Swap Rates and Credit Quality

Supplementary Results

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This version contains supplementary results regarding general cases not covered in the short version of the paper!

Abstract: The impact of credit quality on swap rates is determined under alternative netting assumptions. With counterparties of different default risk, swap valuation is non-linear in the underlying promised exchange of cash flows. The impact of credit risk asymmetry and of netting is presented through both theory and numerical examples, which include interest rate and currency swaps.

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1. Introduction

This paper presents a model for valuing claims subject to default by both contracting parties. This extends the valuation model for defaultable claims proposed by Duffie and Singleton (1994) to cases in which the two counterparties have asymmetric default risk. The extension permits a re-examination of the impact of credit risk on swap rates. While the valuation model applies to all forms of contingent claims in which both contracting parties are at risk to default, such as forward contracts, we focus on swaps for purposes of illustration.

For example, consider a 5-year interest rate swap between a given party paying floating LIBOR rates and another counterparty paying a fixed rate. Replacing the given fixed-rate counterparty with a "lower-quality" counterparty whose bond yields are 100 basis points higher increases the swap rate by roughly 1 basis point, using our model and typical parameters for LIBOR rate processes. This credit impact on swap rates is approximately linear within the range of normally encountered credit quality. For a 5-year currency swap, given a foreign exchange rate with 15 percent volatility, our model shows the impact of credit risk asymmetry on the market swap rate to be roughly 10-fold greater than that for interest rate swaps, that is, approximately 10 basis points in swap rate per 100 basis points in bond yield credit spread. The main goal of this paper is to provide a simple and theoretically consistent model allowing such computations.

The basic idea behind our model is that the impact of credit risk on swap rates depends on the probability distribution of the path taken by the value of the swap itself. When the swap value is positive for a given counterparty, it is the default characteristics (default hazard rate and fractional loss given default) of the *other counterparty* that are relevant for the backward recursive computation of the current swap value given its value at the next point in time. The basic idea for this recursion was developed by Rendleman (1992). Rendleman's model, however, is based on the impact of the swap value on the balance sheets of the counterparties, and considers the direct implications for structural insolvency (liabilities exceed assets). In order to address the problem of determining market swap rates, for which one cannot normally analyse the financial statements of the counterparties on a case-by-case basis with any degree of ease or accuracy, we develop a reduced-form model in which the default characteristics of the counterparties are directly estimated in terms of credit spreads.

The discrete-time intuition behind our model is as follows. At any given time t, the current market value of the swap, assuming that it has not yet defaulted, is denoted V_t. We suppose that V_t is the value to counterparty A, and therefore that $-V_t$ is the value to the other counterparty, B. If $V_t > 0$ then, in usual cases, counterparty A is at risk to the default of counterparty B between t and $t+1$. Thus, under risk-neutral probabilities, V_t is the probability that B defaults between t and $t+1$ multiplied by value given default by B, plus the probability that B does not default between t and $t+1$, multiplied by the market value given no default by B. The market value given no default is the expected present value of receiving V_{t+1} at $t+1$, plus any net dividends paid to A by B between t and $t+1$, under the terms of the swap. The market value given default is some fraction, associated with the credit quality of B, of the market value given no default. If, on the other hand, $V_t < 0$, then this recursive method for computing V_t from V_{t+1} is the same, except for the fact that B is at risk to default by A in this case, so the probability of default and fractional recovery on default used in the recursion are those of A. (We explore several other conventions regarding the nature of the losses given default, including those based on alternative standards that have been considered for pre-termination settlement of swap contracts set by the International Swap Dealers Association (ISDA).)

The effective credit quality of a counterparty in our model is the spread S in the short rate of interest that applies for debt of that counterparty, over the usual (default-free) short rate r. In continuous-time, this spread was demonstrated by Duffie and Singleton (1994) to be $S_t = (1 - \varphi_t)h_t$, where φ is the stochastic process for fractional recovery rates given default and h is the process describing the hazard rate for default. In effect, $h_t \Delta t$ is approximately equal to the conditional probability at time t of default over the next interval of "small" length Δ_t. The default-adjusted effective short rate is $R = r + S$. Typical term-structure models for default-free bonds are valid for defaultable bonds when substituting the default-adjusted short rate R for the usual short rate r.

We consider cases in which counterparty A is always of higher credit quality than counterparty B, in the sense that $S^A < S^B$, where S^A is the short credit spread for A and S^B is the short credit spread for B. In these cases, we show that netting across swap portfolios always increases the market value of the portfolio for the higher-quality counterparty A (and therefore reduces the market value for the lower quality counterparty B). Of course, the diversification of credit risk associated with netting is usually beneficial to both counterparties. We also show that the party of higher credit quality prefers to delay the release of information that may have an impact on swap values. We provide a relatively explicit formula for the marginal impact of an increase in the credit-risk asymmetry $S^B - S^A$ on the market value of a swap. The distinguishing feature of this formula is the appearance of an expectation of an integral over time of $(S_t^A - S_t^B)V_t^+$, the credit spread multiplied by the positive part of the market value of the swap contract, showing the importance to both counterparties of the volatility of the market value of the swap. Indeed, in an example involving a currency swap, we are able to exploit the Black-Scholes formula to compute this marginal impact of credit risk asymmetry, and thereby deduce the marginal impact on the swap rate.

In Section 2, we present and characterize a two-counterparty defaultable claim valuation model in a general setup, extending results from Duffie and Singleton (1994) and Duffie, Schroder, and Skiadas (1993). In Section 3, we apply our model to the case of interest-rate swaps and calculate the impact on swap rates of asymmetric credit quality. In Section 4, we apply the model to a portfolio of swaps and calculate the impact of netting provisions on swap rates. In Section 5, we apply the model to foreign currency swaps. All proofs are in Appendix A.

2. Valuation of Defaultable Swaps

We are interested in the valuation of a contingent claim (or contract) between two counterparties, allowing for the possibility that either counterparty could default before the maturity of the contract. If the claim is an asset (that is, has positive value) to one party throughout the life of the contract, the valuation problem is normally reduced to that involving a single defaultable party, since only default by the party of liability affects the value of the claim. Such one-party defaultable claim valuation problems have been studied extensively by many authors; see Artzner and Delbaen (1991); Duffie, Schroder, and Skiadas (1993); Duffie and Singleton (1994); Hull and White (1992a); Jarrow, Lando, and Turnbull (1993); Jarrow and Turnbull (1992); Lando (1993, 1994); Longstaff and Schwartz (1993); Nielsen, Saá-Requejo, and Santa-Clara (1993); Ramaswamy and Sundaresan (1986); and

Sundaresan (1991). Our interest lies in the valuation of two-party contingent claims (or contracts) that can be either an asset or a liability to each party during the life of the contract. Without loss of generality, this problem is equivalent to the valuation of defaultable swaps.

Since swap default settlement rules are based on market values (obtained through market quotations) at the time of default (as will be specified later in this section), the valuation of the default risk of swaps can be studied within the framework of defaultable claims valuation proposed by Duffie and Singleton (1994), as extended by Duffie, Schroder, and Skiadas (1993), which assumes that the payoff upon default is a function of the market value of the claim just prior to the time of default (or equivalently, to the market value of un-defaulted claims that are otherwise identical). The above two studies, however, focus on defaultable claims with one-party default risk and apply to defaultable swaps only when the two counterparties have identical default risk. In this section, we use the same framework to study the valuation of swaps involving counterparties with asymmetric default risk.

2.1. Basic Setup

We begin with a probability space $(\Omega, \mathcal{F}, P)$ and a family $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the *usual* conditions. (See, for example, Protter (1990) for technical details.) The filtration $\mathbb{F}$ represents the arrival of information over time. We also assume the existence of a short rate process r (progressively measurable and integrable), so that an investor can place one unit of account in riskless deposits at any time t and roll over the proceeds until time $s \geq t$ for a (time-s) market value of $\exp(\int_t^s r_u \, du)$.

2.2. Swaps and Default Characteristics

Consider two counterparties denoted, respectively, as party 1 and 2. Following Duffie, Schroder, and Skiadas (1993), we model the stochastic default time of party i ($i = 1, 2$) as an $\mathbb{F}$-stopping time τ^i valued in $[0, \infty]$.[1] The default time for the swap is defined as $\tau = \tau^1 \wedge \tau^2$, the minimum of τ^1 and τ^2. The event $\{\tau > T\}$ is then the event of no default. In this setup, a *swap*[2] with maturity T initiated at time zero between these two counterparties is formally defined by

[1] Throughout, we use superscripts 1 and 2 to denote counterparties. The event $\tau^i = \infty$ means no default.

[2] Our definition of a swap includes a forward contract as a special case.

(1) a pre-default payment by party 2 to party 1 of a cumulative dividend process $\{D_t : 0 \leq t \leq T\}$, where D is a semimartingale of finite variation[3] such that[4]

$$\mathbf{E}_Q \left[\int_0^T \exp\left(- \int_0^t r_u \, du \right) |dD_t| \right] < \infty;$$

(2) a settlement payoff to party 1 in case of default at time t by one or both parties: party 1 receives $Z^1(\omega, t)$ when party 1 defaults at state ω and time t and $Z^2(\omega, t)$ when party 2 defaults at state ω and time t. The processes Z^1 and Z^2 are assumed to be predictable and may depend endogenously on the valuation of the swap, as described below. Of course, both parties could default simultaneously, but in practice it is unlikely that they will do so with positive probability. For completeness, we consider the possibility of simultaneous default in Appendix B.

Suppose, for example, that parties 1 and 2 are engaged in a fixed-for-floating interest rate swap in which party 1 exchanges a fixed interest payment for a floating interest payment (on the same constant notional amount) with party 2, semiannually until maturity. Party 1's cumulative dividend D_t is then the total floating interest payment by party 2 up until time t, minus the total fixed interest payment by party 1 up until time t.

The predictability of the default settlement payoff means roughly that, were the time of default known, the payoff upon default would be known just prior to default. The default time, however, may be a "surprise," that is, *totally inaccessible*.[5] Intuitively, a totally inaccessible stopping time represents an event that cannot, with positive probability, be foreseen immediately before it occurs. Madan and Unal (1994) argued that, since a predictable stopping time for default may force the default spread near zero for small maturities, the empirical fact that spreads can be substantial for even short maturities makes it more reasonable to assume that default times are surprises. Of course, this argument assumes that

[3] See, for example, Protter (1990) for a technical definition of a semimartingale. We always assume without loss of generality that a semimartingale is RCLL (right-continuous with left limits).

[4] This integrability condition is satisfied if the market value of the promised gross payment of each counterparty, if assumed to be default-free, is finite.

[5] A stopping time T is *totally inaccessible* if for every predictable stopping time S, $P(\{\omega : T(\omega) = S(\omega) < \infty\}) = 0$. A *predictable stopping time* S is a special case of an *accessible stopping time*, and is defined as the limit of an increasing sequence of stopping times $(S_n)_{n \geq 1}$ that are strictly smaller than S on $\{S > 0\}$.

empirical spreads are due to default risk. See Grinblatt (1994) and Duffie and Singleton (1994) for a discussion of alternative determinants of spreads, with references.

There exist at least two kinds of default settlement rules in the swap market. Under the so-called "fault" or "one-way payment" rule, specified in early standard swap documentation supported by the International Swap Dealers Association (ISDA), the payment due the non-defaulting party is the higher of the market value of its position or zero. In other words, the non-defaulting party is not obligated to compensate the defaulting party if the remaining market value of the swap is positive for the defaulting party. The current standard practice in swaps markets (supported by current standard documentation by ISDA), however, is the so-called "no fault," or "two-way payment" rule, which obligates the party with negative remaining market value in the swap to compensate the other party, based on the remaining market value of the swap, regardless of the identity of the defaulting party. Standard swap contracts also often include netting provisions across swap portfolios, requiring only the net market value of all swaps between two counterparties to be paid in the event of default.

