Risk and Valuation of Collateralized Debt Obligations

Darrell Duffie and Nicolae Gârleanu

Graduate School of Business
Stanford University*

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Abstract: This paper addresses the risk analysis and market valuation of collateralized debt obligations (CDOs). We illustrate the effects of correlation and prioritization for the market valuation, diversity score, and risk of CDOs, in a simple jump-diffusion setting for correlated default intensities.

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*Duffie and Gârleanu are with the Graduate School of Business, Stanford University, Stanford, CA 94305-5015. Email: duffie@stanford.edu and ngarlea@leland.stanford.edu. This work is supported in part by a grant from the Gifford Fong Associates Fund at the Graduate School of Business, Stanford University. We are grateful for discussions with Ken Singleton of Stanford University and Sergio Kostek of Morgan Stanley Dean Witter. Helpful comments and research assistance were provided by Jun Pan. We are also grateful for several helpful suggestions for improvement by Mr. Kazuhisa Uehara of the Fuji Research Institute Corporation.

1 Introduction

This paper addresses the risk analysis and market valuation of collateralized debt obligations (CDOs). We illustrate the effects of correlation and prioritization for the market valuation, diversity score, and risk of CDOs, in a simple jump-diffusion setting for correlated default intensities.

A CDO is an asset-backed security whose underlying collateral is typically a portfolio of bonds (corporate or sovereign) or bank loans. A CDO cash-flow structure allocates interest income and principal repayments from a collateral pool of different debt instruments to a prioritized collection of CDO securities, which we shall call *tranches*. While there are many variations, a standard prioritization scheme is simple subordination: Senior CDO notes are paid before mezzanine and lower-subordinated notes are paid, with any residual cash flow paid to an equity piece. Some illustrative examples of prioritization are provided in Section 4.

A *cash-flow* CDO is one for which the collateral portfolio is not subjected to active trading by the CDO manager, implying that the uncertainty regarding interest and principal payments to the CDO tranches is determined mainly by the number and timing of defaults of the collateral securities. A *market-value* CDO is one in which the CDO tranches receive payments based essentially on the mark-to-market returns of the collateral pool, as determined in large part by the trading performance of the CDO manager. In this paper, we concentrate on cash-flow CDOs, avoiding an analysis of the trading behavior of CDO managers.

A generic example of the contractual relationships involved in a CDO is shown in Figure 1, taken from Schorin and Weinreich [1998]. The collateral manager is charged with the selection and purchase of collateral assets for the SPV. The trustee of the CDO is responsible for monitoring the contractual provisions of the CDO. Our analysis assumes perfect adherence to these contractual provisions. The main issue that we address is the impact of the joint distribution of default risk of the underlying collateral securities on the risk and valuation of the CDO tranches. We are also interested in the efficacy of alternative computational methods, and the role of "diversity scores," a measure of the risk of the CDO collateral pool that has been used for CDO risk analysis by rating agencies.

Our findings are as follows. We show that default time correlation has a significant impact on the market values of individual tranches. The priority of the senior tranche, by which it is effectively "short a call option" on the performance of the underlying collateral pool, causes its market value to decrease with the risk-neutral default-time correlation, fixing the (risk-neutral) distribution of individual default times. The value of the equity piece, which resembles a call option, increases with correlation. There is no clear Jensen effect, however, for intermediate tranches. With sufficient over-collateralizaion, the option "written" (to the lower tranches) dominates, but it is the other way around for sufficiently low levels of over-collateralization. Spreads, at least for mezzanine and senior tranches, are not especially sensitive to the "lumpiness" of information arrival regarding credit quality, in that replacing the contribution of diffusion with jump risks (of various types), holding constant the degree of mean reversion and the term structure of credit spreads, plays a relatively small role. Regarding alternative computational methods, we show that if (risk-neutral) diversity scores can be evaluated accurately, which is computationally simple in the framework we propose, these scores can be used to obtain good approximate market valuations for reasonably well collateralized tranches. Currently the weakest link in the chain of CDO analysis is the availability of empirical data that would bear on the correlation, actual or risk-neutral, of default.

