Floating-Fixed Credit Spreads[1]

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Abstract

We study the term structure of yield spreads between floating-rate and fixed-rate notes of the same credit quality and maturity. Floating-fixed spreads are theoretically characterized in some practical cases, and quantified in a simple model, in terms of maturity, credit quality, yield volatility, yield-spread volatility, correlation between changes in yield spreads and default-free yields, and other determining variables.

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1 Introduction

We study the term structure of yield spreads between floating-rate and fixed-rate notes of the same credit quality and maturity. Floating-fixed spreads are theoretically characterized in some practical cases, and quantified in a simple model, in terms of maturity, credit quality, yield volatility, correlation between changes in yield spreads and default-free yields, and other determining variables.

We show that if the issuer's default risk is risk-neutrally independent of interest rates, then the sign of floating-fixed spreads is determined by the term structure of the risk-free forward rate. For example, if the term structure of default-free rates is increasing up to some maturity, then spreads on floating-rate debt are larger than spreads on fixed-rate debt. Conversely, under the same independence assumption, if the default-free term structure is inverted, then floating-rate spreads are smaller than fixed-rate spreads.

Intuitively, if the term structure is upward sloping, investors anticipate that floating-rate coupons are likely to increase with time. Default risk for a given issuer increases with time, for the issuer cannot survive to time t unless it also survives to each time $s < t$. As the higher anticipated coupon payments of later dates are also the more likely to be lost to default, investors must be compensated by a floating spread that is slightly larger than the fixed-rate spread.

In terms of magnitude, however, in most practical cases floating-fixed spreads are small, typically a few basis points at most, as will be shown by example.[2] Our persistent queries to market practioners have generated no examples in which market paricipants make a distinction between par floating rate spreads and par fixed rate spreads, except for certain cases in which one of these forms of debt is viewed as "more liquid" than another, an issue that we do not pursue.

For example, consider an issuer whose credit quality implies a fixed-rate spread on 5-year par-coupon debt of 100 basis points over the rate on default-

[2]Previous work on this topic by Cooper and Mello (1988) pointed to differences between fixed- and floating-rate spreads that are at least an order of magnitude larger than found here. One possible explanation is the artificial definition of a floating-rate note used for illustration by Cooper and Mello. Longstaff and Schwartz (1995) provide some model results for floating- and fixed-rate pricing, but not in a format that would allow a direct calculation of the spread between floating- and fixed-rate debt of the same maturity and issuer.

free 5-year par-coupon fixed-rate debt. Suppose changes in credit quality are not correlated with state prices (in a sense to be made precise). In a typical upward-sloping term-structure environment, based on the steady-state behavior of a two-factor CIR model fitted[3] to LIBOR swap rates recorded during the 1990s, floating-rate debt of the same credit-quality and maturity would be issued at a spread of roughly 101 basis points. This is of course not to say that the issuer should prefer to issue fixed over floating debt, but rather that a slightly higher credit spread is required to compensate investors paying par for floating-rate debt.

As suggested by this example, the magnitude of the floating-fixed spread associated with default risk is sufficiently small that one could safely attribute any non-trivial differences that may exist in actual fixed and floating rates of the same credit quality to institutional differences between the fixed-and floating-rate note markets.

For our model, the floating-fixed spread is roughly linear in the issuer's fixed-rate credit spread, roughly linear in the slope of the yield curve, roughly linear in the level of the yield curve, and roughly linear in the correlation between changes in default-free yields and fixed-rate yield spreads. The floating-fixed spread is non-linear in maturity. There is essentially no dependence in the level of the yield curve, holding slope constant. The floating-fixed spread is greatest at high yield-spread volatility and high correlation between yield spread and default-free yields.

Our methodology for valuing defaultable debt is that of Duffie and Singleton (1999). Our numerical examples are based on three-factor term-structure models. Two of the three state variables determine a default-free term structure model estimated from LIBOR swap data, while the expected default-loss rate process is based on all three factors, allowing for correlation between default risk and default-free rates. For purposes of studying the effects of correlation between yields and credit spreads, we move from our multi-factor CIR setting to a "quadratic-gaussian" credit-spread model.

2 Getting Started

We begin for simplicity in a discrete-time setting. We let $\Gamma_{m,n}$ denote the time-m price of a default-free zero-coupon bond maturing at time $n > m$.

[3]The parameters of the model are as estimated by Duffie and Singleton (1997).

The one-period default-free floating rate coupon $c(n)$ at time n is

$$c(n) = (\Gamma_{n-1,n})^{-1} - 1.$$

The coupon rate $C(N)$ at time 0 for fixed-rate par-valued default-free debt maturing at time N is determined by

$$C(N) = \frac{1 - \Gamma_{0,N}}{\sum_{n=1}^{N} \Gamma_{0,n}}.$$

The in-n-for-1 forward rate for maturity n is defined by

$$f(n) = \frac{\Gamma_{0,n-1} - \Gamma_{0,n}}{\Gamma_{0,n}} = \frac{\Gamma_{0,n-1}}{\Gamma_{0,n}} - 1.$$

We will later use the relationship

$$\sum_{n=1}^{N} \Gamma_{0,n}(f(n) - C(N)) = 0. \tag{1}$$

We also let $\pi(n)$ denote the state-price density[4] for time-n contingent claims, so that, for example, $\Gamma_{0,n} = E[\pi(n)]$.

For an issuer of given credit quality, Pye (1974) and, in a setting of uncertain interest rates and credit quality, Duffie and Singleton (1999), show simple conditions under which one may price a defaultable claim by treating the claim as default-free after an additional discount $D_{0,n} = \prod_{i=0}^{n-1}(1+s(n))^{-1}$ for contingent cash flows at time n, where $s(n) \geq 0$ is the (state-dependent) short default spread, conditional on information at time n.

For example, letting $\Lambda_{m,n}$ denote the price at time m of a zero-coupon bond maturing at time n of the given issuer quality, we have

$$\Lambda_{0,n} = E[D_{0,n}\pi(n)].$$

We adopt this defaultable valuation model here. For simplicity, we assume that the short default spread $s(n)$ does not vary among the claims of the given issuer that we consider.[5]

[4]We fix a probability space. The existence of a state-price density, a positive random variable sometimes called a state-price deflator, or state-price kernel, is implied by the absence of arbitrage and mild integrability conditions.