The actual default settlement *payoff* can be short of that given by the default settlement *rule*. The default settlement payoffs to party 1, given that the default time is t and that the market value of the swap just prior to the default (the left limit) is some number v, are given by

$$
\begin{aligned}
Z^1(\omega,t) &= \varphi^1(v,\omega,t)\, v\, \mathbf{1}_{\{v<0\}} + \bar{\varphi}^2(v,\omega,t)\, v\, \mathbf{1}_{\{v\geq 0\}}; \\
Z^2(\omega,t) &= \varphi^2(v,\omega,t)\, v\, \mathbf{1}_{\{v\geq 0\}} + \bar{\varphi}^1(v,\omega,t)\, v\, \mathbf{1}_{\{v<0\}},
\end{aligned}
\tag{2.1}
$$

where, for each i, the functions φ^i and $\bar{\varphi}^i$ belong to the class

$$\Lambda = \{\lambda : \mathbb{R} \times \Omega \times [0,T] \to \mathbb{R},\ \lambda \text{ is measurable and } \lambda(v,\cdot,\cdot) \text{ is predictable for all } v\}.$$

Here, φ^i represents the fraction of market value paid by the defaulting party i when the swap has negative net market value for i, and $\bar{\varphi}^i$ represents the fraction of market value paid by the non-defaulting party i when the swap has negative net market value for i. Note that $\bar{\varphi}^i = 0$ represents the "one-way payment" settlement rule, while $\bar{\varphi}^i = 1$ represents the "two-way payment" settlement rule. The case of $0 < \bar{\varphi}^i < 1$ can represent either a mixture of these settlement rules, or the risk faced by the defaulting party in collecting the remaining market value of the swap from the non-defaulting party.

The above formulation of defaultable swaps can be applied to a portfolio of swap contracts written with a netting provision, under which the cumulative dividend process D to party 1 represents the net cash flow exchange (in the event of no default) across all swaps in the portfolio. An example of the impact of netting on swap valuation appears in Section 4.

2.3. Recursive Valuation of Defaultable Swaps

To characterize the market value of a defaultable swap, we formally define a process V with the property that, if there has been no default by time t, then V_t is the market value S_t of the swap to counterparty 1. The value $V(\omega, t)$ for $t \geq \tau(\omega)$ need not be uniquely defined. We will show, however, that V is uniquely defined up to the default time. By *market* value, we mean the price to any investor, of a claim that pays off the same cash flow that counterparty 1 receives from the swap contract.

We study the valuation directly under an equivalent martingale measure, denoted Q, relative to the short rate process r. This means that, for any security defined by a cumulative dividend process $\{X_t : 0 \leq t \leq \infty\}$ (adapted, RCLL, with finite variation), the market value of the security at time t is

$$S_t = \mathbf{E}_Q \left[\int_t^\infty \exp\left(- \int_t^s r_u \, du \right) dX_s \,\middle|\, \mathcal{F}_t \right], \tag{2.2}$$

where $\mathbf{E}_Q$ denotes expectation under Q. We do not deal directly with the existence of an equivalent martingale measure, a property essentially equivalent to the absence of arbitrage, as shown by Harrison and Kreps (1979), nor with the identification of some particular equivalent martingale measure from market prices.

Some conditions are placed on the default times τ^1 and τ^2. We assume that, under Q, the default time of each counterparty is totally inaccessible and that the associated hazard rates are well defined and bounded. More explicitly, we introduce, for each i, the *default indicator functions*, $H_t^i = \mathbf{1}_{\{t \geq \tau^i\}}$, a stochastic process that is equal to one if default by party i has occurred, and zero otherwise. The Doob-Meyer decomposition implies that H^i can be uniquely decomposed as $H^i = A^i + M^i$, where A^i is a predictable and right-continuous increasing process with $A_0^i = 0$, and M^i is a Q-martingale. We assume that the default time of each counterparty is totally inaccessible, so that A^i is continuous.[6] Indeed,

[6] See Lemma 2 in Appendix A, where we show that A^i is continuous if and only if the stopping time τ^i is totally inaccessible.

we assume that A^i is absolutely continuous, in that there is a (progressively measurable) non-negative integrable process h^i, called the default hazard rate of counterparty i, such that

$$A_t^i = \int_0^t h_s^i \mathbf{1}_{\{s<\tau_i\}}\, ds, \quad t \geq 0.$$

Artzner and Delbaen (1994) give technical conditions under which a hazard rate exists under one probability measure if and only if a hazard rate exists under an equivalent probability measure.

We allow each counterparty's default hazard rate h^i to depend on the value of the swap contract. That is, we take $h_t^i(\omega) = \lambda^i(V_t(\omega), \omega, t)$, where λ^i is in Λ.

The cumulative dividend process X of the swap for counterparty 1 is given by

$$X_t = \int_0^t \mathbf{1}_{\{s<\tau\}}\, dD_s + \mathbf{1}_{\{s\leq\tau\}} \left(Z_s^1\, dH_s^1 + Z_s^2\, dH_s^2 \right), \qquad t \leq T. \tag{2.3}$$

The first term is the prearranged swap payment before default. The second term is the settlement payoff in two different default scenarios: party 1 defaults or party 2 defaults. Appendix B considers the extension of (2.3) allowing for simultaneous default of the two counterparties. Here, this event is implicitly assumed to generate zero payment. Equivalently, we can assume that this event has zero probability and therefore has zero payment almost surely.

A valuation formula can be obtained by substituting (2.3) into (2.2), using the above decompositions of the default indicator functions, and then applying the formula for integration by parts for discontinuous semimartingales given, for example, by Protter (1990). The result is a recursive integral equation of the general form

$$V_t = \mathbf{E}\left[\int_t^T f(V_s, \omega, s)\, ds + dD_s \,\middle|\, \mathcal{F}_t \right], \quad t \leq T, \tag{2.4}$$

for some f in Λ that is given explicitly below. This equation (2.4) is to be solved for the stochastic process V representing the market value of the swap prior to default. Conditions for existence and uniqueness of solutions to similar equations are given by Duffie and Epstein (1992) and Antonelli (1993), and extended slightly in Appendix A to handle (2.4).

Formally, a process V is a *pre-default value process* if the swap value process S is well defined by (2.2) and (2.3), and is indistinguishable from V before τ, in that $S_t = V_t \mathbf{1}_{\{t<\tau\}}$ for all t.

For defaultable swap valuation, we will consider the version of (2.4) given by

$$V_t = \mathbf{E}_Q \left[\int_t^T -R_s(V_s, \omega) V_s \, ds + dD_s \;\middle|\; \mathcal{F}_t \right], \quad t \leq T. \tag{2.5}$$

where

$$R_t(v, \omega) = r_t(\omega) + s_t^1(v, \omega) \mathbf{1}_{\{v<0\}} + s_t^2(v, \omega) \mathbf{1}_{\{v \geq 0\}}, \tag{2.6}$$

with

$$\begin{aligned} s_t^1(v, \omega) &= \left(1 - \varphi_t^1(v, \omega)\right) h_t^1(v, \omega) + \left(1 - \bar{\varphi}_t^1(v, \omega)\right) h_t^2(v, \omega); \\ s_t^2(v, \omega) &= \left(1 - \varphi_t^2(v, \omega)\right) h_t^2(v, \omega) + \left(1 - \bar{\varphi}_t^2(v, \omega)\right) h_t^1(v, \omega). \end{aligned} \tag{2.7}$$

Extending from Duffie and Singleton (1994) and Duffie, Schroder, and Skiadas (1993), we may think of R as the short rate after adjustment for the effect of default risk. We call a function λ in Λ *uniformly varying* if $v \mapsto \lambda(v, \omega, t)v$ is uniformly Lipschitz[7].

PROPOSITION 1. *Suppose that r is bounded and that R, h^1, and h^2 are uniformly varying. Then the swap price process S exists and is unique. Moreover, there exists a unique solution V for (2.5). Furthermore, if the jump of V at τ is zero almost surely, then V is a pre-default value process; that is, $S_t = V_t \mathbf{1}_{\{t<\tau\}}$ for all t.*

If the short rate r is unbounded, the result remains valid with appropriate, but slightly cumbersome, changes of numeraire, and mild technical integrability conditions.

In practice, the requirement that V does not jump at τ (almost surely) may not be met, since V is discontinuous on dates of lump-sum payments when, in general, there may exist a positive probability of inability of a counterparty to make the promised payment. This difficulty can be avoided by taking into account[8] the positive default probability and lump-sum dividends on these deterministic dates with boundary conditions and applying (2.5) between these dates.

[7] $v \mapsto f(v, \omega, t)$ is *uniformly Lipschitz* if there exists some constant k such that $|f(u, \omega, t) - f(v, \omega, t)| \leq k|u - v|$ for all (ω, t) and all $(u, v) \in \mathbb{R}^2$.

[8] For example, at a coupon date t, let V_- denote the pre-default value at $t-$ and V denote the pre-default value at t. Letting $p(\omega, t, V_-)$ denote the conditional probability under the equivalent martingale measure of default at t (given no default up to t, taking into account the lump-sum dividend payment $\Delta D(\omega, t)$), one has the equation $V_- = p(\omega, t, V_-)\varphi(\omega, t, V_-)V_- + \left(1 - p(\omega, t, V_-)\right)\left(\Delta D(\omega, t) + V\right)$ to solve for V_-. Regularity conditions on p and φ then allow one to solve for V_s for $s < t$, using a recursive integral equation for the period between the previous coupon date and t, with terminal boundary condition V_{t-} given.

Aside from lump-sum payment dates, there is no obvious reason to expect V to jump at default. For example, we can rule out any jumps for V at default times in diffusion-style models such as those treated in Section 3.

Although (2.5) is useful for proving the existence of a unique solution for the swap price S and for its characterization (as will be shown later), it is sometimes more intuitive to rewrite (2.5), through Ito's lemma, as

$$V_t = \mathbf{E}_Q \left[\int_t^T e^{-\int_t^s R_u \, du} \, dD_s \, \middle| \, \mathcal{F}_t \right], \quad t \le T. \tag{2.5$'$}$$

We call R_t the *effective discount rate* and s^i the *default spread* for counterparty i. The discount rate R_t has a switching-type dependence on the swap value V_t. It is the riskless interest rate plus the default spread of that counterparty with negative swap value. When the two parties have asymmetric default risk (that is, different default spreads), the value of the swap does not depend on the *promised* cash exchange D in a linear fashion.[9] We explore this non-linearity in more detail in the following proposition, which shows that, if party 1 always has a lower default spread than party 2, then the value of a swap portfolio with a netting provision (to party 1) is always weakly higher than that of the same swap portfolio without a netting provision. Furthermore, the value of the swap portfolio with a netting provision (to party 1) is strictly higher than that of the same portfolio without a netting provision if, given the information available at any t, the values of swaps in the portfolio (calculated separately without netting provision) can, with positive probability, offset each other in the future.

