + p_i^I(T)\,\theta_i(uV_i(\lambda_0, R_0)),$$

recalling that $\theta_i(u) = E(\exp(-iuL_i))$.

With our analytic approximation of the Fourier transform of W_T, we can now calculate tail distribution of W_T by Fourier inversion. The error-control technique is similar to that established in Section 3, controlling for both discretization and truncation error. For our small time horizon T, and for the cases to be examined in the next section, we find that our approximations are more than adequate, as indicated by comparison with simulation-based results. (See Tables 4 and 5.)

6 Example: Credit-Risk Exposures

As examples, we now consider the *VaR* of a loan portfolio and of an OTC option portfolio, with credit risk. As we shall see, the tail *VaR*s are dominated by credit risk by the loan portfolios, and by equity index risk for the options portfolios.

6.1 A Loan Portfolio with Credit Risk

Consider an illustrative estimate of the value at risk at the time horizon T for a portfolio of m loans contracted to m respective counterparties, each with maturity $T_i > T$ and principal F_i. Given that firm i survives to the *VaR* horizon T, the market value of its loan is then

$$V_i(R_T, \lambda_T) = F_i\, E_T^*\left[\exp\left(-\int_T^{T_i}\left(r(t) + \overline{L}\lambda_i^*(t)\right)\,dt\right)\right], \tag{6.1}$$

where E_T^* denotes risk-neutral expectation given information at time T, λ_i^* is is the risk-neutral default intensity process for counterparty i, $\overline{L}$ is the risk-neutral mean of the fractional loss L_i of market value at default, and r is the default-free short rate process. (For this calculation, and supporting technical conditions, see Duffie and Singleton [1999].) We will take $r(t) = \bar{r} + R_1(t) + R_2(t)$ where $\bar{r}$ is a constant and where R_1 and R_2 are independent CIR processes. For simplicity in exposition, we assume no risk premia so that the distribution of $((\lambda_1, \tau_1), \ldots, (\lambda_m, \tau_m), R)$ coincides with its risk-neutral distribution. Similar to the derivation of (5.4), we then have

$$V_i(R_T, \lambda_T) = F_i\,\exp\left[-\bar{r}(T_i - T) + a + b_1 R_i(T) + b_2 R_2(T) + c\lambda_i(T)\right], \tag{6.2}$$

where the coefficients a, b_1, b_2, and c are determined by Riccati equations, as usual for a multi-factor CIR setting. In this example, we fix the model parameters for the risk-free short rate to those reported in Duffie and Singleton [1997], and set the initial levels of R_1 and R_2 to their respective long-run means. The parameters for each default intensity λ_i are set for illustration at $\kappa = 0.25$, $\bar{\lambda} = \bar{\lambda}^I + p\bar{\lambda}^C = 0.03$, and a volatility parameter σ set so that the initial instantaneous volatility of intensity is 100%. Default intensities are initiated at their long-run means by letting $\lambda_i^I(0) = \bar{\lambda}^I$ and $\lambda^C(0) = \bar{\lambda}^C$. Keeping κ, $\bar{\lambda}$, and σ fixed, and assuming a constant fractional loss of $L_i = \overline{L} = 50\%$ at default, two cases with different degrees of correlation in counterparty credit risk are considered:

High Correlation: $p = 0.8$ and $\bar{\lambda}^I/\bar{\lambda} = 20\%$.

Low Correlation: $p = 0.2$ and $\bar{\lambda}^I/\bar{\lambda} = 80\%$.

The distribution of individual default times is identical for these two cases.

The 320 borrowers have 1-year loans of equal principal. The percentage value-at-risk, as a fraction of the initial market value of the loan portfolio, is shown in Table 4. As opposed to the case of the options portfolio considered later, there is a substantive contribution of credit risk, if one examines sufficiently far out into the tail of the distribution. For example, for both high and low correlation cases, the percentage value-at-risk increases significantly from the 0.2% level to the 0.1% level. This difference is due to the likelihood of a market-wide credit event before the *VaR* horizon T, which, at our parameters with $\lambda^C = 0.03$, is 0.12%, indeed lying between 0.1% and 0.2%.

Table 4: Total 2-Week Value-at-Risk for A Loan Portfolio

	High Correlation		Low Correlation		No Credit Risk	
Prob(%)	Analytical	Simulation	Analytical	Simulation	Analytical	Simulation
0.1	38.53	38.49 (0.18)	8.49	8.58 (0.16)	0.64	0.65 (0.00)
0.2	0.87	0.87 (0.01)	0.77	0.77 (0.01)	0.59	0.59 (0.00)
0.3	0.79	0.78 (0.00)	0.70	0.70 (0.00)	0.55	0.55 (0.00)
0.4	0.74	0.74 (0.00)	0.65	0.65 (0.00)	0.52	0.52 (0.00)
0.5	0.70	0.70 (0.00)	0.62	0.62 (0.00)	0.50	0.50 (0.00)
0.6	0.67	0.67 (0.00)	0.60	0.59 (0.00)	0.49	0.49 (0.00)
0.7	0.65	0.65 (0.00)	0.58	0.57 (0.00)	0.47	0.47 (0.00)
0.8	0.63	0.63 (0.00)	0.56	0.56 (0.00)	0.46	0.46 (0.00)
0.9	0.61	0.61 (0.00)	0.54	0.54 (0.00)	0.45	0.45 (0.00)
1.0	0.60	0.60 (0.00)	0.53	0.53 (0.00)	0.44	0.44 (0.00)
2.0	0.49	0.49 (0.00)	0.43	0.43 (0.00)	0.36	0.36 (0.00)
3.0	0.43	0.43 (0.00)	0.38	0.38 (0.00)	0.31	0.31 (0.00)
4.0	0.38	0.38 (0.00)	0.34	0.33 (0.00)	0.28	0.28 (0.00)
5.0	0.34	0.34 (0.00)	0.30	0.30 (0.00)	0.25	0.25 (0.00)

The sample standard errors (in parentheses) are calculated using 10 simulated subsamples of 500,000 scenarios each. The VAR estimate based on simulation is the sample mean of the 10 sub-sample estimates.