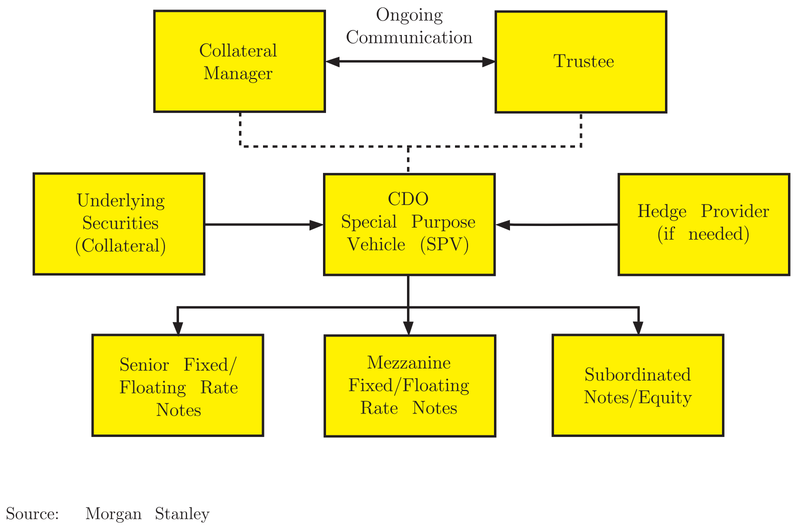

Figure 1: Typical CDO Contractual Relationships

2 Some Economics of CDO Design and Valuation

In perfect capital markets, CDOs would serve no purpose; the costs of constructing and marketing a CDO would inhibit its creation. In practice, CDOs address some important market imperfections. First, banks and certain other financial institutions have regulatory capital requirements that make it valuable for them to securitize and sell some portion of their assets, reducing the amount of (expensive) regulatory capital that they must hold. Second, individual bonds or loans may be illiquid, leading to a reduction in their market values. Securitization may improve liquidity, and thereby raise the total valuation to the issuer of the CDO structure.

In light of these market imperfections, at least two classes of CDOs are popular. The *balance-sheet CDO*, typically in the form of a collateralized loan obligation (CLO), is designed to remove loans from the balance sheets of banks, achieving capital relief, and perhaps also increasing the valuation of the assets through an increase in liquidity.[1] An *arbitrage* CDO, often underwritten by an investment bank, is designed to capture some fraction of the likely difference between the total cost of acquiring collateral assets in the secondary market and the value received from management fees and the sale of the associated CDO structure. Balance-sheet CDOs are normally of the cash-flow type. Arbitrage CDOs may be collateralized bond obligations (CBOs), and have either cash-flow or market-value structures.

Among the sources of illiquidity that promote, or limit, the use of CDOs are adverse selection, trading costs, and moral hazard.

With regard to adverse selection, there may be a significant amount of private information regarding the credit quality of a junk bond or a bank loan. An investor may be concerned about being "picked off" when trading such instruments. For instance, a potentially better-informed seller has an option to trade or not at the given price. The value of this option is related to the quality of the seller's private information. Given the risk of being picked off, the buyer offers a price that, on average, is below the price at which the asset would be sold in a setting of symmetric information. This reduction in price due to adverse selection is sometimes called a "lemon's premium" (Akerlof [1970]). In general, adverse selection cannot be eliminated by securitization of assets in a CDO, but it can be mitigated. The seller achieves a higher total valuation (for what is sold and what is retained) by designing the CDO structure so as to concentrate into small subordinate tranches the majority of the risk about which there may be fear of adverse selection. A large senior tranche, relatively immune to the effects of adverse selection, can be sold at a small lemon's premium. The issuer can retain, on average, significant fractions of smaller subordinate tranches that are more subject to adverse selection. For models supporting this design and retention behavior, see DeMarzo [1998], DeMarzo [1999], and DeMarzo and Duffie [1999].

For a relatively small junk bond or a single bank loan to a relatively obscure borrower, there can be a small market of potential buyers and sellers. This is not unrelated to the effects of adverse selection, but also depends on the total size of an issue. In order to

[1]A synthetic CLO differs from a conventional CLO in that the bank originating the loans does not actually transfer ownership of the loans to the special-purpose vehicle (SPV), but instead uses credit derivatives to transfer the default risk to the SPV. The direct sale of loans to SPVs may sometimes compromise client relationships or secrecy, or can be costly because of contractual restrictions on transferring the underlying loans. Unfortunately, regulations do not always provide the same capital relief for a synthetic CLO as for an standard balance-sheet CLO. See Punjabi and Tierney [1999].)

sell such an illiquid asset quickly, one may be forced to sell at the highest bid among the relatively few buyers with whom one can negotiate on short notice. Searching for such buyers can be expensive. One's negotiating position may also be poorer than it would be in an active market. The valuation of the asset is correspondingly reduced. Potential buyers recognize that they are placing themselves at the risk of facing the same situation in the future, resulting in yet lower valuations. The net cost of bearing these costs may be reduced through securitization into relatively large homogeneous senior CDO tranches, perhaps with significant retention of smaller and less easily traded junior tranches.