PROPOSITION 2. *Suppose that, for each party i, s^i is bounded and does not depend on the swap value directly: $s_t^i(v,\omega) = \hat{s}_t^i(\omega)$ for all $(v,\omega,t) \in I\!R \times \Omega \times [0,T]$. Let V^a, V^b, and V^{ab} be, respectively, the value processes (to party 1) of swaps with cumulative dividend processes D^a, D^b, and $D^a + D^b$. If $\hat{s}^1 \le \hat{s}^2$, then $V^{ab} \ge V^a + V^b$. Furthermore, for given*

[9] Since the *promised* payoff D of a defaultable security is not its actual payoff X, this nonlinearity does not violate the simple consequence of absence of arbitrage which states that the value of a security is linearly related to its payoff.

t, $V_t^{ab} > V_t^a + V_t^b$ *on event* B, *where*

$$A = \{(\omega, u) : u \geq t, \hat{s}_u^1(\omega) < \hat{s}_u^2(\omega), V_u^a(\omega)V_u^b(\omega) < 0\}$$

$$B = \left\{\omega : \mathbf{E}_Q\left[\int_t^T \mathbf{1}_A(\,\cdot\,, u)\, du \,\middle|\, \mathcal{F}_t\right] > 0\right\}.$$

We note that Proposition 2 and, more generally, the non-linear relationship between the value process V of a swap portfolio with netting provisions and the promised cumulative dividend process D of the swaps in the portfolio implies that it is inappropriate to value such a portfolio between two counterparties by valuing each swap (with default risk considered) separately and then adding these values. Indeed, without some convention for treating a swap in the context of the portfolio within which it is netted, the "price" or "rate" for the swap is not a well defined concept. One can, however, define the *marginal value* of a single swap to a netted portfolio by the difference between the value of the portfolio including the swap and that of the portfolio without the swap. We will study, numerically, the impact of netting on the marginal values of swaps in Section 4.

Proposition 2 also captures the value of the favorable industry practice of attempting, if possible, to arrange swaps of offsetting default risk in a portfolio with netting provisions, particularly with the same counterparty of a lower credit rating. (See, for example, Ruml (1992).) An incorrect assumption of linear dependence of V on D (as in the case of valuing each swap separately) would suggest that this kind of activity has no impact on market value. Of course, swap diversification is also a useful means of risk management, independently of its impact on market values.

Since the promised cash flow exchange in a single swap is always netted, Proposition 2 also implies that it is inappropriate to value a single defaultable swap by pricing the default risk of the promised gross payment from each counterparty (for example, the fixed interest payment by one party and the floating interest payment by another party in a coupon swap) separately and then adding the two together (as done, for example, by Longstaff and Schwartz (1993) and Sundaresan (1991)). Because the values of the two gross payments have strictly opposite signs for party 1, Proposition 2 shows that this method of calculating default risk would exaggerate the default risk of the swap by underestimating the value of the swap to the party with higher credit quality (that is, the party with lower default spread).

The above implication of Proposition 2 for the valuation of coupon swaps also helps explain the "puzzle," posed by Litzenberger (1992): Market term swap rates (the fixed rate on a term swap against a floating rate of LIBOR flat) do not reflect credit rating differences between counterparties to the extent that corporate bond yields do. One commonly recognized explanation is that the notional amounts of the swap are not involved in an interest rate swap, which therefore does not have the extent of default risk of corporate bonds. Furthermore, only the net amount of the exchanged cash flow, not the floating or fixed coupon separately, is exposed to default risk. This reduced exposure to default risk for coupon swaps (as opposed to an exchange of a floating-rate corporate bond with a fixed-rate corporate bond between the same two counterparties) is widely recognized as the second important reason for the small swap-rate spreads for counterparties of different credit quality (compared with yield spreads in the corporate bond market); see, for example, Hull and White (1992a) and Sundaresan (1991). Pricing the default risk of the floating payment and the fixed payment separately and then adding the two, however, does not capture the pricing implication of this reduced default risk exposure for interest rate swaps, and exaggerates the required credit spreads for swap rates. For example, Sundaresan (1989, 1991), using a CIR term-structure model with reasonable parameters, finds that a credit spread of about 100 basis points in the bond market translates into a credit spread of about 30 to 60 basis points in term swap rates.[10] Longstaff and Schwartz (1993), under their parameter assumptions, gave 10 to 35 basis points for the swap credit spreads for 10-year interest rate swaps between counterparties with asymmetric credit risks. Both of these results are much larger than those observed in the coupon swap market, in which swap rates are often not adjusted at all by some investment banks for their investment grade clients; see Litzenberger (1992). This market behavior is consistent with our approach of discounting the *net* value of the swap with the discount rate of the party with negative market value. We will study this issue in detail, numerically, in the next section.

We emphasize that the default payoff ratios φ^i and $\bar{\varphi}^i$ as well as the default hazard rates h^i could depend on the swap's value V. This introduces another source of non-linear

[10] Sundaresan's formula appears in the 1991 published form of his paper. These numerical results, however, are from the 1989 pre-publication version of the paper.

dependence of the swap value V on the prearranged cash exchange amount D. We allow for, but do not characterize, the impact of this additional source of non-linearity.

2.4. Early Resolution of Information and Default Spread Asymmetry

Nabar, Stapleton, and Subramanyam (1988) and Duffie, Schroder, and Skiadas (1993) have pointed out that the future timing of resolution of information may influence the current market price of a defaultable claim whose default hazard rate or payoff upon default may depend on the price of the claim. Our defaultable swaps valuation model provides an example of such an effect. Simply put, the party with a lower default spread prefers later resolution of uncertainty because it causes the swap value to deviate from its initial (zero) value more slowly and therefore leaves the two parties exposed to default risk for a shorter period of time.

To illustrate this effect, we compare defaultable swap prices in two markets. Market F, with filtration $\mathbb{F}$, is the one that we have been studying. Market G, with filtration $\mathbf{G} = \{\mathcal{G}_t : t \in [0,T]\}$, is identical to market F except that it has earlier resolution of uncertainty. That is, $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all t while $\mathcal{F}_0 = \mathcal{G}_0$. The equivalent martingale measure Q is assumed to apply to both markets, as we are interested in the pure effect of information. (One can imagine, for example, a setting with risk-neutral investors.) Consider a swap of a cumulative pre-default dividend process D, a semimartingale of integrable variation with respect to both $\mathbb{F}$ and $\mathbf{G}$. The following proposition shows that, if party 1 always has a lower default spread, the time-zero value of the swap (to party 1) in market F is higher than it is in market G. The proposition can be deduced from Duffie, Schroder, and Skiadas (1993), whose technical convexity assumption is naturally satisfied with the structure we have here of asymmetric default risk. The converse result, for higher default risk by counterparty 1, is easy to deduce.

Proposition 3. *Suppose that, for each party i, s^i is bounded and does not depend on the swap value directly: $s^i_t(v,\omega) = \hat{s}^i_t(\omega)$ for all $(v,\omega,t) \in \mathbb{R} \times \Omega \times [0,T]$. Suppose that r, $\hat{s}^1$, and $\hat{s}^2$ are adapted to filtrations $\mathbb{F}$ and $\mathbf{G}$. Suppose that $\hat{s}^2 \geq \hat{s}^1$. Let V^F and V^G denote, respectively, the values in markets F and G respectively (to party 1) of a swap of a given cumulative pre-default dividend process D, which is a semimartingale of integrable variation with respect to filtrations $\mathbb{F}$ and $\mathbf{G}$. Then $V^F_0 \geq V^G_0$.*

Next, we study the price impact of default-spread asymmetry. As one might expect, each party's market value for the swap is monotonically decreasing with respect to the other party's default spread.

We again assume that, for each party i, the default spread s^i does not depend on the swap value directly: $s_t^i(v, \omega) = \hat{s}_t^i(\omega)$ for all $(v, \omega, t) \in I\!\!R \times \Omega \times [0, T]$. To formally describe this monotonicity, we will compute the *Gateaux derivative* of the swap value with respect to $\eta \equiv \hat{s}^2 - \hat{s}^1$, the default-spread asymmetry of the two parties, at a given asymmetry $\bar{\eta}$. This derivative is defined (when it exists) as a process $\nabla V(\bar{\eta}; \eta)$ such that

$$\lim_{\epsilon \downarrow 0} \sup_t \left| \nabla V_t(\bar{\eta}; \eta) - \frac{V_t(\bar{\eta} + \epsilon\eta) - V_t(\bar{\eta})}{\epsilon} \right| = 0,$$

for any bounded predictable process η, where $V(\eta)$ denotes the pre-default value process, that process solving (2.5) for spread asymmetry η.

COROLLARY 1. *Suppose that $\hat{s}^1$ and $\hat{s}^2$ are bounded and predictable. Let $\bar{\eta} = \hat{s}^2 - \hat{s}^1$. For any bounded predictable η, the Gateaux derivative $\nabla V(\bar{\eta}; \eta)$ exists and is given by*

$$\nabla V_t(\bar{\eta}; \eta) = -\mathbf{E}_Q \left[\int_t^T \exp\left[-\int_t^s \left(r_u + \hat{s}_u^1 + \bar{\eta}_u \mathbf{1}_{\{V_u(\bar{\eta}) \geq 0\}} \right) du \right] \max(V_s(\bar{\eta}), 0) \eta_s \, ds \,\middle|\, \mathcal{F}_t \right]$$
$$\leq 0. \tag{2.8}$$

This derivative, giving the marginal impact of changing the default spread, shows that increasing the default spread of counterparty 2 relative to that of counterparty 1 reduces the swap value to counterparty 1. As expected, the impact of default spread is more dramatic in more volatile markets, given the appearance of $\max(V_s, 0)$ in (2.8).

In Section 5, we derive an explicit expression of $\nabla V_t(0; \eta)$ for an example involving fixed-coupon currency swaps. This formula is then used to estimate default spreads for currency swaps.

Hull and White (1992a) treat the case in which counterparty 1 is default-free and arrive at an impact of default risk on swap values similar in spirit to (2.8), with $\hat{s}^1 = 0$. The two models, however, are quite different.

2.5. Effective Discount Rates In A Few Cases

The default-adjusted short rate R in (2.6) has a general form that applies in several particular situations:

(i) *One-party defaultable claims*: If the value of the financial claim to party 1 is always positive (for example, if D is increasing), then the problem is reduced to that of a one-party defaultable claim problem. Assume that the claimholder (party 1) never defaults (that is, $h_t^1 = 0$) or, if it does, the default does not affect its cash flow from the claim (that is, $\bar{\varphi}_t^2 = 1$). Then $R_t(v,\omega) = r_t(\omega) + \big(1 - \varphi_t^2(v,\omega)\big)h_t^2(v,\omega)$. This agrees with Duffie and Singleton (1994) and Duffie, Schroder, and Skiadas (1993).

(ii) *One-way payment (fault) rule*: In this case, $\bar{\varphi}_t^1 = \bar{\varphi}_t^2 = 0$, and we have

$$R_t = r_t + \big[(1-\varphi_t^1)h_t^1 + h_t^2\big]\mathbf{1}_{\{V_t<0\}} + \big[(1-\varphi_t^2)h_t^2 + h_t^1\big]\mathbf{1}_{\{V_t\geq 0\}}.$$

We see that the party with the lower probability of default (that is, lower hazard rate h_t^i) could actually have a *higher* discount rate, for example, when the default payoff ratios of the two parties are equal (that is, $\varphi^1 = \varphi^2$). This is caused by the nature of the "one-way" payment rule, which acts against the counterparty of lower credit quality, since that party is the more likely to default, and when it does default the swap value may be positive. It is therefore theoretically possible that the market interest rate swap term rates for higher credit quality parties may actually be higher than those for lower credit quality parties under the "one-way" payment rule.

(iii) *Two-way payment (no-fault) rule*: In this case, $\bar{\varphi}_t^1 = \bar{\varphi}_t^2 = 1$, and we have

$$R_t = r_t + (1-\varphi_t^1)h_t^1\mathbf{1}_{\{V_t<0\}} + (1-\varphi_t^2)h_t^2\mathbf{1}_{\{V_t\geq 0\}}. \tag{2.9}$$

Under the two-way payment rule, investors in a given firm's liabilities should use the same instantaneous discount rate for swap positions (with negative value to that firm) as for its corporate bonds (that are given the same priority in default settlement). This provides a framework for interpreting the relationship between corporate bond yield spreads and market swap-rate spreads, the subject of Section 3. This approach is similar to that of Hull and White (1992a, 1992b), who first pointed out the advantage of using bond yield credit spread data to estimate the impact of default risks on market prices of OTC derivatives.