Figure 3 shows the impact on the *VaR* of increasing the number m of borrowers. We also compare the *VaR* with that of a parallel model with no common credit events. Specifically, in the parallel model, there is a common factor $p\lambda^C$ of intensities as before, but conditional on the intensities $\lambda_1^I, \ldots, \lambda_m^I, \lambda^C$, all default times are independent. The intensity processes for this parallel model have the same joint distribution (same individual default risk model, and same correlation among default intensities) as for our original model, but there are no common default-generating events.

From case (a) of Figure 3, we also see that the market-wide credit event hits all three portfolios with a probability of approximately 0.2%. Again, one can show that the arrival intensity of such a market-wide credit event is $\lambda^C = 0.06$, which corresponds to an event with probability 0.23% over the two-week horizon. It is also interesting to see the extra structure associated with the *VaR* of the portfolio with 32 counterparties. This is evident from the "enlargement" shown in Figure 4, from which we see small "platforms" corresponding to the number of individual firms hit by a credit event. These are visible only in the case of $N = 32$ borrowers, whose diversification benefit is limited relative to the case of $N = 320$ or $N = 1600$.

Figure 5 shows the effect of increasing the volatility of interest rates, at both the 2-week and 16-week *VaR* time horizons. Figure 6 shows the effect of increasing the correlation of default risk, in two ways: (a) through the likelihood p of default by a given borrower at a common credit event, and (b) through correlation in the intensity processes. In both cases, the individual credit quality is held constant. Overall, one may see that the presence of joint credit events may significantly increase *VaR*, holding fixed the joint distribution of default intensity processes.

6.2 An Options Portfolio with Credit Risk

Consider a portfolio of options on 32 respective equity indices, with credit exposure to 320 firms. Specifically, there is a total of 320 options, all of which are at-the-money, European-style, and with a time to exercise of $T_i = 1$ year. Each of the 320 option contracts is associated with a particular equity index. Given no default by the corresponding counterparty up to time T, the market value of the option at that time is

$$V_i(R_T, \lambda_T) = C_i(R_T) E_T^* \left[\exp\left(-\overline{L} \int_T^{T_i} \lambda_i(t)\, dt \right) \right], \tag{6.3}$$

where $\overline{L} = E^*(L_i)$ is the risk-neutral mean fractional loss of market value at default. The first component, $C_i(R_T)$, is the usual option pricing formula in the absence of credit risk. With our particular specification for the default intensity λ, the second component is of an exponential-affine form, which can be computed explicitly along the lines of (5.6).

For each counterparty i, the default intensity λ_i is characterized by a mean-reversion parameter $\kappa = 0.25$, a long-run mean $\bar\lambda = \bar\lambda^I + p\bar\lambda^C = 0.03$, and a volatility parameter σ that is set so that the initial instantaneous volatility is 100%. Again, we assume no risk premia throughout. Each intensity is initiated at its long-run mean by letting $\lambda_i^I(0) = \bar\lambda^I$ and $\lambda^C(0) = \bar\lambda^C$. Keeping κ, $\bar\lambda$, and σ fixed, and assuming a constant mean fractional default loss of $\overline{L} = 50\%$, two cases with different degrees of correlation are considered:

High Correlation: $p = 0.8$ and $\bar\lambda^I / \bar\lambda = 20\%$.

Low Correlation: $p = 0.2$ and $\bar{\lambda}^I / \bar{\lambda} = 80\%$.

Table 5: Total 2-Week Value-at-Risk for An Options Portfolio

	High Correlation		Low Correlation		No Credit Risk	
Prob(%)	Analytical	Simulation	Analytical	Simulation	Analytical	Simulation
0.1	45.88	45.58 (0.14)	44.38	44.33 (0.11)	44.32	44.28 (0.13)
0.2	43.05	42.98 (0.11)	42.19	42.17 (0.09)	42.14	42.08 (0.13)
0.3	41.42	41.37 (0.09)	40.79	40.78 (0.06)	40.74	40.69 (0.12)
0.4	40.24	40.21 (0.07)	39.73	39.70 (0.06)	39.69	39.64 (0.13)
0.5	39.30	39.30 (0.06)	38.88	38.84 (0.06)	38.84	38.80 (0.12)
0.6	38.52	38.51 (0.04)	38.15	38.13 (0.07)	38.11	38.10 (0.10)
0.7	37.84	37.84 (0.05)	37.51	37.49 (0.07)	37.48	37.46 (0.10)
0.8	37.24	37.23 (0.07)	36.95	36.93 (0.05)	36.91	36.90 (0.11)
0.9	36.71	36.67 (0.06)	36.44	36.42 (0.05)	36.40	36.39 (0.10)
1.0	36.22	36.18 (0.07)	35.97	35.95 (0.05)	35.94	35.93 (0.10)
2.0	32.75	32.76 (0.05)	32.61	32.59 (0.03)	32.58	32.57 (0.08)
3.0	30.48	30.49 (0.05)	30.38	30.37 (0.04)	30.35	30.34 (0.06)
4.0	28.72	28.74 (0.05)	28.65	28.64 (0.03)	28.63	28.62 (0.06)
5.0	27.27	27.29 (0.05)	27.21	27.20 (0.03)	27.19	27.19 (0.05)

The sample standard errors (in parentheses) are calculated using 10 simulated sub-samples of 500,000 independent scenarios each. The reported estimate of VAR based on simulation is the sample mean of the 10 subsamples.

The percentage value at risk is shown for various cases in Table 5. Figure 7 compares the sensitivity of the risk of the option portfolio to increasing the mean default intensity of counterparties to the analogous effect on the loan portfolio of the previous example. Clearly, at the 2-week *VaR* horizon, the risk of the option portfolio due to variation of the underlying equity indices dominates the effect of credit risk. Even at the 16-week *VaR* horizon, as shown by Figure 8, the relative contribution of credit risk to the *VaR* of the the option portfolio is small.