[5]This is an assumption of most reduced-form defaultable valuation models, such as that of Duffie and Singleton (1999) or Jarrow and Turnbull (1995).

The spread $k(N)$ at time zero on defaultable floating-rate debt of maturity N is defined by matching to 1 the price of a defaultable note that obliges the issuer, so long as solvent, to pay $c(n) + k(N)$ at each time $n < N$, and to pay $1 + c(N) + k(N)$ at time N. We then have

$$k(N) = \frac{1 - \Lambda_{0,N} - \sum_{n=1}^{N} E[D_{0,n}\pi(n)c(n)]}{\sum_{n=1}^{N} \Lambda_{0,n}}. \tag{2}$$

The fixed-rate spread $K(N)$ on defaultable debt of maturity N is similarly determined by

$$K(N) = \frac{1 - \Lambda_{0,N} - C(N)\sum_{n=1}^{N} \Lambda_{0,n}}{\sum_{n=1}^{N} \Lambda_{0,n}}. \tag{3}$$

The difference $\Delta(N)$ between the floating and fixed spreads is then

$$\Delta(N) \equiv k(n) - K(N) = \frac{\sum_{n=1}^{N}\Big(\Lambda_{0,n}C(N) - E[\pi(n)D_{0,n}c(n)]\Big)}{\sum_{1}^{N} \Lambda_{0,n}}. \tag{4}$$

Proposition: Suppose, for all n, that the state-price density $\pi(n)$ and the default discount $D_{0,n}$ are uncorrelated. Suppose, moreover, that there exists some n_0 such that $f(n) \leq C(N)$ for $n \leq n_0$ and $f(n) \geq C(N)$ for $n < n_0 \leq N$. (It is enough for this that the forward rate $f(n)$ is increasing in n up to time N.) Then the floating-fixed spread $\Delta(N)$ is non-negative. If, in addition, the short default spread $s(n)$ is greater than 0 with positive probability for each time n before default, and if $f(n)$ is not constant in n, then $\Delta(N) > 0$.

We give a continuous-time version of the proposition, with a proof, in the appendix. The proof of the above discrete version is similar. The intuition for the result is given in the introduction. A similar result applies to obtain a negative $\Delta(N)$ for an "inverted" forward-rate curve.

3 Floating-Rate Debt in an Affine Setting

In order to work with an econometrically estimated model of the term structure and to provide for sufficient analytical tractability, we move to a traditional continuous-time setting in which there a short-rate process r and a "risk-neutral" probability measure Q, defined by the property that any contingent claim paying X at some time T has a price at any time $t < T$ given by

$$E_t^Q \left[\exp\left(-\int_t^T r_s \, ds \right) X \right],$$

where E_t^Q denotes expectation under Q conditional on information[6] available to investors at time t. For example, the default-free zero-coupon bond price in this setting is given by

$$\Gamma_{0,n} = E_0^Q \left[\exp\left(-\int_0^n r_t \, dt \right) \right]. \tag{5}$$

Duffie and Singleton (1999) provide conditions under which, for the issuer's given credit quality, there is a default-risk-adjusted short-rate process $R \geq r$ such that the price at time t of a defaultable claim to X at time T is given by

$$E_t^Q \left[\exp\left(-\int_t^T R_s \, ds \right) X \right].$$

That is, one can apply the standard formula for pricing default-free claims to defaultable claims provided the default-free short rate r is replaced by the risk-adjusted short rate R. For example, the defaultable zero-coupon bond price is given by

$$\Lambda_{0,n} = E^Q \left[\exp\left(-\int_0^n R_t \, dt \right) \right].$$

The continuous-time analogue of the zero-correlation assumption given in the above proposition is that the short spread process $S = R - r$ is independent of r under Q. We extend that proposition in the appendix, and here will explore cases in which this assumption does not necessarily hold.

In order to tractably value floating-rate debt in a flexible parametric setting, we work with some "state" process X valued in $I\!R^k$ that (under Q) is a k-dimensional affine jump-diffusion, in the sense of Duffie and Kan (1996). That is, X is valued in some appropriate domain $D \subset I\!R^k$, with

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dB_t + dJ_t,$$

[6]Underlying the model is a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions, as stated for example in Protter (1990). The probability measure Q is equivalent to P, and integrability is assumed as required for the analysis shown. Expectation under Q given $\mathcal{F}_t$ is denoted E_t^Q. The short-rate process r is assumed to be progressively measurable.

where B is a standard brownian motion in $I\!R^d$ under Q, J is a pure jump process with jump-arrival intensity under Q of $\{\kappa(X_t) : t \geq 0\}$, and with a jump-size distribution under Q of ν, on $I\!R^k$, and where $\kappa : D \to [0, \infty)$, $\mu : D \to I\!R^k$, and $\Sigma \equiv (\sigma\sigma^\top) : D \to I\!R^{k \times d}$ are affine functions.[7] We delete time dependencies in the coefficients for notational simplicity only; the approach outlined below extends to the case of time-dependent coefficients in a stragihtforward manner.

A classical special case is the "multi-factor CIR state process" X valued in $D = I\!R^k_+$, for which $X^{(1)}, X^{(2)}, \ldots, X^{(k)}$ are Q-independent processes of the "square-root" type[8] introduced into term-structure modeling by Cox, Ingersoll, and Ross (1985).

We can take advantage of the affine setting for pricing defaultable floating-rate and fixed-rate debt if we suppose that the default-adjusted short rate process R of a given issuer is of the affine form, in that

$$R(t) \quad = \quad A + B \cdot X(t), \tag{6}$$

where A is a real number and $B \in I\!R^k$. For analytical approaches based on the affine structure just described, one can repeatedly use the following calculation, regularity conditions for which are provided by Proposition 1 of Duffie, Pan, and Singleton (1999). For given times t and $s > t$, and given coefficients $a \in I\!R$ and $b \in I\!R^k$, let

$$g(X_t, t) = E^Q \left[\exp\left(\int_t^s -R(u)\, du \right) e^{a + b \cdot X(s)} \;\middle|\; X_t \right]. \tag{7}$$

Under technical conditions, there are ordinary differential equations (ODEs)

[7] This is made precise by defining the generator $\mathcal{D}$ for X, by

$$\mathcal{D}f(x) = f_x(x)\mu(x) + \frac{1}{2}\sum_{ij} \Sigma_{ij}(x) f_{x_i\, x_j}(x) + \kappa(x) \int [f(x+z) - f(x)]\, d\nu(z),$$

for any C^2 function f with compact support. One may add time dependencies to these coefficients. Conditions must be imposed for existence and uniqueness of solutions, as indicated by Duffie and Kan (1996). Generalizations are discussed in Duffie, Pan, and Singleton(1999).