2.6. Valuation of Defaultable Swaps in Markovian Settings

In a Markovian setting, the above valuation framework can be simplified. For example, we take Y to be a continuous-time Markov process in some state space $\mathcal{Y}$ with differential generator[11] $\mathcal{D}$ and assume that:

1) the prearranged swap cumulative dividend process D is made up of lump-sum payments at a finite set $\mathcal{T}_D = \{t_1, t_2, \ldots, t_N = T\}$ of (deterministic) times:

$$D_t = \sum_{t_n \in \mathcal{T}_D} \mathbf{1}_{\{t \geq t_n\}} \delta_n(Y_{t_n})$$

for technically well-behaved functions $\delta_1, \ldots, \delta_n$;

2) $r_t(\omega) = \bar{r}(Y_t, t)$ for a technically well-behaved function $\bar{r}$;

3) all contemporaneous default hazard rates and default settlement payoff ratios depend only on V_t, Y_t, and t so that we have, for well-behaved functions $\bar{s}^1$ and $\bar{s}^2$,

$$R(v, \omega, t) = \bar{r}(Y_t(\omega), t) + \bar{s}^1(v, Y_t(\omega), t)\mathbf{1}_{\{v<0\}} + \bar{s}^2(v, Y_t(\omega), t)\mathbf{1}_{\{v \geq 0\}}. \tag{2.10}$$

Then, under technical conditions,[12] we can take $V_t(\omega) = J(Y_t(\omega), t)$, where $J : \mathcal{Y} \times [0, T] \to \mathbb{R}$ solves the partial differential equation (PDE)

$$\mathcal{D}J(y,t) - \left[\bar{r}(y) + \bar{s}^1(J(y,t), y, t)\mathbf{1}_{\{J(y,t)<0\}} + \bar{s}^2(J(y,t), y, t)\mathbf{1}_{\{J(y,t)\geq 0\}}\right] J(y,t) = 0, \quad \text{for} \quad (y,t) \in \mathcal{Y} \times \big([0,T] \backslash \mathcal{T}_D\big). \tag{2.11}$$

The terminal boundary condition is

$$J(y, T) = 0, \qquad y \in \mathcal{Y}. \tag{2.12}$$

Interim boundary conditions are given by computing the cum dividend value $J(y, t_n-)$ in terms of the ex dividend value $J(y, t_n)$ by

$$J(y, t_n-) = J(y, t_n) + \delta_n(y), \qquad y \in \mathcal{Y} \text{ and } n \in \{1, \ldots, N\}. \tag{2.13}$$

[11] For example, if Y solves a stochastic differential equation (SDE), $\mathcal{D}$ is given by Ito's Lemma. More generally, Y may be of a jump-diffusion variety.

[12] The technical conditions are to ensure the existence of a unique solution to (2.11)–(2.13), so that the Feynman-Kac representation implies that this solution uniquely solves (2.5) with $V_T = 0$. For the case in which Y solves an SDE, technical conditions for the existence of a unique solution to (2.11)–(2.13), the first boundary value problem of a semilinear parabolic differential equation, can be found, for example, in Ivanov (1984), pp. 170–171.

The solution to such a PDE can be computed by a finite-difference algorithm. An example follows in the next section.

3. Term Swap Rate Credit Spreads versus Corporate Bond Yield Spreads

In this section, we apply the general framework of valuation of defaultable swaps developed in the last section to study the pricing of interest rate swaps involving an exchange of floating and fixed interest payments between two counterparties with asymmetric default risk. In particular, we are interested in the quantitative relationship between the term swap rates spread that counterparties with different credit ratings face in the fixed-for-floating swap market (against the same counterparty), and the yield spreads they face in the corporate bond market.

3.1. Coupon Swap and Model Specification

We select as our object of study a "plain vanilla" coupon swap with semiannual exchanges of fixed-rate payments for floating-rate payments on a constant notional amount. We assume that counterparty 1 pays the six-month LIBOR rate and counterparty 2 pays the fixed term swap rate. For our purpose, we fix counterparty 1 and vary the credit quality of counterparty 2. We then study how the swap credit spread is related to the credit spreads of the two parties in the bond market.

To be consistent with current standard market practice, we assume the "no fault" or "two-way" payment rule for swaps. In order to focus on the pricing implications of netting, we make the following simplifying assumptions.

First, we assume that each counterparty's credit spread s^i does not depend on the value of the swap. With this simplification, for example, the term swap rates are the same for two swap contracts that have different notional amounts but are otherwise the same. We can therefore assume for convenience that the notional amount is one unit. Second, we assume that the credit quality of counterparty 1 is such that its effective discount rate, $r_t + s_t^1$, is always equal to the short term LIBOR rate, which we represent by ρ_t. (We call such a counterparty a LIBOR party.) In fact, the LIBOR rate is a "replenished" AA rate, that is, the current AA rate. The fact that a given party may diverge from AA is ignored here. Third, we assume that the default spread η_t between counterparty 2 and the LIBOR party is a function of the spot short term LIBOR rate, that is, $\eta_t = \bar{\eta}(\rho_t)$. Finally, we assume that the short term LIBOR rate is modeled in the same way that Cox, Ingersoll, and Ross (1985) modeled the short rate r_t. That is,

$$d\rho_t = \kappa(\mu - \rho_t)\, dt + \sigma\sqrt{\rho_t}\, dB_t, \tag{3.1}$$

where κ, μ, and σ are positive constants and B is a standard Brownian motion and an $\mathbb{F}$-martingale relative to the equivalent martingale measure Q. This is the form of the LIBOR rate process studied empirically by Duffie and Singleton (1994).

3.2. Method of Calculation

Valuation can be done in a Markovian setting. With payment in arrears, one would use two state variables, the LIBOR rate $\hat{\rho}_t$ on the last reset date and the current LIBOR rate ρ_t. Note that $\hat{\rho}_t$ is constant between a reset date and the associated payment date. For the case of symmetric default risk for the two counterparties (including default-free valuation as a special case), a standard trick avoids the use of $\hat{\rho}_t$ by having dividends paid at reset dates according to appropriately discounted amounts. The non-linear nature of valuation in the case of counterparties with asymmetric default risk prohibits such a simplification.

In order to avoid the use of two state variables, we numerically study the simpler case without payment-in-arrears. For completeness, in Appendix C we outline a numerical valuation method for defaultable swaps with payment-in-arrears.

The general valuation equations (2.11)–(2.13) can now be written

$$\frac{1}{2}\sigma^2 y J_{yy} + \kappa(\mu - y)J_y + J_t - \left[y + \bar{\eta}(y)\mathbf{1}_{\{J\geq 0\}}\right] J = 0, \quad y \geq 0, \quad t_n \leq t < t_{n+1}, \tag{3.2}$$

where $0 < t_1 < \cdots < t_N = T$ define the coupon dates. The boundary conditions are

$$J(y,T) = 0, \qquad y \in [0, \infty) \tag{3.3}$$

and

$$J(y, t_n-) = J(y, t_n) + \delta(y), \tag{3.4}$$

where $\delta : \mathbb{R}_+ \to \mathbb{R}$ describes the dependence of the net payment to counterparty 1 on the spot short term LIBOR rate. For a coupon swap with semiannual exchange of a fixed rate C with the market six-month LIBOR rate on payment date t_n, we have

$$\delta(y) = \frac{C}{2} - \left(\frac{1}{p(y, 0.5)} - 1\right), \tag{3.5}$$

where $p(y,t)$ denotes the price of a zero-coupon LIBOR bond with time to maturity t and current LIBOR rate y. From Cox, Ingersoll, and Ross (1985),

$$p(y,t) = \bar{\alpha}(t) \exp[-\bar{\beta}(t)y], \tag{3.6}$$

with

$$\begin{aligned} \bar{\alpha}(t) &= \left[\frac{2\gamma e^{(\gamma+\kappa)t/2}}{(\gamma+\kappa)(e^{\gamma t}-1)+2\gamma} \right]^{2\kappa\mu/\sigma^2} \\ \bar{\beta}(t) &= \frac{2(e^{\gamma t}-1)}{(\gamma+\kappa)(e^{\gamma t}-1)+2\gamma}, \end{aligned} \tag{3.7}$$

for $\gamma = (\kappa^2 + 2\sigma^2)^{1/2}$.

Equation (3.2) can be solved by any of several finite-difference methods. We use the Crank-Nicholson method for our calculation. (See, for example, Duffie (1992) for an illustration of this method applied to pricing contingent claims in a CIR model.) For a given initial LIBOR rate ρ_0, the term swap rate is then obtained by searching for the fixed rate $\bar{C}(\rho_0)$ that makes the initial swap value $J(\rho_0, 0)$ equal to zero.

Our goal in this section is to study the quantitative relationship between the swap rate credit spread and the corporate bond yield credit spread of two companies. Given the default-spread asymmetry $\bar{\eta}(\rho_t)$, we can use the above procedure to calculate the company's term swap rate $\bar{C}(\rho_0)$. We can also calculate the yield of the company's zero-coupon bond with the same maturity as the swap and establish the relationship between the swap credit spread and the bond yield spread of a company against a LIBOR party. This numerical relationship depends on the functional form of $\bar{\eta}(\cdot)$. Here, we study the relationship for:

1) $\bar{\eta}(\rho_t) = c$ (constant);
2) $\bar{\eta}(\rho_t) = \bar{c}\rho_t$ for some constant $\bar{c}$.

The price (and yield) of a zero-coupon bond issued by a company with these two kinds of default-spread asymmetries against a LIBOR party can be calculated analytically as follows. Let $p(y,t;\kappa,\mu,\sigma)$, given by (3.6) and (3.7), denote the current price of a zero-coupon bond issued by a LIBOR party when the current instantaneous LIBOR rate is y, the time to maturity is t, and (κ,μ,σ) are the CIR parameters describing the dynamics of the LIBOR rate process ρ. Then the price of a zero-coupon bond issued by a party with default-spread asymmetry $\bar{\eta}(\rho_t) = c$ against a LIBOR party is given by $e^{-ct}p(y,t;\kappa,\mu,\sigma)$, given that the current LIBOR rate is y and the time to maturity is t. The price of a zero-coupon bond issued by a party with default-spread asymmetry $\bar{\eta}(\rho_t) = \bar{c}\rho_t$ against a LIBOR party is given by $p(y,(1+\bar{c})t; \frac{\kappa}{1+\bar{c}}, \mu, \frac{\sigma}{\sqrt{1+\bar{c}}})$, given that the current LIBOR rate is y, the time to maturity is t, and (κ, μ, σ) are the CIR parameters describing the dynamics of the LIBOR rate process ρ.

3.3. Results and Discussion

Throughout this section, we assume that the CIR model parameters are $\kappa = 0.4$, $\mu = 0.1$, and $\sigma = 0.06$. These parameters are not empirical estimates, but are not atypical; see Pearson and Sun (1990), Gibbons and Ramaswamy (1993), Chen and Scott (1993), or Duffie and Singleton (1994).

We report the results for cases (1) and (2) above in Tables 1 and 2 for a 5-year coupon swap. For case (1) (or (2)), we calibrate the credit-spread parameter c (or $\bar{c}$) so that the bond yield spread is given by the number in the first row of Table 1 (or 2). We then use the same constant to calculate the swap credit spread. The initial LIBOR rate is chosen as $\rho_0 = 10.1818\%$. (The results of Tables 1 and 2 are not sensitive to ρ_0 within reasonable ranges.) In addition to the swap credit spread, we show the "pseudo" swap credit spread calculated by treating the default risk of the fixed-rate payment and the floating-rate payment separately. This is shown to exaggerate swap credit spreads dramatically. All credit spreads are assumed to be against a LIBOR party. Yield spreads are shown in continuously compounding form.