Appendix

Table 6: Correlations of Equity Indices of 32 Countries.

	ARS	ATS	AUD	BEF	CAD	CHF	DEM	DKK	EMB	ESP	FIM	FRF	GBP	HKD	IDR	IEP
ARS	1.0															
ATS	0.4	1.0														
AUD	0.2	0.4	1.0													
BEF	0.4	0.6	0.4	1.0												
CAD	0.6	0.4	0.2	0.5	1.0											
CHF	0.4	0.6	0.3	0.8	0.6	1.0										
DEM	0.4	0.7	0.4	0.8	0.5	0.8	1.0									
DKK	0.3	0.6	0.4	0.6	0.4	0.6	0.7	1.0								
EMB	0.4	0.1	0.2	0.3	0.2	0.2	0.1	0.1	1.0							
ESP	0.5	0.7	0.3	0.7	0.6	0.8	0.8	0.7	0.3	1.0						
FIM	0.4	0.7	0.4	0.7	0.5	0.8	0.8	0.7	0.2	0.7	1.0					
FRF	0.5	0.7	0.3	0.8	0.6	0.9	0.8	0.7	0.2	0.8	0.8	1.0				
GBP	0.4	0.7	0.4	0.8	0.5	0.9	0.8	0.7	0.2	0.7	0.8	0.9	1.0			
HKD	0.2	0.4	0.5	0.4	0.3	0.4	0.4	0.4	−0.0	0.4	0.5	0.4	0.5	1.0		
IDR	0.1	0.2	0.3	0.2	0.1	0.1	0.2	0.1	0.2	0.3	0.2	0.2	0.2	0.4	1.0	
IEP	0.3	0.6	0.5	0.7	0.5	0.7	0.7	0.6	0.2	0.6	0.8	0.7	0.7	0.4	0.2	1.0
ITL	0.4	0.6	0.3	0.7	0.5	0.8	0.8	0.7	0.2	0.8	0.8	0.9	0.8	0.4	0.2	0.7
JPY	0.2	0.4	0.5	0.4	0.2	0.4	0.4	0.4	0.2	0.3	0.5	0.4	0.5	0.3	0.2	0.5
KRW	0.2	0.3	0.3	0.4	0.3	0.3	0.4	0.2	0.2	0.3	0.4	0.4	0.4	0.4	0.3	0.4
MXN	0.7	0.3	0.2	0.4	0.5	0.4	0.4	0.3	0.2	0.4	0.4	0.3	0.3	0.2	-0.1	0.3
MYR	0.1	0.3	0.4	0.3	0.0	0.3	0.2	0.2	0.2	0.2	0.3	0.2	0.3	0.3	0.2	0.2
NLG	0.4	0.7	0.4	0.9	0.5	0.9	0.9	0.7	0.3	0.8	0.8	0.8	0.9	0.4	0.3	0.7
NOK	0.4	0.7	0.5	0.6	0.4	0.6	0.7	0.8	0.0	0.6	0.8	0.6	0.7	0.5	0.1	0.7
NZD	0.2	0.3	0.4	0.4	0.2	0.2	0.3	0.3	0.5	0.3	0.3	0.2	0.3	0.2	0.3	0.4
PHP	0.2	0.1	0.5	0.1	0.2	0.1	0.2	0.1	0.2	0.2	0.2	0.1	0.2	0.5	0.3	0.2
PTE	0.4	0.7	0.3	0.8	0.5	0.8	0.7	0.6	0.2	0.8	0.6	0.8	0.6	0.3	0.2	0.6
SEK	0.4	0.6	0.3	0.7	0.5	0.8	0.8	0.7	0.1	0.7	0.8	0.8	0.8	0.4	0.1	0.7
SGD	0.3	0.4	0.5	0.3	0.3	0.3	0.4	0.3	0.2	0.4	0.4	0.3	0.4	0.6	0.5	0.4
THB	0.1	0.2	0.5	0.2	0.2	0.2	0.2	0.1	0.2	0.2	0.2	0.1	0.3	0.5	0.6	0.2
TWD	0.0	0.2	0.1	0.1	0.1	0.1	0.2	-0.0	0.2	0.1	0.1	0.0	0.1	0.1	0.2	0.1
USD	0.7	0.3	0.1	0.4	0.8	0.5	0.4	0.3	0.2	0.5	0.5	0.5	0.4	0.2	0.0	0.4
ZAR	0.5	0.7	0.5	0.7	0.5	0.6	0.7	0.6	0.3	0.7	0.7	0.6	0.7	0.5	0.3	0.6

	ITL	JPY	KRW	MXN	MYR	NLG	NOK	NZD	PHP	PTE	SEK	SGD	THB	TWD	USD	ZAR
ITL	1.0															
JPY	0.5	1.0														
KRW	0.3	0.4	1.0													
MXN	0.3	0.2	0.1	1.0												
MYR	0.3	0.3	0.3	0.1	1.0											
NLG	0.8	0.4	0.4	0.3	0.3	1.0										
NOK	0.7	0.4	0.2	0.3	0.2	0.7	1.0									
NZD	0.2	0.2	0.3	0.2	0.3	0.4	0.3	1.0								
PHP	0.1	0.1	0.3	0.2	0.0	0.2	0.2	0.3	1.0							
PTE	0.7	0.4	0.4	0.4	0.2	0.7	0.5	0.4	0.2	1.0						
SEK	0.8	0.4	0.4	0.4	0.2	0.8	0.7	0.2	0.1	0.6	1.0					
SGD	0.3	0.4	0.3	0.2	0.2	0.4	0.4	0.3	0.5	0.4	0.3	1.0				
THB	0.1	0.2	0.4	0.1	0.3	0.3	0.1	0.2	0.5	0.2	0.1	0.7	1.0			
TWD	0.0	-0.0	0.3	-0.1	0.1	0.1	0.0	0.4	0.3	0.2	-0.1	0.2	0.1	1.0		
USD	0.4	0.1	0.3	0.6	-0.1	0.4	0.3	0.1	0.2	0.4	0.4	0.2	0.1	0.1	1.0	
ZAR	0.7	0.5	0.4	0.4	0.3	0.7	0.7	0.4	0.3	0.7	0.6	0.5	0.4	0.1	0.4	1.0

Source: RiskMetrics, November 20, 1998.