[8] That is,

$$dX_t^{(i)} = \kappa_i(\overline{x}_i - X_t^{(i)})\, dt + \sigma_i \sqrt{X_t^{(i)}}\; dB_t^{(i)},$$

for some given constants $\kappa_i > 0$, $\overline{x}_i > 0$, and σ_i.

for $\alpha : [0, s] \to I\!R$ and $\beta : [0, s] \to I\!R^k$ such that

$$g(x, t) = \exp\left(\alpha(t) + \beta(t) \cdot x\right), \tag{8}$$

with boundary conditions $\alpha(s) = a$ and $\beta(s) = b$. Provided the Laplace transform of the distribution of the jump size of X is given explicitly, the ODEs for α and β are easily and routinely solved by numerical methods such as Runge-Kutta. Details, with illustrative numerical examples and empirical applications, can be obtained in Duffie, Pan, and Singleton (1999). For the special multi-factor CIR case, explicit closed-form solutions for α and β can be deduced from Cox, Ingersoll, and Ross (1985).

Now, suppose there is a reference discrete-tenor floating rate, such as LIBOR (the London Interbank Offering Rate), on which an individual issuer's floating-rate payments are based. For an inter-coupon time interval of length δ, such as one-half year, the reference rate $L(t)$ paid at time t on floating rate loans is the simple rate of interest set at time $t - \delta$ for loans maturing at t, defined by the fact that the price $p_L(t-\delta, t)$ of a zero-coupon reference-quality bond sold at time $t - \delta$ for maturity at time t satisfies

$$1 + L(t) = \frac{1}{p_L(t - \delta, t)}. \tag{9}$$

(We emphasize that $L(t)$ is set at time $t - \delta$ and paid at time t.) If the default risk of an issuer of the reference (say LIBOR) quality is captured by a default-adjusted short-rate process of the form $R_L = A_L + B_L \cdot X$, where $A_L \in I\!R$ and $B_L \in I\!R^k$ are fixed for simplicity, then

$$p_L(t, s) = E^Q\left[\exp\left(\int_t^s -R_L(u)\, du\right) \,\middle|\, X_t\right]. \tag{10}$$

Under the technical regularity conditions of Duffie, Pan, and Singleton (1999), Proposition 1, from (8) we have

$$p_L(t - \delta, t) = e^{\alpha_L + \beta_L \cdot X(t-\delta)},$$

for fixed coefficients α_L and β_L that are easily calculated. Then, from (9),

$$L(t) = e^{-\alpha_L - \beta_L \cdot X(t-\delta)} - 1. \tag{11}$$

Now, consider a non-reference issuer with default-adjusted short-rate process $R = A + B \cdot X$. Let $V(t, \delta, K, n)$ denote the price at time t of a floating-rate note, of the same inter-coupon period δ as that of the reference rate L, with spread K to the reference floating rate, and with a time to maturity of $n\delta$, for some integer number $n \geq 1$ of coupon periods. This floating-rate note is a defaultable claim to a total coupon payment of $L(t+\delta j)+K$ at coupon date $t+\delta j$, for each $j \leq n$, and a claim to the principal of 1 at the n-th (maturity) coupon date. We therefore have

$$V(t,\delta,K,n) = p(t,t+n\delta) + \sum_{j=1}^{n} q(t,t+j\delta), \tag{12}$$

where, for any s,

$$p(t,s) = E^Q\left[\exp\left(\int_t^s -R(u)\,du\right) \,\middle|\, X_t\right]$$

is the market value of a zero-coupon bond of this quality to maturity date s, and

$$q(t,t+j\delta) = E^Q\left[\exp\left(\int_t^{t+j\delta} -R(u)\,du\right)[L(t+j\delta)+K] \,\middle|\, X_t\right] \tag{13}$$

is the market value at time t of the j-th floating-rate coupon.

We now show how to calculate $p(t,s)$ and $q(t,s)$ for any s, thereby providing a calculation of the value $V(t,\delta,K,n)$ of the floating-rate note. From (7)-(8),

$$p(t,s) = e^{c(s-t)+C(s-t)\cdot X(t)},$$

for some coefficients $c(s-t) \in \mathbb{R}$ and $C(s-t) \in \mathbb{R}^d$ that depend only on $s-t$. Substituting (11) into (13),

$$q(t,s) = (K-1)p(t,s) + u(t,s)$$

where

$$u(t,s) = E^Q\left[\exp\left(\int_t^s -R(u)\,du\right) e^{-\alpha_L-\beta_L\cdot X(s-\delta)} \,\middle|\, X_t\right].$$

Now, by the law of iterated expectations,

$$u(t,s) = E^Q\left[e^{\int_t^{s-\delta} -R(u)\,du} E^Q\left[e^{\int_{s-\delta}^{s} -R(u)\,du} e^{-\alpha_L-\beta_L\cdot X(s-\delta)} \,\middle|\, X_{s-\delta}\right] \,\middle|\, X_t\right].$$

Because

$$\begin{aligned} E^Q\left[e^{\int_{s-\delta}^{s} -R(u)\,du} e^{-\alpha_L - \beta_L \cdot X(s-\delta)} \;\middle|\; X_{s-\delta} \right] &= e^{-\alpha_L - \beta_L \cdot X(s-\delta)} p(s-\delta, s) \\ &= e^{c(\delta) - \alpha_L + (C(\delta) - \beta_L)\cdot X(s-\delta)}, \end{aligned}$$

another application of (7)-(8), again under the technical conditions of Duffie, Pan, and Singleton (1999), Proposition 1, implies that we can calculate new coefficients $\tilde{\alpha}(t,s)$ and $\tilde{\beta}(t,s)$ so that

$$u(t,s) = e^{\tilde{\alpha}(t,s) + \tilde{\beta}(t,s)\cdot X(t)}.$$

Thus,

$$q(t,s) = (K-1)e^{c(s-t) + C(s-t)\cdot X(t)} + e^{\tilde{\alpha}(t,s) + \tilde{\beta}(t,s)\cdot X(t)}. \tag{14}$$

Finally, both $p(t,s)$ and $q(t,s)$ are explicit (and easily calculated), and we have $V(t,\delta,K,n)$ from (12). The par-floating rate spread at time t for a time to maturity of $n\delta$ is that spread K with the property that $V(t,\delta,K,n) = 1$. That spread is normally expressed at the annualized rate K/δ.