Table 1. Swap Credit Spread versus Corporate Zero-Coupon Bond Yield Spread for $\bar{\eta}(\rho_t) = c$ (all spreads in basis points)

Bond Yield Spread	0	100.00	200.00	300.00
Swap Credit Spread	0	0.95	1.90	2.84
Pseudo-Swap Credit Spread	0	26.37	53.19	80.46

Table 2. Swap Credit Spread versus Corporate Zero Coupon Bond Yield Spread for $\eta = \bar{c}\rho_t$ (all spreads in basis points)

Bond Yield Spread	0	100.00	200.00	300.00
Swap Credit Spread	0	0.76	1.53	2.29
Pseudo-Swap Credit Spread	0	26.52	53.51	80.97

Two conclusions can be drawn from these results. First, the netting of fixed against floating payments in interest rate swaps significantly reduces the impact of default risk on swap rates. With our parameter choices, a credit spread of 100 basis points in the bond market translates into a credit spread of less than one basis point in the coupon swap market. Second, one can drastically exaggerate the impact of default risks on swap rates if one does not deal with the nonlinear effect of netting the two streams of payments in swaps. A large swap spread error is produced by treating fixed-rate payments and floating-rate payments separately, and adding values. These conclusions are supported by both tables. The functional form of $\bar{\eta}(\cdot)$, at least among those considered here, does not heavily influence our main conclusions.

One may note that the swap credit spreads are smaller in Table 2 than those in Table 1 for the same bond yield spreads. For the results in Table 2, counterparty 1 pays the floating rate and is thus exposed to the default risk of counterparty 2 only when the LIBOR rate ρ_t is low, which is also when the default-spread asymmetry $\bar{c}\rho_t$ is low. The impact of default-spread asymmetry on the swap price is further reduced by this monotonic relationship between ρ_t and $\bar{\eta}(\rho_t)$.

Subsequent to the initiation of a swap, the credit spread of the swap may become higher (or lower) if market interest rate changes favor (or act against) the counterparty with higher quality. An interest-rate swap that is 100 basis points off the market in favor of the LIBOR party (defined as a swap that would be marked-to-market if the fixed rate is lowered by 100 basis points) but is otherwise identical to that studied in Table 1 has a credit spread of 2.9 basis points for a yield spread of 100 basis points in the bond market.

The above swap credit spread becomes 0.2 basis points if the swap is off the market against the LIBOR party by 100 basis points while everything else remains the same.

In the above example of coupon swaps, we assumed that the fixed-rate payment dates match the floating-rate payment dates. Many coupon swaps, however, involve a fixed-rate payment once (or twice) a year, but a floating-rate payment four times per year. (We call these coupon swaps 4-for-1 (or 4-for-2).) Compared with the case of matched payment dates, the precedence of the floating-rate payments over the fixed-rate payments reduces the effect of netting and generally results in a larger swap credit spread when the floating-rate payer has higher credit quality. To illustrate this effect, we consider two coupon swaps between a LIBOR party and a second party with a default-adjusted short rate that is 100 basis points higher than that of the LIBOR party. The LIBOR rate dynamics and the initial LIBOR rate are those underlying Table 1. The first swap, a 1-for-1 swap in which the LIBOR party pays a one-year LIBOR rate annually against a fixed rate by the other party, has a credit spread of 1.0 basis point. This result is roughly the same with that of the 2-for-2 swap in Table 1. The second swap, a 4-for-1 swap in which the LIBOR party exchanges 4 quarterly payments of the 3-month LIBOR rate against a year-end fixed-rate payment by the other party, has a credit spread of 4.4 basis points. This higher credit spread reflects the LIBOR party's additional exposure to default risk by the fixed-rate payer, for example at the point at which 3 quarterly payments by the LIBOR party have been made, while the offsetting annual fixed rate payment is yet to be made.

The above results are consistent with the recent practice started of some investment banks who, when on the floating side of a swap, request that the floating-rate payments be compounded and paid on the fixed-rate payment dates. These requests are usually met without a change of swap rate. Our calculation indicates that this practice increases the value of the swap to the floating-rate payer, in addition to reducing the exposure to credit risk.

We next present some new results. First, we show how the swap credit spread can depend on the slope of the yield curve. We consider the same swap as studied in Table 1. We vary the initial spot rate and the long term mean of the CIR model such that the fixed rate for a second party with the same risk quality as the libor party stays the same. We then calculate the swap credit spread for a party with a constant default-spread asymmetry

η = 100 basis points against a LIBOR party, which implies a bond yield spread of 100 basis points. Intuitively, we expect that, as the yield curve becomes less upward-sloping (or more downward-sloping), the exposure of the float-rate payer should increase, which in turn causes the swap credit spread to increase. This is confirmed by the result shown in the following table.

Table 3. Effect of Yield Curve Slope on the Swap Credit Spread for $\bar{\eta}(\rho_t) = c$ (all spreads in basis points)

Short LIBOR rate (in %)	9.78	10.18	10.60	11.05
5-Year LIBOR rate (in %)	10.00	10.04	10.07	10.11
Long Term Mean (in %)	10.25	10.00	9.73	9.47
Fixed Rate for $c = 0$ (in %)	10.2922	10.2922	10.2922	10.2922
Fixed Rate for $c = 1\%$ (in %)	10.3006	10.3017	10.3029	10.3043
Swap Spread for $c = 1\%$ (in bp)	0.85	0.95	1.08	1.21

Second, we assume that the default-spread asymmetry is linearly increasing in time, with $\eta(t) = ct$ and that the swap is otherwise the same with that studied in Table 1. If we calibrate c such that the two parties has 100 basis points in bond yield spread, then the swap credit spread is 0.84 basis points.

Third, we consider the fact that the credit quality of the fixed-rate payer might get worse as the floating rate gets lower in time relative to his fixed rate. We model this by a default-spread asymmetry of the form $\bar{\eta}(\rho_t) = a - b\rho_t$. We let $a = 2\%$ and calibrate b so that the two parties have 100 basis points in bond yield spread. The swap credit spread is shown to be 1.14 basis points and, as expected, higher than that in Table 1.

Finally, we consider the possibility that both parties can have positive default spread against the LIBOR rate. Assume that the effective instantaneous discount rates of the "libor party" and other party are, respectively, 100 and 200 basis points higher than the spot libor rate while keeping all other factors the same as in Table 1. Then the bond yield spread of the two parties is still 100 basis points and the swap credit spread is shown to be 0.95 basis points, the same with the result in Table 1.

4. The Effect of Netting Provisions On Values of Swap Portfolios

In the last section, we studied the impact on coupon swap term rates of netting the fixed and floating payments of a single swap. As Proposition 2 shows, netting among swaps in a swap portfolio also influences the value of the portfolio. In this section, we use a simple example of two coupon swaps to illustrate the effect of netting on the valuation of swap portfolios.

Our first approach is to compare the difference between the value of the portfolio with a netting provision and that of the same portfolio without a netting provision. The difference represents the financial benefit of netting to the counterparty with higher credit quality (in addition to risk-management benefits).

Alternatively, one can think of the effect of netting as follows. Suppose that counterparty 1, the party with higher credit quality, is about to enter into a swap contract, called *the new swap*, with counterparty 2. If the new swap is not to be netted with an existing swap portfolio, then the term rate of the new swap should be set such that the value of the new swap is zero at initiation. If, however, the new swap is to be netted with some existing swap portfolio between the two parties, then counterparty 1 may set a slightly lower rate for counterparty 2 because, for any given promised payment, the marginal value of the new swap with netting is higher than it is without netting. The amount of discount depends on the extent to which the credit exposure of the new swap offsets that of the existing swap portfolio.

To illustrate this effect, we consider the following example. (See Table 3 for an illustration of the setup and all definitions. Each swap (or swap portfolio) in the table is assumed to be marked to market at $t = 0$.) Let counterparty 1 be the LIBOR party and let counterparty 2 have a constant default spread, $s^2 - s^1$, of 100 basis points against counterparty 1. At a fractional payment on default of 50 percent, for example, this translates into an annual probability of default by counterparty 2 that is roughly 2 percent more than that of counterparty 1. We take the new swap to be exactly the same as the swap studied in the last section, in which the LIBOR party, semiannually, exchanges the six-month LIBOR rate for a fixed rate with party 2, until maturity in five years. The dynamics of the short term LIBOR rate is again assumed to be described by the CIR model with the parameters assumed in Section 3. The term rate of the new swap for counterparty 2, without a netting provision for the portfolio of swaps, is denoted $\bar{C}(\rho_0)$ and is calculated as in the last section.

We assume that the new swap is to be netted with an existing swap, called *the old swap*, between the two parties. The old swap is taken to be an *inverse floater* against fixed, in which party 2 exchanges, semiannually, a fixed rate for a floating rate by party 1 of

$$L(\rho_0) - k\left[L(\rho_t) - L(\rho_0)\right],$$

where k is a constant and $L(y)$ denote the six-month LIBOR rate when the spot short term LIBOR rate is given by y. With the short term LIBOR rate described by the CIR model, we have

$$L(y) = \frac{1}{p(y, 0.5)} - 1,$$

where $p(y,t)$ is given by (3.6)-(3.7). (If $k > 1$, we refer to the inverse floater as an *inverse super floater*.) To simplify the problem, we assume that the old swap has the same maturity as the new one and that counterparty 2 pays the fixed rate $\hat{C}(\rho_0, k)$ with the property that the initial value of the old swap is zero.

Table 3. Semiannual Floating and Fixed Payments of the Old Swap, the New Swap, and the Netted Portfolio

	Old Swap (no netting)	New Swap (no netting)	Old and New Swap (netted portfolio)
Floating Payment	$L(\rho_0) - k\left[L(\rho_t) - L(\rho_0)\right]$	$L(\rho_t)$	$(1-k)L(\rho_t) + (1+k)L(\rho_0)$
Fixed Payment	$\frac{1}{2}\hat{C}(\rho_0, k)$	$\frac{1}{2}\bar{C}(\rho_0)$	$\frac{1}{2}\left[\hat{C}(\rho_0, k) + \bar{C}_k(\rho_0)\right]$

The term rate of the new swap depends on the extent to which the payments in the old and new swaps offset each other. In other words, with netting, the term rate $\bar{C}_k(\rho_0)$ of the new swap to party 2 depends on the parameter k of the inverse floater between the two parties. If $k \leq 0$, the payoffs of the new swap and those of the old swap are linearly related and perfectly correlated. There is, therefore, no impact on swap rates of netting in this case. That is, $\bar{C}_k(\rho_0) = \bar{C}(\rho_0)$ for $k \leq 0$. For $0 < k < 1$, the floating rate of the old swap partially offsets the floating rate of the new swap, with the greatest offset occurring at $k = 1$. Consequently, the term rate $\bar{C}_k(\rho_0)$ of the new swap is smaller than $\bar{C}(\rho_0)$ and decreases with increasing k. If the inverse floater is an inverse super floater, that is, $k \geq 1$, then the floating rate of the new swap is fully offset by the inverse floater and the term rate of the new swap to be netted with an inverse super floater is the same as that of the new swap to be netted with an inverse floater with $k = 1$, that is, $\bar{C}_k(\rho_0) = \bar{C}_1(\rho_0)$ for $k \geq 1$. This is illustrated in Figure 1. The linear dependence on k for $0 < k < 1$, shown in Figure 1, is demonstrated in Appendix D.

(Please insert Figure 1 here.)

Figure 1. Impact of Netting on Swap Rates

We provide the numerical result for one example. Suppose that the constant asymmetry of default spread between the two parties is 100 basis points. For $\rho_0 = 10.1818\%$, we have

$$\bar{C}_0(10.1818\%) = 10.3017\%, \quad \bar{C}_1(10.1818\%) = 10.2835\%,$$

and

$$\bar{C}_0(10.1818\%) - \bar{C}_1(10.1818\%) = 1.82 \text{ basis points.}$$

The effect of netting on the value of a portfolio of coupon swaps is small for realistic parameters because netting the fixed and floating payments, within a single swap, already reduces the credit spread to about 1 basis point (at least for our typical term struture model and parameters). The netting effect of a master swap agreement, however, should be quantitatively more significant for other forms of contracts, such as foreign exchange swaps or forwards, with larger credit risk exposure. This is illustrated in the next section.