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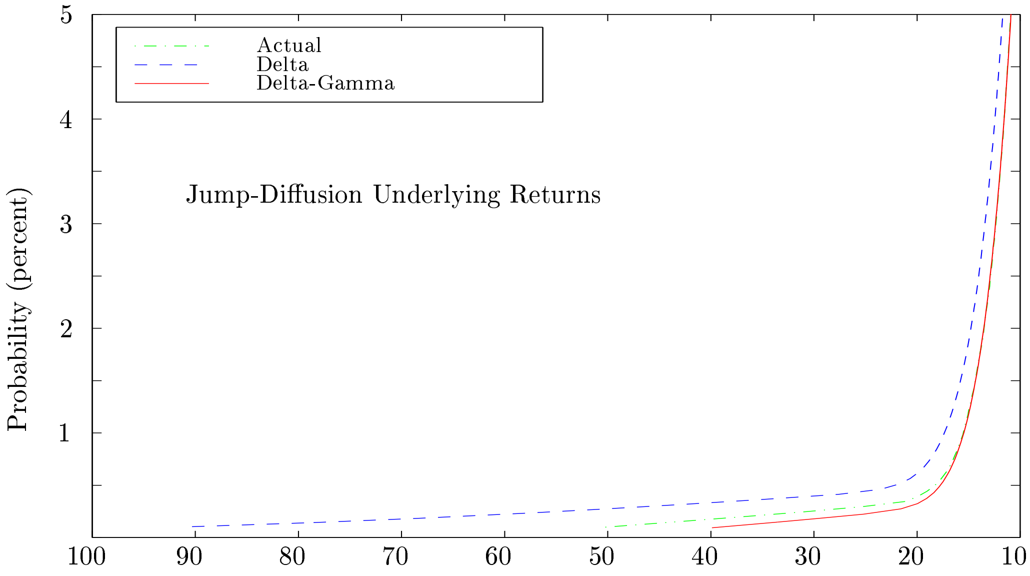
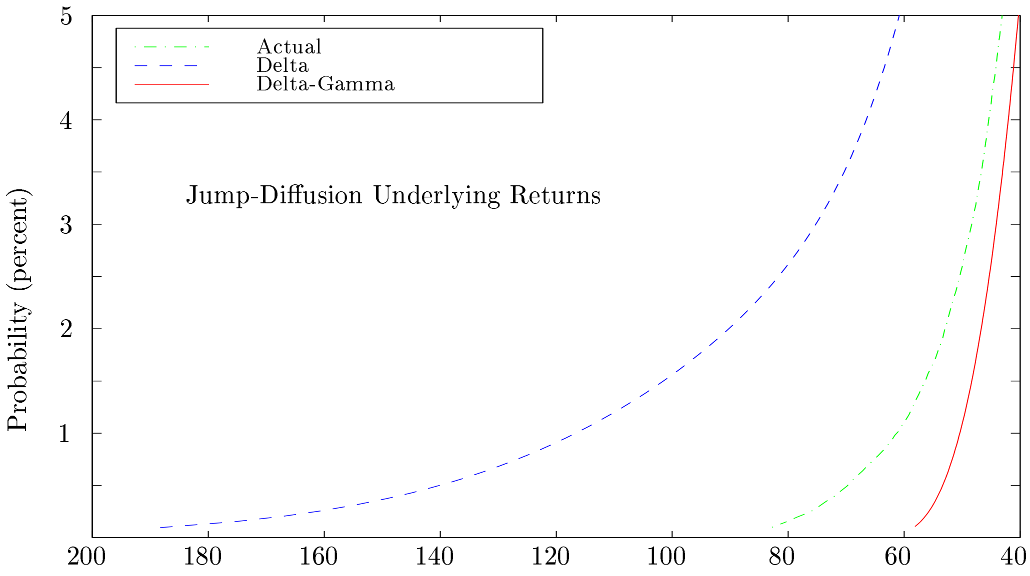
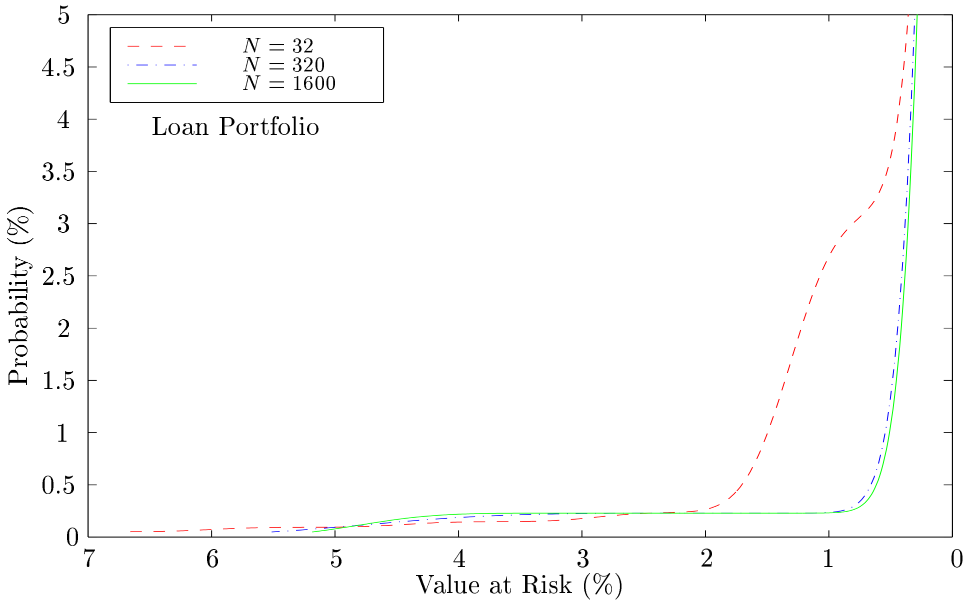