4 Computational Examples

In this section, we give a concrete example. The state process $X = (X_1, X_2, X_3)'$ is made up of 3 independent "CIR" processes. That is, for each i,

$$dX_{it} = (\kappa_i \theta_i - (\kappa_i + \lambda_i) X_{it})\,dt + \sigma_i \sqrt{X_{it}}\;dW_{it},$$

for given coefficients $\kappa_i, \lambda_i, \theta_i$, and σ_i, where $W = (W_1, W_2, W_3)$ is standard Brownian motion in $\mathbb{R}^3$ under[9] Q. We assume that

$$r = X_1 + X_2 - \bar{y},$$

and show in Table 1 estimates of the coefficients κ_i, θ_i, λ_i and σ_i for $i \in \{1,2\}$ that were estimated from LIBOR swap data at several maturities by Duffie

[9]The risk-premium coefficients λ_1, λ_2, and λ_3 can be used to determine the behavior under the original probability measure P, as in the conventional model of Cox, Ingersoll, and Ross (1985), but we have no need for that here.

Table 1: Parameters of the Model for Risk-Free Term Structure

κ_1	θ_1	σ_1	λ_1
0.544	0.374	0.023	–0.036
κ_2	θ_2	σ_2	λ_2
0.003	0.258	0.019	–0.004

and Singleton (1997) (The coefficient $\bar{y}$ was estimated to be 0.58). As for the short-spread process, we assume that

$$S = \gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 X_3,$$

where γ_1, γ_2, and γ_3 are coefficients that we adjust, along with the coefficients and initial condition of X_3, in order to obtain various alternative credit-spread behaviors.

As all zero-coupon default yields and yield spreads are in closed form for this model, we can easily set up the model for given[10]

- 3-month zero-coupon yield (base case 10 percent).
- 10-year yield minus 3-month yield ("slope") (base case 1.5 percent).
- 5-year credit spread, the difference between the 5-year zero-coupon defaultable yield and the 5-year zero-coupon default-free yield (base case 100 basis points).
- Conditional volatility of 5-year credit spread (base case 48 %).

For the CIR model, one can compute par defaultable fixed and floating rate spreads explicitly, as shown in an appendix. For the calculations that follow, we have taken fixed and floating coupon payments to be made continuously in time, so as to simplify the calculations, as shown in an appendix. Our numerical results are roughly the same as for discrete coupon payments, except for maturities close to zero. For these results, we have kept to the base-case parameters described above, with the exception of the parameter whose level is varied in each case.

[10]The base case parameters for S are $\gamma_1 = 0.005$, $\gamma_2 = 0.01$, $\gamma_3 = 1$,$\lambda_3 = 0$, $\kappa_3 = 0.01$, $\theta_3 = 0.005$, $\sigma_3 = 0.0015$. We adjust $X_3(0)$ for the desired 5-year zero-coupon yield spread.

Figure 1 shows the relation between maturity and the differential (floating-fixed) spread. As maturity goes to zero, the defaultable fixed and floating spreads of course both approach the difference between the default-adjusted short rate R_t and risk-free short rate r_t, so the floating-fixed spread approaches zero. The long-maturity behavior in the figure is determined essentially by the shape of the default-free yield curve.

Figure 2 shows the dependence of the floating-fixed spread on the slope of the yield curve. For changes in the level of the yield curve of up to 15%, holding slope constant, there is at most a 0.05 basis-point impact on the floating-fixed spread. Figure 3 shows the dependence, which is close to linear, on the 5-year zero-coupon yield spread of the issuer.

CIR models, in fact even general affine affine term-structure models of the sort introduced by Duffie and Kan (1996), have limited flexibility with regard to the correlation between yield spread and default-free yield. For example, it appears that one cannot have this correlation negative, within this class, while guaranteeing that yields and yield spreads remain positive. In order to explore the implications of negative correlation for floating-fixed spreads, and only for that purpose, we therefore use a "Quadratic-Gaussian" term-structure model suggested by El Karoui, Myneni and Viswanathan (1992). We take