5. The Effect of Netting Provisions on Currency Swap Rates

Currency swaps typically involve an exchange of principals, and are therefore subject to more exposure to default risks than are interest rate swaps. In this section, we calculate the impact of default risks on currency swap rates.

We use "dollar" and "yen" to denote, respectively, the units of the domestic and foreign currencies. We first describe a specific foreign exchange swap contract of interest. Suppose that counterparties 1 and 2 are engaged in a fixed-for-fixed foreign currency swap with P_d and P_f denoting, respectively, the principal amounts of domestic currency (in dollars) and foreign currency (in yen). Counterparty 1 exchanges a fixed coupon payment of $\frac{1}{2}c_d P_d$ dollars for a fixed coupon payment of $\frac{1}{2}c_f P_f$ yen with counterparty 2, semiannually until maturity at time T, where c_d and c_f are constant coupon rates. At maturity, counterparty 1 exchanges P_d dollars for P_f yen with counterparty 2.

Since the volatility of the market value of the above fixed-for-fixed currency swap depends mostly on the volatility of the currency exchange rate, we simplify by taking constant domestic and foreign interest rates, r_d and r_f, respectively. The foreign exchange rate W, with W_t defined as the market value (in dollars) of one yen at time t, is taken to be a geometric Brownian motion under the equivalent martingale measure Q. That is,

$$dW_t = (r_d - r_f)W_t\,dt + \sigma_w W_t\,dB_t, \tag{5.1}$$

where σ_w is a constant, B is a Brownian motion under Q, and the drift term $(r_d - r_f)W_t$ ensures that the gain process associated with rolling over one yen in short term riskless lending, discounted by the domestic interest rate, is a martingale under Q.

One way to estimate the impact of default risks on currency swap rates is to apply the PDE (2.11)–(2.13), taking W as the state variable. In this section, however, we use the

Gateaux derivative $\nabla V_t(0;\eta)$, given in (2.8), to estimate the impact of default risks on swap rates of the default-spread asymmetry η between the two parties. The resulting first order approximation provides sufficient accuracy for most practical applications. The advantage of this approximation method is that the Gateaux derivative $\nabla V_t(0;\eta)$ can be computed relatively explicitly.

Consistent with common practice, we assume for our example that P_d dollars have the same initial market value as P_f yen; that is, $P_d = W_0 P_f$. Second, we assume that the domestic and foreign interest rates are equal; that is, $r_d = r_f$. Third, we assume that the default spread s^1 of counterparty 1 is a constant, so that $R^1 = r_d + s^1$ is a constant. Finally, we assume that the default-spread asymmetry η is a constant c. These assumptions are made for analytical tractability and should not heavily influence the numerical relationship between the swap credit spread and the default-spread asymmetry η, which depends essentially on the exchange rate volatility.

We do the first order approximation from a reference point of $\eta = 0$ for credit-spread asymmetry. For $\eta = 0$, the predefault value process $V(\eta)$ (to counterparty 1) can be calculated using (2.5′), and is given by

$$V_t(0) = P_d \left[\left(\frac{W_t}{W_0} - 1 \right) e^{-R^1(T-t)} + \left(c_f \frac{W_t}{W_0} - c_d \right) \sum_{t_n > t} e^{-R^1(t_n - t)} \right], \tag{5.2}$$

where T is the time of maturity and t_n is, for each n, a coupon date. If $c_d = c_f$, then $V_0(0) = 0$. Our first step is to calculate, for the case of $c_f = c_d$, the impact of a small constant default-spread asymmetry c on the market value of the swap.

According to (2.8), the Gateaux derivative of the initial market value of such a swap with respect to the default-spread asymmetry η at $\bar{\eta} = 0$ is

$$\begin{aligned}
\nabla V_0(0;c) &= -\mathbf{E}_Q \left[\int_0^T e^{-R^1 t} \max(V_t(0), 0)\, c\, dt \right] \\
&= -c P_d \mathbf{E}_Q \left[\int_0^T e^{-R^1 t} \left[e^{-R^1(T-t)} + c_d \sum_{t_n > t} e^{-R^1(t_n - t)} \right] \max\left(\frac{W_t}{W_0} - 1, 0 \right) dt \right] \\
&= -c P_d \Bigg(e^{-R^1 T} \int_0^T \mathbf{E}_Q \left[\max\left(\frac{W_t}{W_0} - 1, 0 \right) \right] dt \\
&\qquad + c_d \sum_{t_n} e^{-R^1 t_n} \int_0^{t_n} \mathbf{E}_Q \left[\max\left(\frac{W_t}{W_0} - 1, 0 \right) \right] dt \Bigg).
\end{aligned} \tag{5.3}$$

Using the Black-Scholes formula and integration by parts, we have

$$
\begin{aligned}
&\int_0^s \mathbf{E}_Q \left[\max\left(\frac{W_t}{W_0} - 1, 0\right)\right] dt \\
&= \int_0^s \left[2N\left(\frac{1}{2}\sigma_w \sqrt{t}\right) - 1\right] dt \\
&= \left(s - \frac{4}{\sigma_w^2}\right)\left[2N\left(\frac{1}{2}\sigma_w \sqrt{s}\right) - 1\right] + \frac{4}{\sqrt{2\pi}}\frac{\sqrt{s}}{\sigma_w}\exp\left(-\frac{\sigma_w^2}{8}s\right),
\end{aligned}
\tag{5.4}
$$

where $N(\cdot)$ is the standard normal cumulative distribution function. Substituting (5.4) into (5.3), we have

$$
\begin{aligned}
\nabla V_0(0;c) = &-cP_d e^{-R^1 T}\left[\left(T - \frac{4}{\sigma_w^2}\right)\left[2N\left(\frac{1}{2}\sigma_w\sqrt{T}\right) - 1\right] + \frac{4}{\sqrt{2\pi}}\frac{\sqrt{T}}{\sigma_w}\exp\left(-\frac{\sigma_w^2}{8}T\right)\right] \\
&- c\, c_d P_d \sum_{t_n} e^{-R^1 t_n}\left[\left(t_n - \frac{4}{\sigma_w^2}\right)\left[2N\left(\frac{1}{2}\sigma_w\sqrt{t_n}\right) - 1\right] + \frac{4}{\sqrt{2\pi}}\frac{\sqrt{t_n}}{\sigma_w}\exp\left(-\frac{\sigma_w^2}{8}t_n\right)\right].
\end{aligned}
\tag{5.5}
$$

Equation (5.5) gives the relationship between the initial swap value and a small default-spread asymmetry for the case of $c_d = c_f$. This result can help us obtain the relationship between the swap credit spread and a small default-spread asymmetry. Let $V_0(\eta; c_d, c_f)$ denote the initial value of a swap with coupon rates c_d and c_f. For each constant default-spread asymmetry c, we fix counterparty 1's coupon rate c_d and search the coupon rate $c_f = C_f(c)$ of counterparty 2 with the property that $V_0(\eta; c_d, c_f) = 0$. For $\eta = 0$, we have $C_f(0) = c_d$. The swap credit spread is given by $C_f(c) - C_f(0)$, which is determined, with accuracy to the first order of c, by

$$
\left[\nabla V_0(0;c) + \big(C_f(c) - C_f(0)\big)\frac{\partial V_0(0; c_d, c_f)}{\partial c_f}\right]\Bigg|_{c_f = c_d} \approx 0,
$$

from which we have

$$
C_f(c) - C_f(0) \approx -\frac{\nabla V_0(0;c)}{\frac{\partial V_0(0;c_d,c_f)}{\partial c_f}}\Bigg|_{c_f=c_d}. \tag{5.6}
$$

Equation (5.6), combined with (5.2) and (5.5), gives the relationship between the currency swap credit spread and a small default-spread asymmetry.

For example, with

$$
\sigma_w = 15\%; \quad R^1 = 6\%; \quad T = 5 \text{ years}; \quad c_d = c_f = 5\%,
$$

we have

$$C_f(c) - C_f(0) \approx 0.087c.$$

For a constant default-spread asymmetry of $c = 100$ basis points, this translates into a bond yield spread of 100 basis points, and a currency swap credit spread of about 8.7 basis points.

This calculation shows that netting can significantly reduce the impact of default risks on the credit spreads for foreign currency swaps. If the above currency swap is not netted, that is, if counterparty 1 exchanges a five year bond (denominated in dollars) with a fixed coupon rate for a five year bond (denominated in yen) with a fixed coupon rate from counterparty 2, then counterparty 1 would demand a credit spread of 100 basis points for a default-spread asymmetry of 100 basis points. Netting reduces the 100 basis-point credit spread to about 8.7 basis points.

The calculation also shows that credit spreads for currency swaps are indeed much higher than those of coupon swaps. This is partly because the principals are exposed to default risks in currency swaps while the notional amounts are not exposed to default risks in coupon swaps. Another factor is the manner of dependence of the volatility of the swap market value on the volatility of the underlying process. This point can be made clear by a comparison between the credit spread of the above currency swap and that of a forward contract on a zero-coupon LIBOR bond. The principals of both contracts are exposed to default risks. The same numerical procedure and parameter choices used in Section 3 show, however, that the credit spread of a five-year forward contract on a zero-coupon LIBOR-quality bond with a five-year maturity (from the date of forward maturity) is only 0.7 basis points for a 100 basis point default-spread asymmetry.

As with interest rate swaps, a major determinant of currency swap spreads is market volatility. For our currency swap example, doubling the volatility parameter σ_w from 15 to 30 percent increases the currency swap spread from approximately 8.7 basis points to approximately 17.2 basis points. These estimates are roughly consistent with those obtained by Hull and White (1992b) for the special case in which one of the counterparties has no default risk.

Appendix A: Technical Lemmas and Proofs

This appendix contains some technical lemmas and proofs of all propositions.

LEMMA 1. *For a given $f \in \Lambda$, an $\mathcal{F}_T$-measurable random variable Y, and a finite variation process $\{D_t : 0 \le t \le T\}$, suppose there is some $p \in [1, \infty)$ such that $\int_0^T |f(0,\omega,t)|\,dt$, Y, and $\int_0^T |dD_t|$ are all in L^p, and suppose that there is some constant $k > 0$ such that f is k-lipschitz in its v argument: $|f(x,\omega,t) - f(y,\omega,t)| \le k|x-y|$ for all (ω,t) and all $(x,y) \in I\!R^2$. Then there exists a unique solution V to the recursive stochastic integral equation*

$$V_t = \mathbf{E}\left[\int_t^T f(V_s,\omega,s)\,ds + dD_s + Y \,\middle|\, \mathcal{F}_t\right], \quad t \le T,$$

in the space $\mathcal{V}^p$ of all RCLL adapted processes that satisfy $\mathbf{E}[(\int_0^T |V_t|\,dt)^p] < \infty$.

PROOF: For $p > 1$, this theorem is a simple generalization of the theorem proved in Appendix A of Duffie and Epstein (1992), with an added $\int_t^T dD_s$ term here. The proof is almost identical. See Antonelli (1994) for the case of $p = 1$. ∎

LEMMA 2. *Let $\tau > 0$ be a stopping time. Define a process U by $U_t = \mathbf{1}_{\{t \ge \tau\}}$. Let A be the unique right-continuous, increasing, predictable process with $A_0 = 0$ such that $U - A$ is a martingale. (The existence and uniqueness of A follows from the fact that U is a special semimartingale.) Then A is continuous if and only if τ is totally inaccessible.*

PROOF: If τ is totally inaccessible, Theorem 11 in Protter (1990) (pp. 99) shows that A is continuous. The converse can be shown as follows.