(a) One-Day Value at Risk

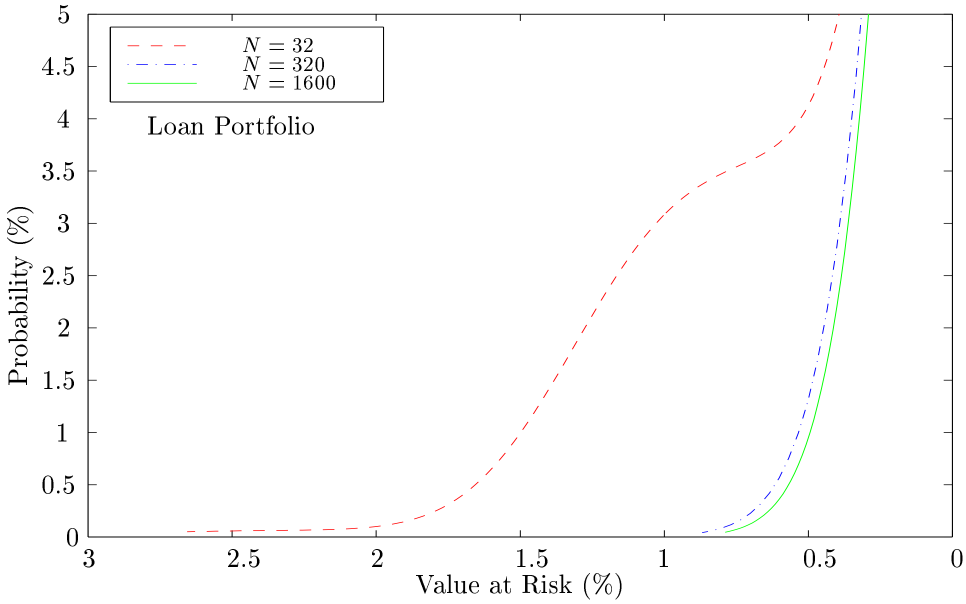


(b) Two-Week Value at Risk

Figure 2: Values at Risk, as a percentage of the market value of a portfolio of call options on 32 equity indices, with a multi-variate jump-diffusion return process, and an expected inter-jump time of one quarter. Half of the total return covariance is contributed by the jump component.



(a) With Common Default Events



(b) Without Common Default Events

Figure 3: Two-week *VaR* as a percentage of initial market value, varying N, for a portfolio of N 1-year loans. Default intensities are CIR with $\bar{\lambda} = 3\%$, $\kappa = 0.25$, and initial volatility of 100%. Intensities are correlated, with 20% contributed by a common credit intensity. For case (a), at a common credit event, each firm defaults with probability 10%. For case (b), there are no common credit events, but intensities have the same joint distribution as case (a).