$$r_t = Y_{1t}^2 + Y_{2t}^2 - \bar{y}$$

and

$$S_t = Y_{3t}^2,$$

where the state process $Y = (Y_1, Y_2, Y_3)'$ is of the Ornstein-Uhlenbeck form

$$dY_t = (\beta - BY_t)\, dt + \Sigma\, dW_t,$$

where B is a diagonal 3×3 matrix, β is a vector in $\mathbb{R}^3$, and

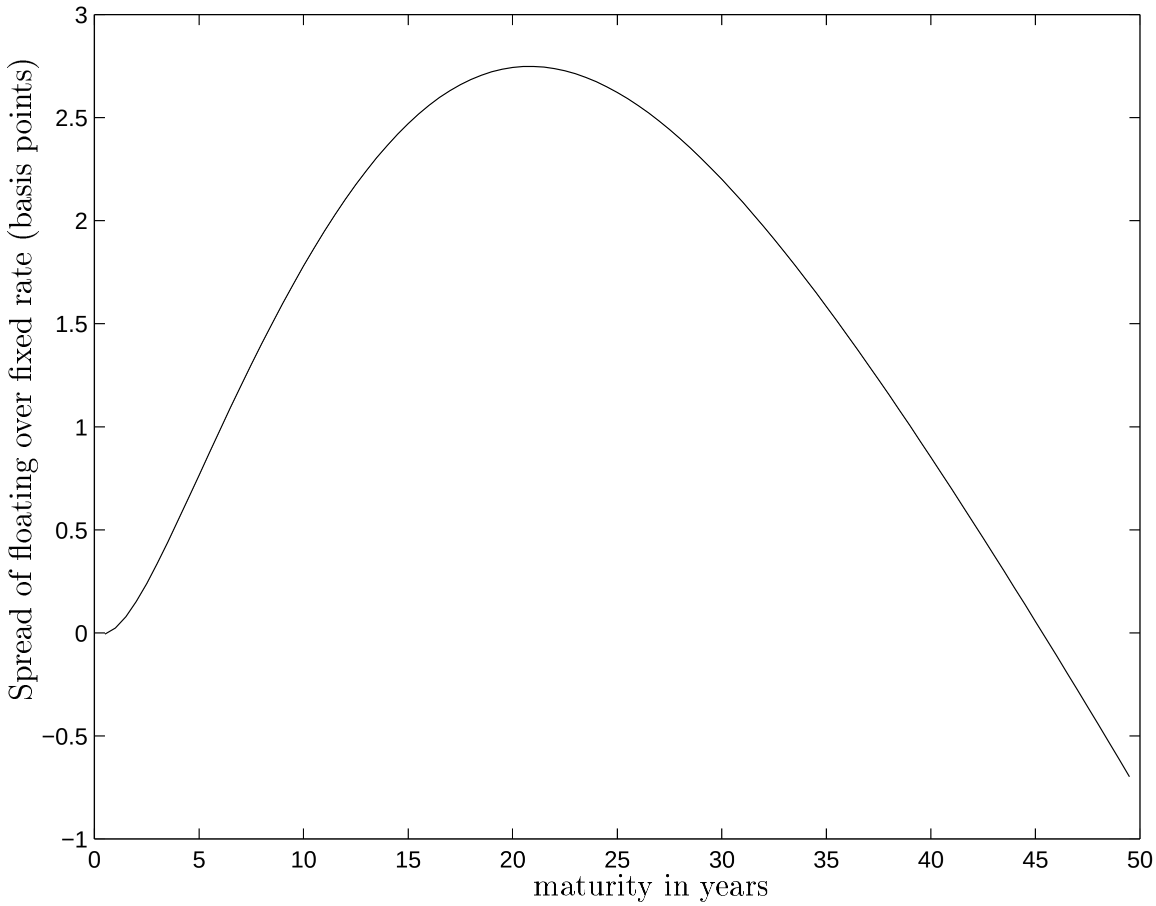
$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ \rho_1\sigma_3 & \rho_2\sigma_3 & \sqrt{1-\rho_1^2-\rho_2^2}\sigma_3 \end{pmatrix}, \tag{15}$$

for given σ_i and ρ_i. For this model, zero-coupon yields and yield spreads of maturity t are of the form $\sum_i [\phi_0(t) + \phi_{1i}(t)Y_i(0) + \phi_{2i}(t)Y_i(0)^2]$, for $\phi_{ji}(t)$ (for $j = 0, 1, 2$ and $i = 1, 2, 3$) solving ordinary differential Riccati equations in t

that are shown in an appendix.[11]

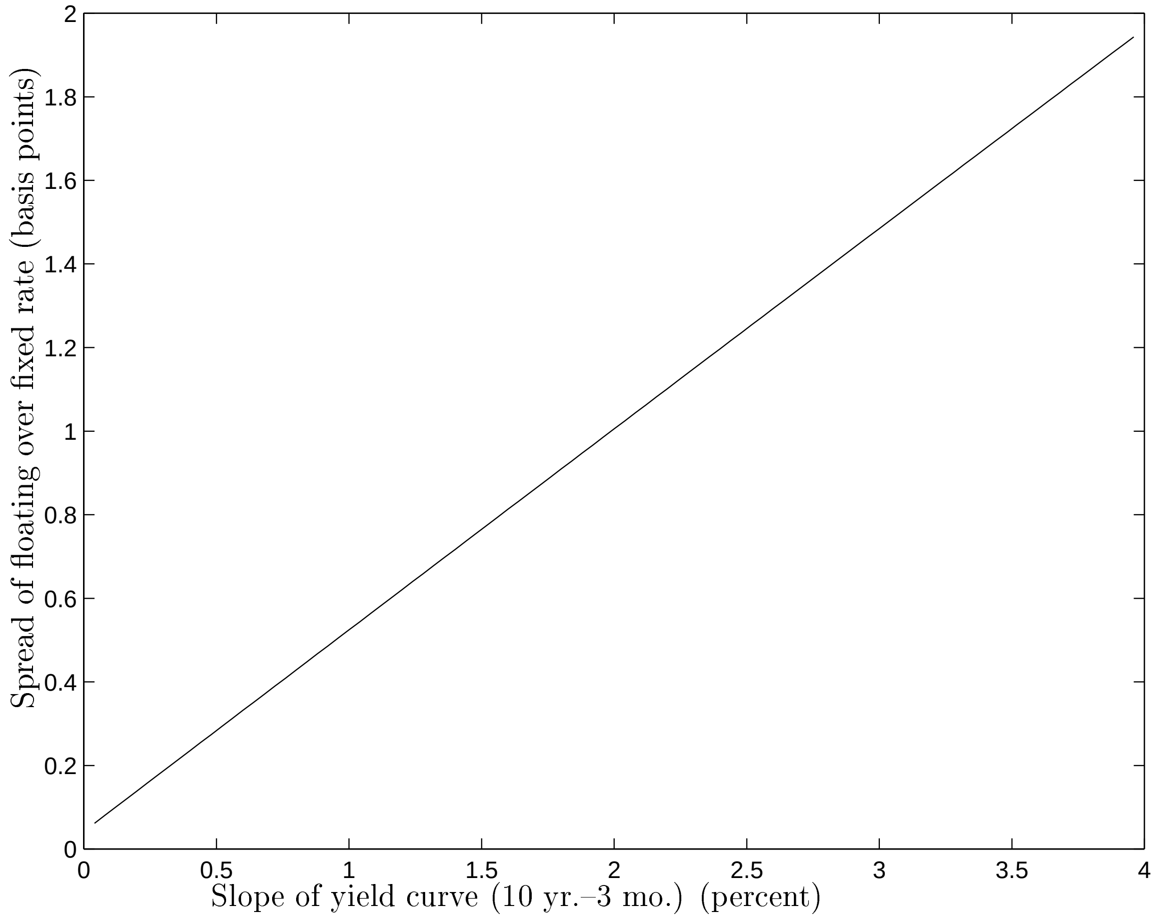
Figure 4 shows the relative impact of differential spread of the correlation between credit spread and changes in the default-free yields. Increasing the correlation between changes in the risk-free term structure and default risk implies that, conditional on the event that the payment on floating debt is high, the probability of default is high. The floating spread is therefore increasing relatively to fixed, intuitively, in this correlation. This effect is indicated in Figure 4, which also shows that the magnitude of the effect, unsurprisingly, grows with the volatility of the default spreads and default-free term structure. For example, increasing the correlation from 0 to 0.4 increases the floating-fixed spread by 6% of its base-case level, or by 22% of its base-case level if the volatilities are also doubled. Our result is consistent with that of Longstaff and Schwartz (1995), who find that the correlation between default risk and the default-free interest rates has a significant effect on the properties of both floating and fixed spreads.