Suppose A is continuous but τ is not totally inaccessible. Then there exists a predictable stopping time S such that $P(\{\omega : \tau(\omega) = S(\omega) < \infty\}) > 0$. Let $B = \{\omega : \tau(\omega) = S(\omega) < \infty\}$. Let a sequence of stopping times $S^{(n)} \uparrow S$, and $S^{(n)} < S$ for $S \ne 0$. Then, since $U - A$ is a right continuous martingale, Doob's Optional Sampling Theorem implies that

$$\begin{aligned}
\mathbf{E}\big[A_S - A_{S^{(n)}}\big] &= \mathbf{E}\big[U_S - U_{S^{(n)}}\big] \\
&\ge \mathbf{E}\big[\big(U_S - U_{S^{(n)}}\big)\mathbf{1}_B\big] \qquad \text{(since } U \text{ is increasing)} \\
&= \mathbf{E}\big[\big(U_\tau - U_{S^{(n)}}\big)\mathbf{1}_B\big] \qquad \text{(since } S = \tau \text{ on } B\text{)} \\
&= \mathbf{E}\big(\mathbf{1}_B\big) \qquad \text{(since } U_\tau = 1 \text{ and } U_{S^{(n)}} = 0 \text{ on } B\text{)} \\
&> 0.
\end{aligned}$$

But $A_S \geq A_S - A_{S^{(n)}} \downarrow 0$ (since A is positive, increasing, and continuous), so the integrability of A_S and the Dominated Convergence Theorem imply that $\lim_{n\to\infty} \mathbf{E}\left[A_S - A_{S^{(n)}}\right] = 0$, a contradiction. Thus τ is totally inaccessible. ▮

For completeness, we restate here a version of the Stochastic Gronwall-Bellman Inequality, due to Costis Skiadas, and as originally stated in Lemma B2 of Duffie and Epstein (1992). We also added a strict inequality result to the lemma. This lemma is used in the proofs of Propositions 2 and 3.

Lemma 3. *Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space whose filtration $\{\mathcal{F}_t : t \in [0,T]\}$ satisfies the usual conditions. Suppose that Y is an integrable optional process, α is a constant, and G is a measurable process. Suppose, for all t, that $s \mapsto Y_s$ is right continuous and $s \mapsto \mathbf{E}(Y_s|\mathcal{F}_t)$ is continuous almost surely. If $Y_T \geq 0$ a.s. and, for all t, $G_t \geq -\alpha|Y_t|$ a.s. and $Y_t = \mathbf{E}[\int_t^T G_s\, ds + Y_T|\mathcal{F}_t]$ a.s. Then, for all t, $Y_t \geq 0$ a.s. Furthermore, for given t, let*

$$A = \{(\omega, u) : u \geq t, G_u(\omega) > -\alpha|Y_u(\omega)|\};$$

$$B = \left\{\omega : \mathbf{E}_Q\left[\int_0^T \mathbf{1}_A(\,\cdot\,, u)\, du \,\middle|\, \mathcal{F}_t\right] > 0\right\} \in \mathcal{F}_t.$$

Then $Y_t > 0$ on B.

Proof: See Lemma B2 in Duffie and Epstein (1992). The added strict inequality part can be proved using a strict version of the Gronwall-Bellman inequality. ▮

The following lemma is a simple generalization of Lemma 1 of Duffie, Schroder, and Skiadas (1993). We use it to simplify the proof of Proposition 1.

Lemma 4. *Let V be a semimartingale satisfying $\mathbf{E}\left(\int_0^T |V_t|\ dt\right) < \infty$, let D be a semimartingale satisfying $\mathbf{E}\left(\int_0^T |dD_t|\right) < \infty$, and let G be a progressively measurable process such that $\mathbf{E}\left(\int_0^T |G_t|\, dt\right) < \infty$. There exists a martingale m such that $dV_t = -G_t\, dt + dD_t + dm_t$, $t \in [0,T]$, if and only if*

$$V_t = \mathbf{E}\left(\int_t^T G_u\, du + dD_u + V_T \,\middle|\, \mathcal{F}_t\right), \quad t \in [0,T].$$

Proof: See Duffie, Schroder, and Skiadas (1993), Lemma 1. ▮

Proof of Proposition 1

PROOF: Substituting the swap cumulative dividend formula (2.3) into the pricing formula (2.2) and making use of the decomposition $H^i = A^i + M^i$, we obtain

$$\begin{aligned} S_t e^{-\int_0^t r_u\, du} &= \mathbf{E}_Q \left[\int_t^T e^{-\int_0^s r_u\ du} \left[\mathbf{1}_{\{s<\tau\}}\, dD_s + \mathbf{1}_{\{s\le\tau\}} \left(Z_s^1\, dH_s^1 + Z_s^2\, dH_s^2 \right) \right] \;\middle|\; \mathcal{F}_t \right] \\ &= -\int_0^t e^{-\int_0^s r_u\, du} \mathbf{1}_{\{s<\tau\}} \left[\left(Z_s^1 h_s^1 + Z_s^2 h_s^2 \right)\, ds + dD_s \right] + m_t, \end{aligned}$$

for some Q-martingale m. Using integration by parts and noting that $S_t = S_t \mathbf{1}_{\{t<\tau\}}$, we have

$$\begin{aligned} dS_t &= -\left(Z_t^1 h_t^1 + Z_t^2 h_t^2 - r_t S_t \right) \mathbf{1}_{\{t<\tau\}}\, dt - \mathbf{1}_{\{t<\tau\}}\, dD_t + d\hat{m}_t \\ &= \left[R_t(S_t,\omega) - h_t^1(S_t,\omega) - h_t^2(S_t,\omega) \right] S_t\, dt - \mathbf{1}_{\{t<\tau\}}\, dD_t + d\hat{m}_t, \end{aligned} \tag{A.1}$$

for some Q-martingale $\hat{m}$. Lemma 4 then implies that

$$S_t = \mathbf{E}_Q \left[\int_t^T - \left[R_u(S_u,\omega) - h_u^1(S_u,\omega) - h_u^2(S_u,\omega) \right] S_u\, du + \mathbf{1}_{\{u<\tau\}}\, dD_u \;\middle|\; \mathcal{F}_t \right], \quad t \le T. \tag{A.2}$$

Lemma 1 shows that there exists a unique solution S to this recursive integral equation.

Suppose that the map $v \mapsto R_t(v,\omega)v$ is uniformly Lipschitz. Then Lemma 1 shows that there exists a unique solution V for (2.5). Lemma 4 implies that

$$dV_t = R_t(V_t,\omega) V_t\, dt - dD_t + dM_t,$$

for some Q-martingale M. Suppose, further, that $\Delta V_\tau = 0$ almost surely. Then, using integration by parts, we have

$$\begin{aligned} d\left(V_t \mathbf{1}_{\{t<\tau\}} \right) &= d\left[V_t (1 - H_t^1)(1 - H_t^2) \right] \\ &= \left(1 - H_{t-}^1\right)\left(1 - H_{t-}^2\right) dV_t - V_{t-} \left[\left(1 - H_{t-}^2\right) dH_t^1 + \left(1 - H_{t-}^1\right) dH_t^2 \right] \\ &= \mathbf{1}_{\{t\le\tau\}} \left[dV_t - \left(h_t^1(V_t,\omega) + h_t^2(V_t,\omega) \right) V_t\, dt \right] + d\bar{m}_t \\ &= \left[R_t(V_t,\omega) - h_t^1(V_t,\omega) - h_t^2(V_t,\omega) \right] \mathbf{1}_{\{t<\tau\}} V_t\, dt - \mathbf{1}_{\{t<\tau\}}\, dD_t + d\hat{M}_t, \quad a.s., \end{aligned} \tag{A.3}$$

for some Q-martingales $\bar{m}$ and $\hat{M}$. Lemma 4 then implies that $V_t \mathbf{1}_{\{t<\tau\}}$ satisfies (A.2) and must be indistinguishable from its unique solution S. ∎

Proof of Proposition 2

PROOF: Define $\rho_t = r_t + \hat{s}_t^1$ as the discount rate for counterparty 1, and let $\eta_t = \hat{s}_t^2 - \hat{s}_t^1 \ge 0$ represent the asymmetry of default spreads between the two parties. For convenience, we change the numeraire as follows. For any swap with cumulative dividend D and value process V, we define $\tilde{D}_t = \int_0^t e^{-\int_0^s \rho_u\, du}\, dD_s$ and $\tilde{V}_t = e^{-\int_0^t \rho_u\, du} V_t$. Then, with Ito's lemma, we can rewrite (2.5) as

$$\tilde{V}_t = \mathbf{E}_Q \left[\int_t^T -\eta_u \mathbf{1}_{\{\tilde{V}_u \geq 0\}} \tilde{V}_u \, du + d\tilde{D}_u \,\middle|\, \mathcal{F}_t \right], \quad t \leq T. \tag{A.4}$$

Applying (A.4) to $\tilde{V}^a$, $\tilde{V}^b$, and $\tilde{V}^{ab}$, we have

$$\tilde{V}_t^{ab} - \tilde{V}_t^a - \tilde{V}_t^b = \mathbf{E}_Q \left[\int_t^T -\eta_u \left[\max(\tilde{V}_u^{ab}, 0) - \max(\tilde{V}_u^a, 0) - \max(\tilde{V}_u^b, 0) \right] du \,\middle|\, \mathcal{F}_t \right].$$

Defining $Y = \tilde{V}^{ab} - \tilde{V}^a - \tilde{V}^b$, we have $Y_T = 0$. Let α denote an upper bound of $|\eta_t|$. Then, using $\eta \geq 0$, we have

$$\begin{aligned} G_t &\equiv -\eta_t \left[\max(\tilde{V}_t^{ab}, 0) - \max(\tilde{V}_t^a, 0) - \max(\tilde{V}_t^b, 0) \right] \\ &\geq -\eta_t \left[\max(\tilde{V}_t^{ab}, 0) - \max(\tilde{V}_t^a + \tilde{V}_t^b, 0) \right] \\ &\geq -\eta_t \max(\tilde{V}_t^{ab} - \tilde{V}_t^a - \tilde{V}_t^b, 0) \\ &\geq -\alpha |\tilde{V}_t^{ab} - \tilde{V}_t^a - \tilde{V}_t^b| \\ &= -\alpha |Y_t|. \end{aligned} \tag{A.5}$$

Applying to Y and G a consequence of the Stochastic Gronwall-Bellman Inequality due to Costis Skiadas that is stated in Lemma 2B of Duffie and Epstein (1992) (and restated in this paper, with an added strict inequality result, as Lemma 3 in Appendix A), we conclude that $Y \geq 0$ and thus that $\tilde{V}^{ab} \geq \tilde{V}^a + \tilde{V}^b$.