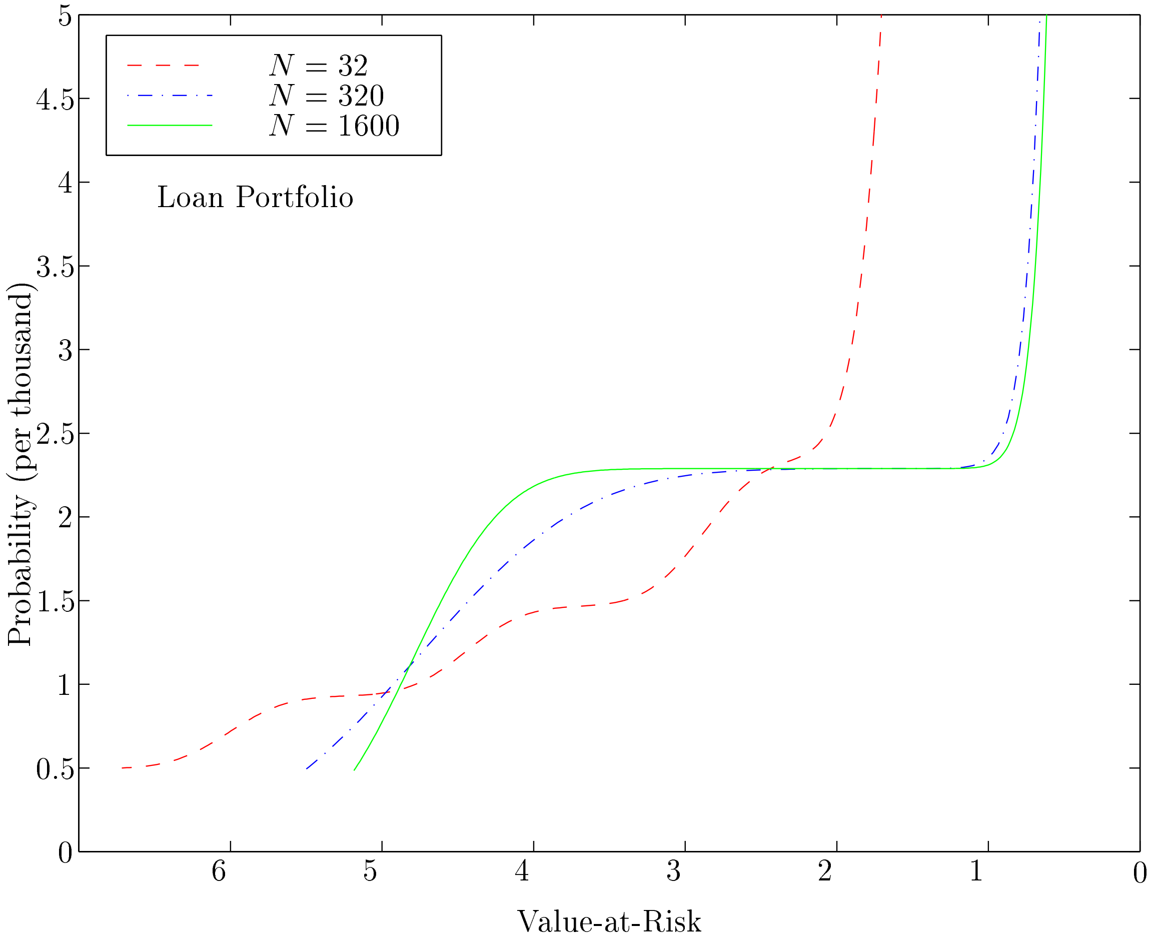
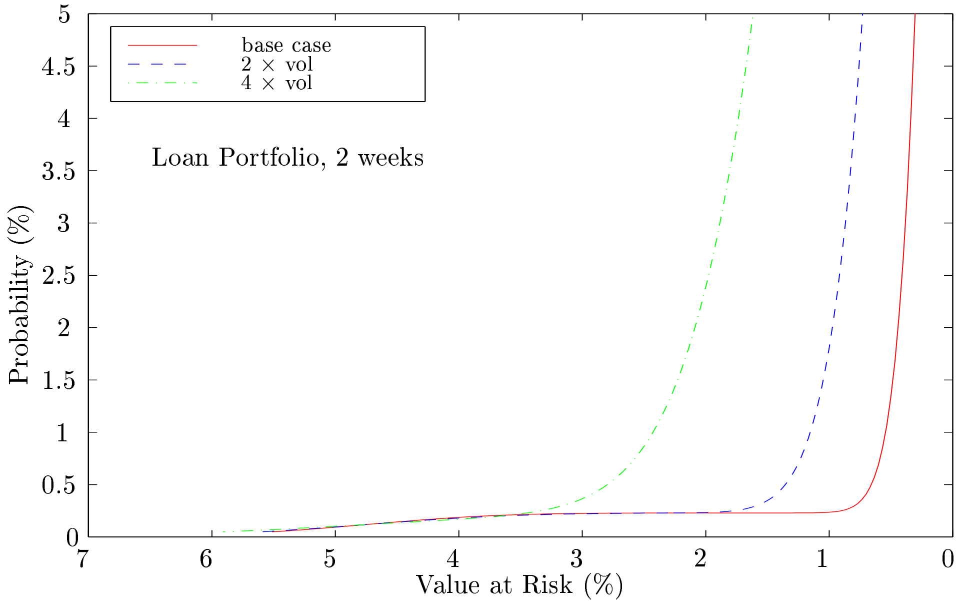
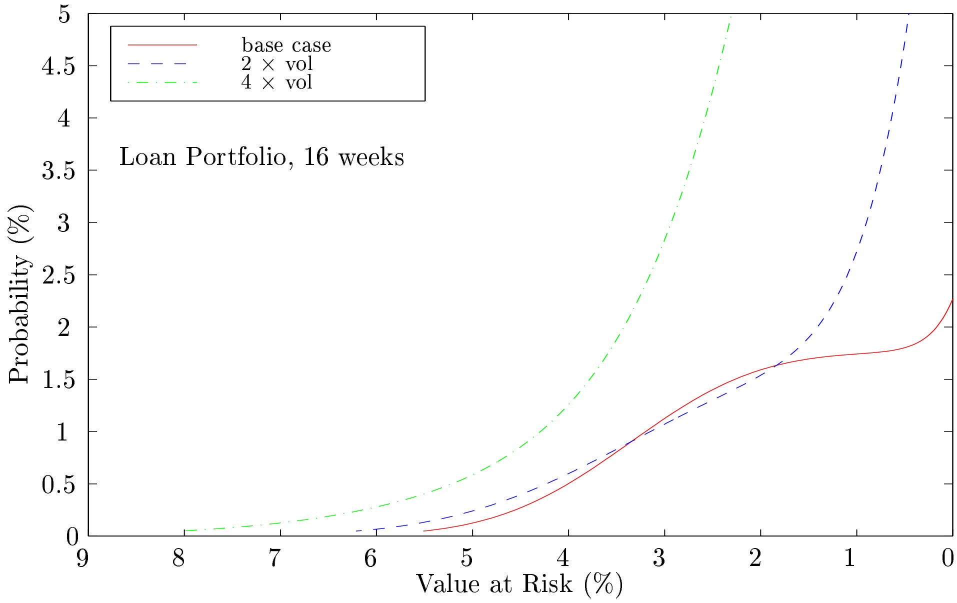


Figure 4: Two-week value-at-risk with varying number of total firms. The loan portfolio is of maturity 1 year, with face value \$100. There are 320 counterparts, whose default intensity are modeled by CIR with long-run mean of 300*bp*, mean-reversion of 0.25, and volatility of 100%. The default intensities are correlated, with 20% coming from a common market-wise credit event, and given its occurrence, each firm defaults with probatility 10%. The default-free interest rate risk is model by a two-factor CIR.