[11]The base-case parameters are $\beta_1 = 0.165$, $B_1 = 0.504$, $\sigma_1 = 0.07$, $\beta_2 = 0.0001$, $B_2 = 0.001$, $\sigma_2 = 0.001$, $\beta_3 = 0.01$, $B_3 = 0.5$, $\sigma_3 = 0.05$, and $\rho_1 = \rho_2 = 0$. These parameters are chosen so as to match by "calibration" to our base-case CIR model. Since Y_i^2 behaves approximately like X_i, we choose σ_i for Y_i to be half of the σ_i for X_i, $B_i = \kappa_i + \lambda_i$, $\beta_i / B_i = (\kappa_i \theta_i / (\kappa_i + \lambda_i))^2$, and $Y_i(0) = \sqrt{X_i(0)}$.



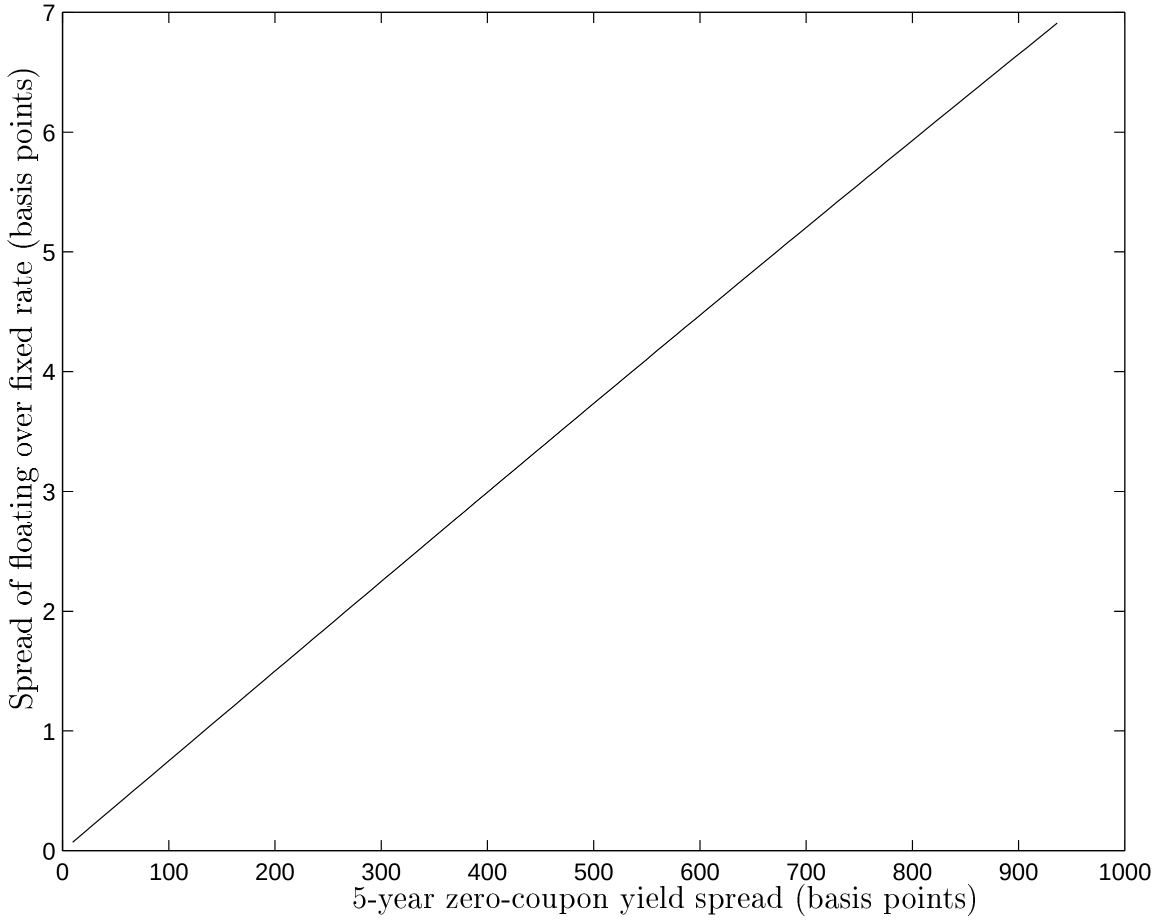
Note: For each maturity, the spread indicated is the difference between the par spread on floating-rate debt less the par spread on fixed-rate debt, at the base-case parameters. At each maturity, the initial short spread $S(0)$ is adjusted (through adjustment of $Y_3(0)$) so as to guarantee a zero-coupon yield spread at each maturity of 100 basis points.

Figure 1: Differential spread as a function of maturity.