If, for given t, $(\omega, u) \in A$, that is, $u \geq t$, $\eta_u(\omega) > 0$, and $\tilde{V}_u^a(\omega)$ and $\tilde{V}_u^b(\omega)$ have opposite signs. Then we have $\max(\tilde{V}_u^a(\omega), 0) + \max(\tilde{V}_u^b(\omega), 0) > \max(\tilde{V}_u^a(\omega) + \tilde{V}_u^b(\omega), 0)$. Inequality (A.5) is then strict on A, implying $G_u > -\alpha|Y_u|$ on A. Applying the strict inequality part of Lemma 3 in Appendix A, we have $Y_t > 0$ on B and thus $\tilde{V}_t^{ab} > \tilde{V}_t^a + \tilde{V}_t^b$ on B. ∎

Proof of Proposition 3

PROOF: Define $\rho_t = r_t + \hat{s}_t^1$ and, for any swap with cumulative dividend D and value process V, define $\tilde{D}_t = \int_0^t e^{-\int_0^s \rho_u\, du}\, dD_s$ and $\tilde{V}_t = e^{-\int_0^t \rho_u\, du} V_t$. Applying (A.4) to $\tilde{V}^F$ and $\tilde{V}^G$, we have

$$\tilde{V}_t^F = \mathbf{E}_Q \left[\int_t^T -\eta_u \max(\tilde{V}_u^F, 0)\, du + d\tilde{D}_u \,\middle|\, \mathcal{F}_t \right], \quad t \leq T; \tag{A.6}$$

$$\tilde{V}_t^G = \mathbf{E}_Q \left[\int_t^T -\eta_u \max(\tilde{V}_u^G, 0)\, du + d\tilde{D}_u \,\middle|\, \mathcal{G}_t \right], \quad t \leq T. \tag{A.7}$$

Define process U by $U_t = \mathbf{E}_Q\left[\tilde{V}_t^G | \mathcal{F}_t\right]$ for all t. Since $-\eta_t \max(v, 0)$ is concave in v, applying conditional Jensen's inequality and conditional version of Fubini's theorem to (A.7), and noting that η_t is measurable with respect to $\mathcal{G}_t$, we have

$$U_t \leq \mathbf{E}_Q\left[\int_t^T -\eta_u \max(U_u, 0)\, du + d\tilde{D}_u \,\middle|\, \mathcal{F}_t\right], \quad t \leq T.$$

Combining this equation with (A.6), we have $\tilde{V}_T^F - U_T = 0$, and

$$\begin{aligned}\tilde{V}_t^F - U_t &= \mathbf{E}_Q\left[\int_t^T -\eta_u \left[\max(\tilde{V}_u^F, 0) - \max(U_u, 0)\right] du \,\middle|\, \mathcal{F}_t\right] \\ &\geq \mathbf{E}_Q\left[\int_t^T -\eta_u \left|\tilde{V}_u^F - U_u\right| du \,\middle|\, \mathcal{F}_t\right], \quad t \leq T.\end{aligned}$$

Lemma 3 then implies that $\tilde{V}_t^F \geq U_t$ for all t, and, at $t = 0$, $V_0^F \geq V_0^G$. ▮

Proof of Corollary 1

PROOF: The computation procedure of $\nabla \tilde{V}_0(\bar{\eta}; \eta)$ is identical to the proof of Theorem 2 in Duffie and Skiadas (1994). ▮

Appendix B: Valuation with Possibility of Simultaneous Defaults

This appendix generalizes the valuation model to include cases in which the two counterparties of a swap contract could default simultaneously with positive probability.

To account for the possibility of simultaneous defaults by the two counterparties, we introduce a *simultaneous default indicator function* $H_t^{12} = \mathbf{1}_{\{t \geq \tau^1\}} \mathbf{1}_{\{\tau^1 = \tau^2\}}$, a stochastic process that is equal to one if simultaneous defaults have occurred, and zero otherwise. (One can show that H^{12} is the indicator function of a (totally inaccessible) stopping time defined by $\tau^1 \mathbf{1}_{\{\tau^1 = \tau^2\}} + \infty \mathbf{1}_{\{\tau^1 \neq \tau^2\}}$ and is an adapted and right-continuous increasing process.) Appealing again to the Doob-Meyer decomposition, we write $H^{12} = A^{12} + M^{12}$, where A^{12} is a predictable and right-continuous increasing process with $A_0^{12} = 0$, and M^{12} is a Q-martingale. Furthermore, A^{12} is continuous (see footnote 8). We again assume that there exists a (progressively measurable) non-negative process h^{12} as the hazard rate for simultaneous defaults such that

$$A_t^{12} = \mathbf{1}_{\{t < \tau\}} \int_0^t h_s^{12}\, ds = \int_0^t h_s^{12} \mathbf{1}_{\{s \leq \tau\}}\, ds, \quad t \geq 0.$$

The boundedness of h^1 and h^2 then implies[13] the boundedness of h^{12}. Again, we allow h^{12} to depend on V_t. That is, we take $h_t^{12}(\omega) = \lambda^{12}(V_t(\omega), \omega, t)$, where λ^{12} is in Λ.

We also need to specify the settlement payoff to party 1 in the event of simultaneous defaults. Given that $\tau^1(\omega) = \tau^2(\omega) = t$ and that the market value $V_{t-}(\omega)$ of the swap just prior to the default is some number v, we assume that party 1 receives

$$Z^{12}(\omega, t) = \psi^1(v, \omega, t)\, v\, \mathbf{1}_{\{v<0\}} + \psi^2(v, \omega, t)\, v\, \mathbf{1}_{\{v\geq 0\}},$$

where ψ^1 and ψ^2 are both in Λ and ψ^i represents the fraction of market value payment by party i in the event of default when it has negative net market value.

The cumulative dividend process of the swap for counterparty 1 can be generally written as

$$X_t = \int_0^t \mathbf{1}_{s<\tau}\, dD_s + \mathbf{1}_{s\leq\tau}\Big[Z_s^1(dH_s^1 - dH_s^{12}) + Z_s^2(dH_s^2 - dH_s^{12}) + Z_s^{12} dH_s^{12}\Big], \quad t \leq T.$$

The first term on the right hand side is the prearranged swap payment before default. The second term is the settlement payoff in three different default scenarios: party 1 alone defaults, party 2 alone defaults, and both parties default simultaneously. Risk-neutral valuation under measure Q again results in Proposition 1 with the effective discount rate taking the following more general form:

$$\begin{aligned} R_t = r_t &+ \Big[(1-\varphi_t^1)(h_t^1 - h_t^{12})(1-\bar{\varphi}_t^1)(h_t^2 - h_t^{12}) + (1-\psi_t^1)h_t^{12}\Big]\mathbf{1}_{\{V_t<0\}} \\ &+ \Big[(1-\varphi_t^2)(h_t^2 - h_t^{12})(1-\bar{\varphi}_t^2)(h_t^1 - h_t^{12}) + (1-\psi_t^2)h_t^{12}\Big]\mathbf{1}_{\{V_t\geq 0\}}. \end{aligned}$$

Appendix C: Markov Valuation for Swaps with Payments in Arrears

For a swap with payments in arrears, we use $\bar{t}_n$ to denote the reset date for the payment date t_n. Usually, the reset date is no earlier than the last payment date. So we assume

$$0 \leq \bar{t}_1 \leq t_1 \leq \bar{t}_2 \leq t_2 \leq \cdots \leq \bar{t}_N \leq t_N = T. \qquad (C.1)$$

To describe the Markov structure of the value process, we need to use two state variables, the short term LIBOR rate $\hat{\rho}_t$ on the reset date and the spot short term LIBOR rate ρ_t, to

[13] $h_t^{12} = \lim_{u\downarrow 0} \frac{Q[t<\tau^1=\tau^2\leq t+u|\mathcal{F}_t]}{u} \leq \min_{i=1,2} \lim_{u\downarrow 0} \frac{Q[t<\tau^i\leq t+u|\mathcal{F}_t]}{u} = \min_{i=1,2} h_t^i.$

describe the value process: $V(\omega, t) = J(\hat{\rho}_t, \rho_t, t)$. Note that, for $t \in [\bar{t}_n, \bar{t}_{n+1})$, $\hat{\rho}(t) = \rho(\bar{t}_n)$ and is constant. The general Markovian setting equations (2.11) – (2.13) for the value process J can now be written as

$$\frac{1}{2}\sigma^2 y J_{yy} + \kappa(\mu - y)J_y + J_t - \left[y + \bar{\eta}(y)\mathbf{1}_{\{J \geq 0\}}\right]J = 0, \quad y \geq 0, t_{n-1} \leq t < \bar{t}_n.$$

The boundary conditions are given by

$$J(\hat{y}, y, T) = 0, \qquad y \in [0, \infty),$$

and, for $1 \leq n \leq N$,

$$J(\hat{y}, y, t_n-) = J(\hat{y}, y, t_n) + \delta_n(\hat{y});$$
$$J(\hat{y}, y, \bar{t}_n-) = J(y, y, \bar{t}_n),$$

where δ_n describes the functional dependence between the net payment to counterparty 1 on payment date t_n and the short term LIBOR rate on the corresponding reset date $\bar{t}_n$ as given in (3.5). Note that, within each sub-interval of the form $[t_n - 1, \bar{t}_n)$, $J(\hat{y}, y, t)$ does not depend on $\hat{y}$.

Appendix D: The Impact on Swap Rates of Netting a Fixed-for-LIBOR Swap Against a Fixed-for-Inverse-Floater Swap

The intuitive discussion leading to Figure 1 can be verified by calculating $\bar{C}_k(\rho_0)$ and studying its dependence on k for a given initial LIBOR rate ρ_0. We also study the impact of netting on the marginal value of the new swap by calculating, numerically, the quantity $\bar{C}_k(\rho_0) - \bar{C}(\rho_0)$.

In order to compute $\bar{C}_k(\rho_0)$, we need to value the netted portfolio composed of the new and old swaps. Table 3 in Section 4 lists all fixed and floating payments when each swap (or swap portfolio) is marked to market at $t = 0$.

The following observations can help simplify the calculation. First, due to netting of the fixed payment with the floating payment, the value of a swap remains unchanged if each party adds the same amount to its promised payment. Second, if a swap has zero initial market value and each party's default spread does not depend on the value of the swap, then multiplying all promised payments of each party by a positive constant leaves the swap at zero initial value. With the help of these observations, we can obtain the new swap term rate (with netting), $\bar{C}_k(\rho_0)$, by comparing the payments of the netted portfolio with those of either the old swap or the new swap:

$$\frac{1}{2}\left[\hat{C}(\rho_0,k)+\bar{C}_k(\rho_0)\right]=\begin{cases}\frac{1}{2}(1-k)\bar{C}(\rho_0)+(1+k)L(\rho_0), & \text{if } k<1;\\ \frac{1}{2}\hat{C}(\rho_0,k-1)+L(\rho_0), & \text{if } k\geq 1.\end{cases} \qquad (D.1)$$

This result can be simplified further by obtaining a simple analytic relationship between $\hat{C}(\rho_0,k)$ and k, for any given ρ_0. Applying the above two observations again, we have

$$\frac{1}{2}\hat{C}(\rho_0,k)=\begin{cases}\frac{1}{2}(-k)\bar{C}(\rho_0)+(1+k)L(\rho_0), & \text{if } k\leq 0;\\ \frac{1}{2}k\hat{C}(\rho_0,1)+(1-k)L(\rho_0), & \text{if } k>0.\end{cases} \qquad (D.2)$$

Combining (D.1) with (D.2), we obtain

$$\bar{C}_k(\rho_0)=\begin{cases}\bar{C}_0(\rho_0), & \text{if } k\leq 0;\\ \bar{C}_0(\rho_0)-k\left[\bar{C}_0(\rho_0)-\bar{C}_1(\rho_0)\right], & \text{if } 0<k<1;\\ \bar{C}_1(\rho_0), & \text{if } k\geq 1,\end{cases} \qquad (D.3)$$

where, for all ρ_0,

$$\begin{aligned}\bar{C}_0(\rho_0)&=\bar{C}(\rho_0);\\ \bar{C}_1(\rho_0)&=4L(\rho_0)-\hat{C}(\rho_0,1).\end{aligned} \qquad (D.4)$$

Equation (D.3) confirms our intuition about the dependence of the new swap term rate on the parameter k of the existing inverse floater swap that is to be netted with the new swap, and shows that, for any given ρ_0, the map $k\mapsto \bar{C}_k(\rho_0)$ can be determined by calculating three numbers, $\bar{C}(\rho_0)$, $L(\rho_0)$, and $\hat{C}(\rho,1)$. The first two of these numbers are calculated in Section 3. The third can be calculated using the same finite-difference algorithm used in Section 3.

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