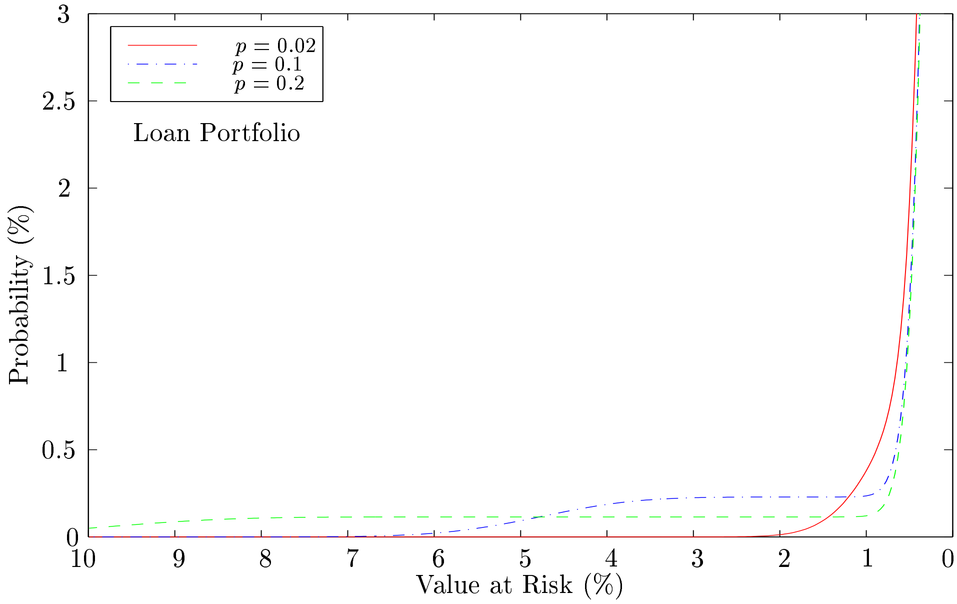


(a) 2-Week Value at Risk of Loan Portfolio

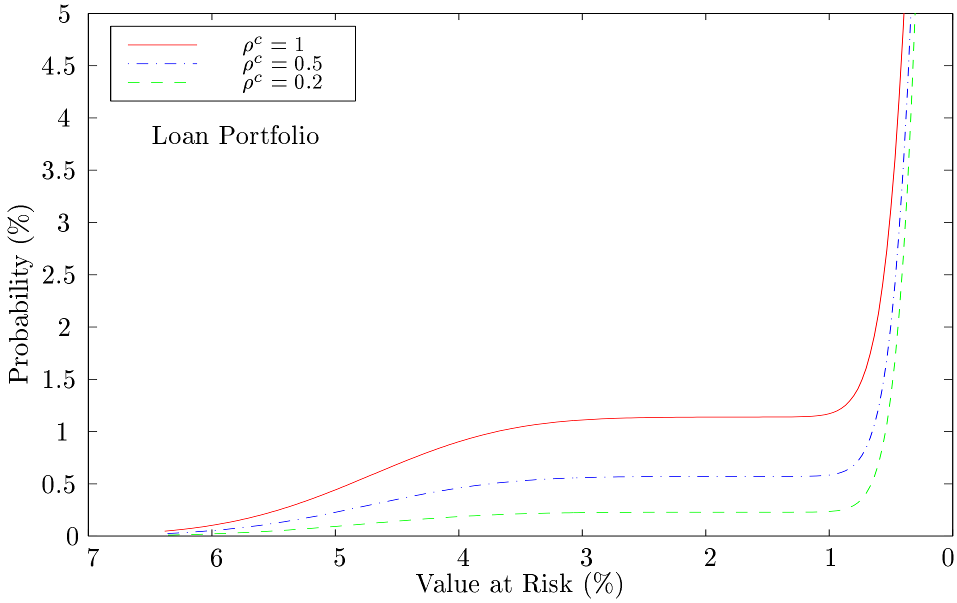


(b) 16-Week Value at Risk of Loan Portfolio

Figure 5: Varying the volatility of the default-free short rate, the percentage *VaR* for a portfolio of 320 loans. Default intensities are CIR with long-run mean $\bar{\lambda} = 3\%$, mean-reversion rate $\kappa = 0.25$, and initial volatility of 100%. Intensities are 20% common. At a common credit event, each borrower defaults with 10% probability.