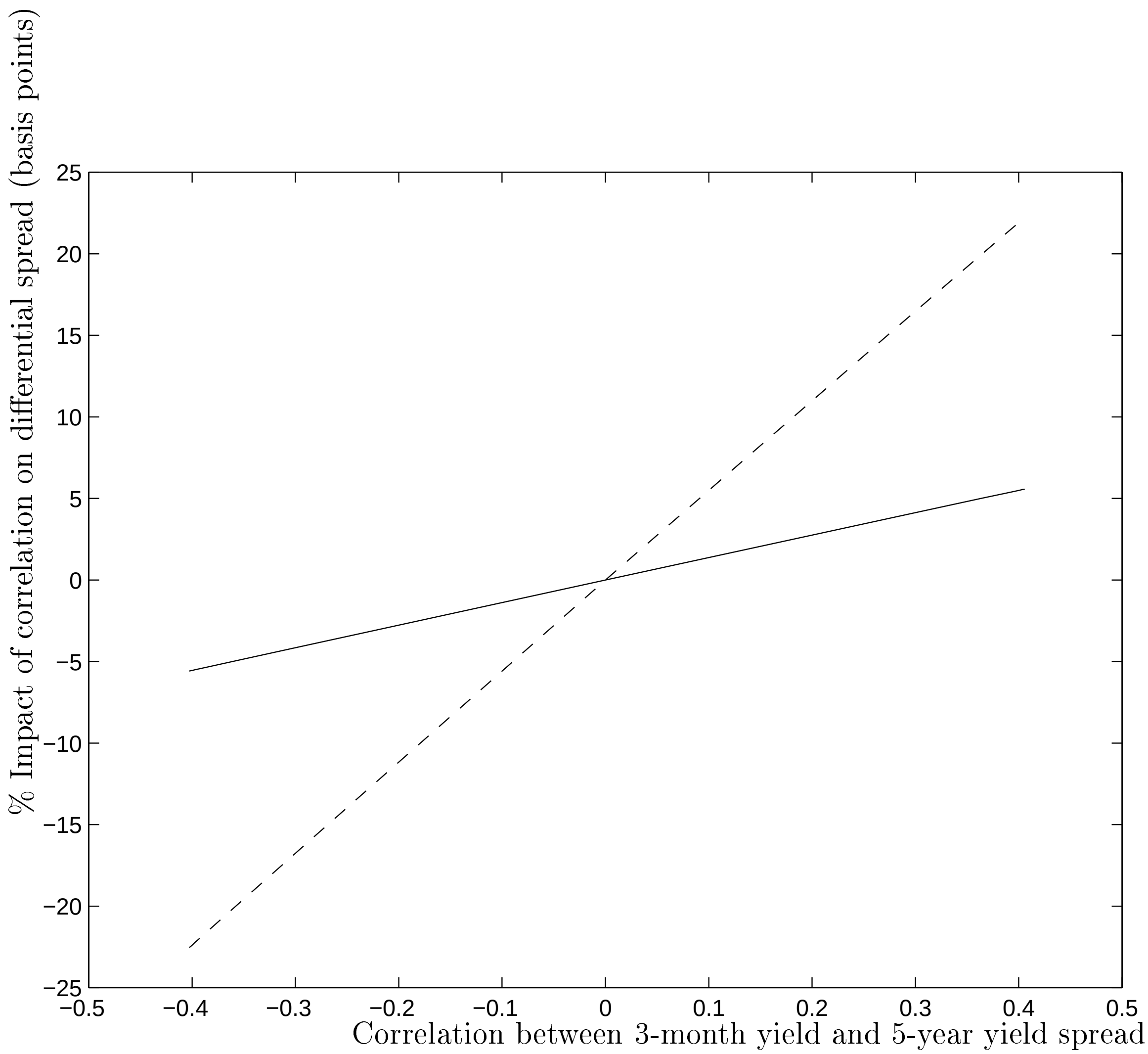
Note: For each slope (10 year-3 month yield spread), the spread indicated is the difference between the par spread on floating-rate debt less the par spread on fixed-rate debt, at the base-case parameters.

Figure 2: Differential spread as a function of slope of yield curve.



Note: For each 5-year zero-coupon yield spread, the spread indicated is the difference between the par spread on floating-rate debt less the par spread on fixed-rate debt, at the base-case paramters.

Figure 3: Differential spread as a function of default spread.



Note: The horizontal axis shows the correlation between "instantaneous" increments of the 3-month yield and the 5-year default spread. The vertical axis shows the percentage difference between the differential spread at the indicated correlation and the differential spread at zero correlation, at the base-case parameters. The dashed line shows the same effect with the diffusion coefficients σ_1, σ_2, and σ_3 all doubled.

Figure 4: Differential spread as a function of correlation between the yield and credit spreads.

Appendix

A. Proof of the Proposition

We assume that both $\Gamma(t)$ and $f(t)$ exist, with instantaneous forward rate $f(t)$ defined by

$$f(t) = -\frac{1}{\Gamma(t)} \frac{d\Gamma(t)}{dt}. \tag{16}$$

The coupon rate $C(T)$ of a par default-free fixed-rate bond of maturity T is defined by

$$C(T) = \frac{E^Q \left[\exp\left(-\int_0^T r_s \, ds\right)\right]}{E^Q \left[\int_0^T \exp\left(-\int_0^t r_s \, ds\right) dt\right]}.$$

We then have

$$\int_0^T \Gamma(t)(f(t) - C(T)) \, dt = 0. \tag{17}$$

The floating-rate spread $k(T)$ for maturity T is given by

$$k(T) = \frac{1 - \Lambda(T) - E^Q \left[\int_0^T \exp\left(-\int_0^t R_s \, ds\right) r_t \, dt\right]}{\int_0^T \Lambda(t) \, dt}. \tag{18}$$

The fixed-rate spread $K(T)$ is given by

$$K(T) = \frac{1 - \Lambda(T) - E^Q \left[\int_0^T \exp\left(-\int_0^t R_s \, ds\right) C(T) \, dt\right]}{\int_0^T \Lambda(t) \, dt}. \tag{19}$$

The floating-fixed spread is then

$$\Delta(T) \equiv k(T) - K(T) = \frac{E^Q \left[\int_0^T \exp\left(-\int_0^t R_s \, ds\right) (C(T) - r_t) \, dt\right]}{\int_0^T \Lambda(t) \, dt}. \tag{20}$$

Proposition: Suppose there exists t_0 such that $f(t) \leq C(T)$ for $t \leq t_0$ and $f(t) \geq C(T)$ for $t > t_0$. (This is true if f is increasing on $[0, T]$.) If S is independent of r under the risk-neutral probability measure Q, then $\Delta(t) \geq 0$

for all $t \leq T$. If, in addition, f is continuous and not constant on $[0,T]$ and S is strictly positive, then $\Delta(t) > 0$.

Proof: Because r and S are Q-independent,

$$\begin{aligned} & E^Q\left[\exp\left(-\int_0^t (r_s+S_s)\,ds\right)(r_t - C(T))\right] \\ &= E^Q\left[\exp\left(-\int_0^t r_s\,ds\right)(r_t - C(T))\right] E^Q\left[\exp\left(-\int_0^t S_s\,ds\right)\right]. \end{aligned} \tag{21}$$

We have

$$E^Q\left[\exp\left(-\int_0^t r_s\,ds\right)(r_t - C(T))\right] = \Lambda(t)(f(t) - C(T))$$

and

$$\begin{aligned} & E^Q\left[\int_0^T \exp\left(-\int_0^t r_s ds\right)(r_t - C(T))\,dt\right] \\ &= \int_0^T \Lambda(t)(f(t) - C(T))\,dt = 0. \end{aligned} \tag{22}$$

Because $S \geq 0$,

$$g(t) \equiv E^Q\left[\exp\left(-\int_0^t S_s\,ds\right)\right]$$

is decreasing in t. It follows that

$$\begin{aligned} & E^Q\left[\int_0^T \exp\left(-\int_0^t (r_s+S_s)\,ds\right)(r_t - C(T))\,dt\right] \\ &= \int_0^T \Lambda(t)(f(t) - C)g(t)dt \\ &= \left(\int_0^{t_0} + \int_{t_0}^T\right)\Lambda(t)(f(t) - C(T))g(t)\,dt. \end{aligned}$$

We have assumed that $f(t) \leq C(T)$ for $t \leq t_0$, and because $S \geq 0$, we know that $g(t) \geq g(t_0)$ for $t \leq t_0$, so

$$\int_0^{t_0} \Lambda(t)(f(t) - C(T))g(t)\,dt \leq \int_0^{t_0} \Lambda(t)(f(t) - C(T))g(t_0)\,dt.$$

Because, for $t \geq t_0$, $f(t) \geq C(T)$ and $g(t) \leq g(t_0)$,

$$\int_{t_0}^{T} \Lambda(t)(f(t) - C(T))g(t)\, dt \leq \int_{t_0}^{T} \Lambda(t)(f(t) - C(T))g(t_0)\, dt.$$

We therefore have

$$E^Q \left[\int_0^T \exp\left(- \int_0^t (r_s + S_s)\, ds \right) (r_t - C(T))\, dt \right]$$
$$\leq \left(\int_0^{t_0} + \int_{t_0}^{T} \right) \Lambda(t)(f(t) - C(T))g(t_0)\, dt = 0.$$

If S is strictly positive, then $g(t)$ is strictly decreasing. If, in addition, $f(t)$ is continuous and not constant, then at least one of the above inequalities is strict, and we obtain $\Delta(T) > 0$.

B. The "Quadratic-Gaussian" Term-Structure Credit-Spread Model

We can write $r = Y^\top \xi Y$ and $R = Y^\top \Xi Y$, for diagonal[12] ξ and Ξ. We will use the fact that the defaultable forward rate $F(t)$ satisfies

$$F(t) = E^Q \left(\exp\left(- \int_0^t R_s\, ds \right) y_t^\top \delta y_t \right),$$

for diagonal δ. One can show that

$$\Lambda_{0,t} = \exp\left[Y_t^\top U(t) Y_t + b(t)^\top Y_t + a(t) \right]$$

and that

$$F(t) = \left[Y_t^\top V(t) Y_t + d(t)^\top Y_t + c(t) \right] \exp\left[a(t) + b(t)^\top Y_t + Y_t^\top U(t) Y_t \right],$$

for time-dependent coefficients U, V, a, b, c, and d. Substituting the above expressions into the PDE satisfied by Λ and F gives the ordinary differential equations

$$\begin{aligned} a' &= \beta^\top b + \frac{1}{2} b^\top \Sigma\Sigma^\top b + \mathrm{tr}(\Sigma\Sigma^\top U) \\ b' &= -B^\top b + 2U^\top \beta + 2U\Sigma\Sigma^\top b \\ U' &= -(B^\top U + UB) + 2U\Sigma\Sigma^\top U - \Xi \end{aligned}$$

[12] One can introduce terms linear in y and a constant term without any difficulty.

and

$$
\begin{aligned}
c' &= \beta^\top d + b^\top \Sigma\Sigma^\top d + \mathrm{Tr}(\Sigma\Sigma^\top V) \\
d' &= -B^\top d + 2V^\top \beta + 2(V\Sigma\Sigma^\top b + U\Sigma\Sigma^\top d) \\
V' &= -(B^\top V + VB) + 2(U\Sigma\Sigma^\top V + V\Sigma\Sigma^\top U),
\end{aligned}
$$

with the initial conditions

$$a(0) = b(0) = U(0) = c(0) = d(0) = 0 \quad \text{and} \quad V(0) = \delta.$$

(These differential equations are solved by Runge-Kutta methods for our examples.)

In order to compute the correlations between the yields, we use the fact that

$$E(Y_{it}) = \bar{y}_{it} \equiv \exp(-B_i t) Y_{i0} + \frac{1}{B_i}(1 - \exp(-B_i t))\beta_i,$$

where B_i denotes the i-th diagonal element of the diagonal matrix B, and

$$\mathrm{cov}(Y_t) = \int_0^t \exp(-B(t-s))\Sigma\Sigma^\top \exp(-B(t-s))\, ds.$$

For our special example of Σ, this covariance matrix $\mathrm{cov}(Y_t) \equiv \Sigma_Y(t)$ is computed as

$$
\begin{pmatrix}
\frac{\sigma_1^2 e^{-2B_1 t}}{2B_1} & 0 & \frac{\rho_1\sigma_1\sigma_3(1-e^{-(B_1+B_3)t})}{B_1+B_3} \\
0 & \frac{\sigma_2^2 e^{-2B_2 t}}{2B_2} & \frac{\rho_2\sigma_1\sigma_3(1-e^{-(B_2+B_3)t})}{B_2+B_3} \\
\frac{\rho_1\sigma_1\sigma_3(1-e^{-(B_1+B_3)t})}{B_1+B_3} & \frac{\rho_2\sigma_1\sigma_3(1-e^{-(B_2+B_3)t})}{B_2+B_3} & \frac{\sigma_3^2(1-e^{-2B_3 t})}{2B_3}
\end{pmatrix}.
$$

Let

$$
\begin{aligned}
w_1(t) &= Y_t^\top U(t) Y_t + b(t)^\top Y_t + a(t) \\
w_2(t) &= Y_t^\top V(t) Y_t + d(t)^\top Y_y + c(t).
\end{aligned}
$$

We then have, suppressing t from the notation,

$$\operatorname{cov}(w_1, w_2) = 2\,\operatorname{tr}(U\Sigma_Y V\Sigma_Y) + (2U\bar{y} + b)^\top \Sigma_y (2V\bar{y} + d).$$

This allows the computation of yield correlations, and thus allows us to "calibrate" coefficients to given yield correlations.

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