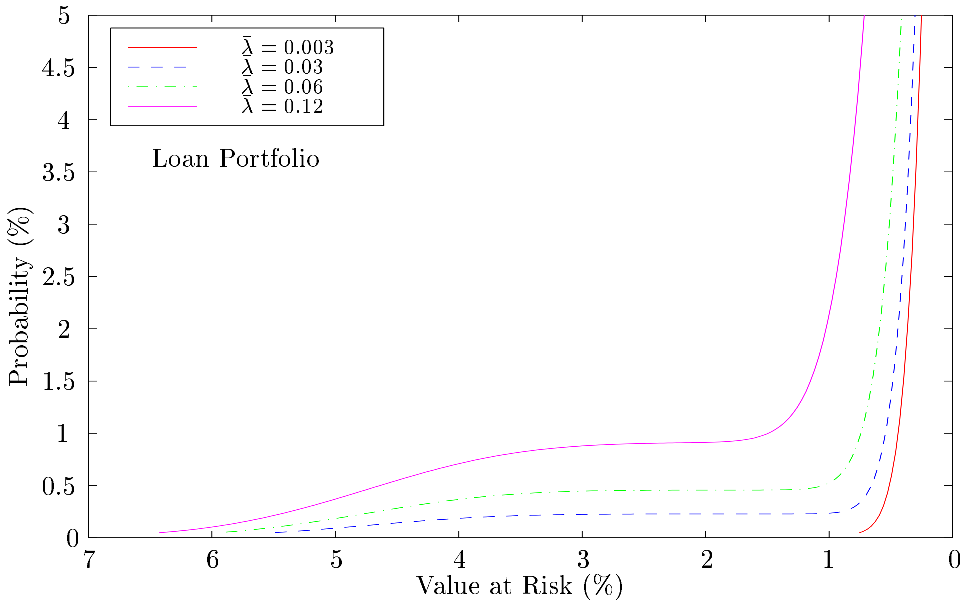


(a) Varying the Probability of Default at Common Events

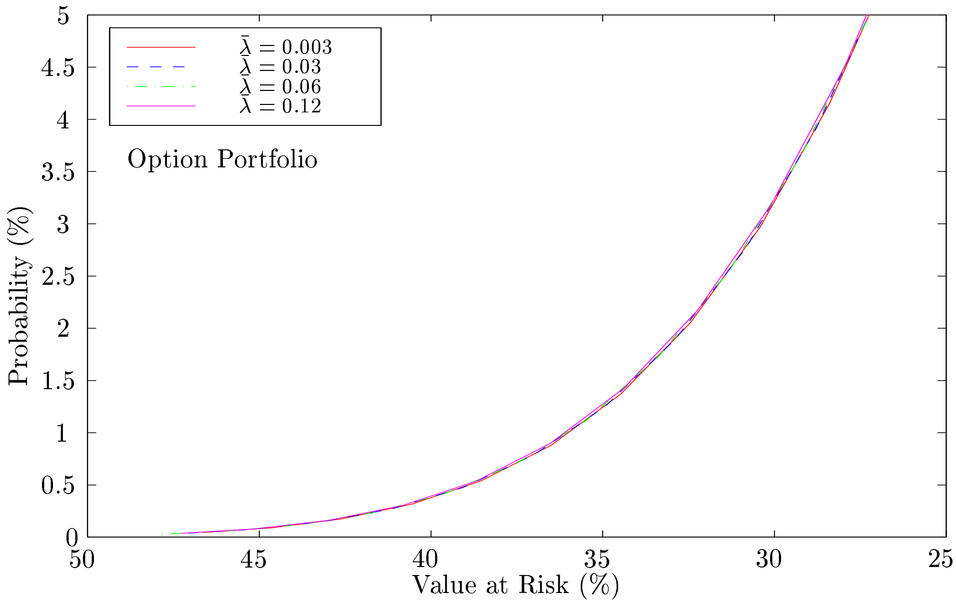


(b) Varying the Correlation of Default Intensities

Figure 6: Two-week *VaR* as a percentage of initial market value, varying (a) the probability p of default conditional on a common credit event, and (b) the fraction ρ^c of each borrower's default intensity that is common, for a portfolio of 320 1-year loans. For case (a), ρ^c is fixed at 20%, while, for case (b), p is fixed at 10%. Default intensities are CIR with $\bar{\lambda} = 3\%$, $\kappa = 0.25$, and an initial volatility of 100%.



(a) Value at Risk of Loan Portfolio



(b) Value at Risk of Option Portfolio

Figure 7: Two-week *VaR* as a percentage of initial market value, varying the long-run mean default intensity $\bar{\lambda}$, for a portfolio of 320 loans, and for a portfolio of 320 at-the-money options on 32 equity indices (covariances from RiskMetrics Nov. 20, 1998). Default intensities are CIR, initiated at $\bar{\lambda}$, with mean-reversion 0.25, and initial volatility of 100%. 20% of intensities are common. At a common credit event, each counterparty defaults with probability 10%.

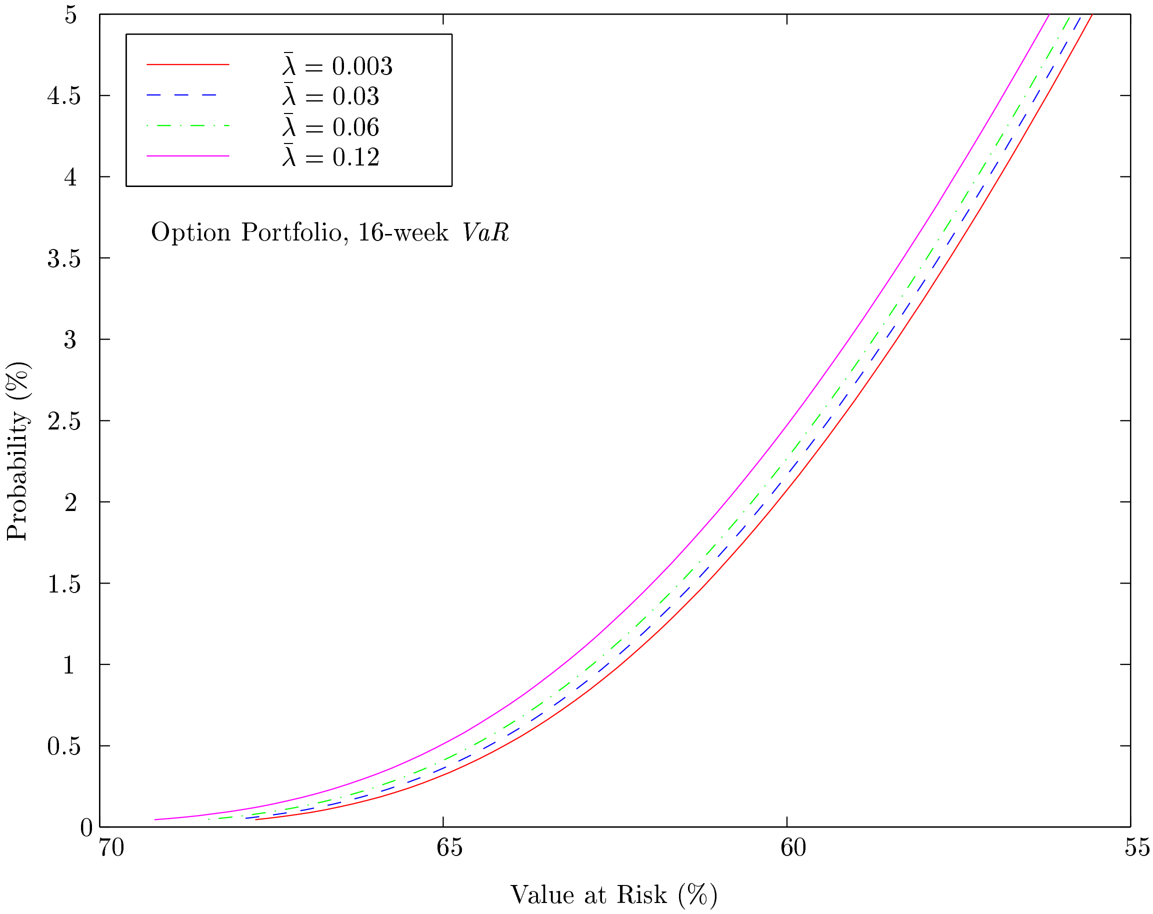


Figure 8: *VaR* as a percentage of initial market value, varying the long-run mean default intensity $\bar{\lambda}$, for a portfolio of 320 at-the-money options on 32 equity indices (covariances from RiskMetrics.) Default intensities are CIR, initiated at $\bar{\lambda}$, with mean-reversion 0.25, and initial volatility of 100%. 20% of intensities are common. At a common credit event, each counterparty defaults with probability 10%.