Moral hazard, in the context of CDOs, bears on the issuer's or CDO manager's incentives to select high-quality assets for the CDO, to engage in costly enforcement of covenants and other restrictions on the behavior of obligors. By securitizing and selling a significant portion of the cash flows of the underlying assets, these incentives are diluted. Reductions in value through lack of effort are borne to some extent by investors. There may also be an opportunity for "cherry picking," that is, for sorting assets into the issuer's own portfolio or into the SPV portfolio based on the issuer's private information. There could also be "front-running" opportunities, under which a CDO manager could trade on its own account in advance of trades on behalf of the CDO. These moral hazards act *against* the creation of CDOs, for the incentives to select and monitor assets promote greater efficiency, and higher valuation, if the issuer retains a full 100-percent equity interest in the asset cash flows. The opportunity to reduce other market imperfections through a CDO may, however, be sufficiently large to offset the effects of moral hazard, and result in securitization, especially in light of the advantage of building and maintaining a reputation for not exploiting CDO investors. The issuer has an incentive to design the CDO in such a manner that the issuer retains a significant portion of one or more subordinate tranches that would be among the first to suffer losses stemming from poor monitoring or asset selection, demonstrating a degree of commitment to perform at high effort levels by the issuer. Likewise, for arbitrage CDOs, a significant portion of the management fees may be subordinated to the issued tranches. (See Schorin and Weinreich [1998].) In light of this commitment, investors may be willing to pay more for the tranches in which they invest, and the total valuation to the issuer is higher than would be the case for an un-prioritized structure, such as a straight-equity pass-through security. Innes [1990] has a model supporting this motive for security design.

One of a pair of CLO cash-flow structures issued by NationsBank in 1997 is illustrated in Figure 2. A senior tranche of $2 billion in face value is followed by successively lower-subordination tranches. The ratings assigned by Fitch are also illustrated. The bulk of the underlying assets are floating-rate NationsBank loans rated BBB or BB. Any fixed-rate loans were hedged, in terms of interest rate risk, by fixed-to-floating interest-rate swaps. As predicted by theory, the majority of the (unrated) lowest tranche was retained.

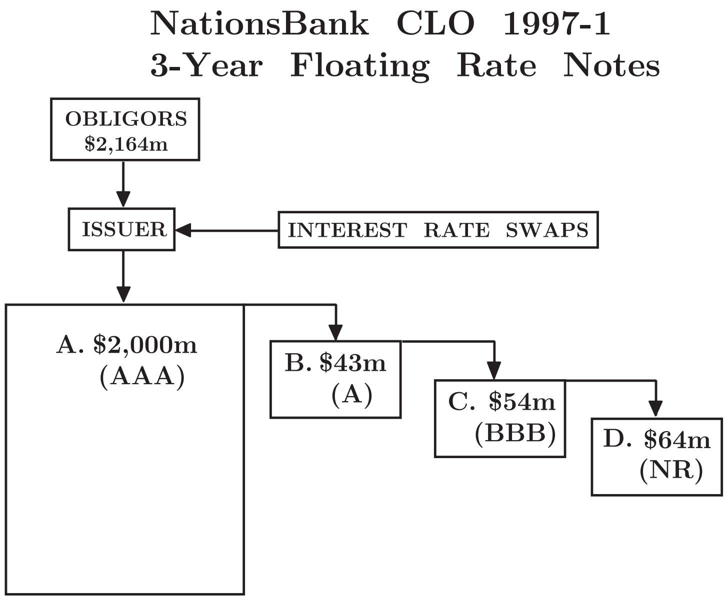

Figure 2: NationsBank 1997-1 CLO Tranches (Source: Fitch)

Our valuation model does not deal directly with the effects of market imperfections. It takes as given the default risk of the underlying loans, and assumes that investors are symmetrically informed. While this is not perfectly realistic, it is not necessarily inconsistent with the roles of moral hazard or adverse selection in the original security design. For example, DeMarzo and Duffie [1999] demonstrate a fully-separating equilibrium in which the sale price of the security or the amount retained by the seller signal to all investors any of the seller's privately-held value-relevant information. As for moral hazard, the efforts of the issuer or manager are, to a large extent, determined by the security design and the fractions retained by the issuer. Once these are known, the default risk of the underlying debt is also determined. Our simple model does not, however, account for the valuation effects of many other forms of market imperfections. Moreover, it is generally difficult to infer separate risk premia for default timing and default-recovery from the prices of the underlying debt and market risk-free interest rates. (See Duffie and Singleton [1999].) These risk premia play separate roles in the valuation of CDO tranches. We are simply taking these risk-premia as given, in the form of parametric models for default timing and recovery distributions under an equivalent martingale measure, as discussed in Section 3.

3 Default Risk Model

This section lays out some of the basic default modeling for the underlying collateral. First, we propose a simple model for the default risk of one obligor. Then we turn to the multi-issuer setting.

3.1 Obligor Default Intensity

We suppose that each underlying obligor defaults at some conditional expected arrival rate. The idea is that, at each time t before the default time τ of the given obligor, the default arrives at some "intensity" $\lambda(t)$, given all currently available information, denoted $\mathcal{F}_t$, in that we have the approximation

$$P\left(\tau < t + \Delta t \;\middle|\; \mathcal{F}_t\right) \simeq \lambda(t)\Delta t, \tag{1}$$

for each "small" time interval $\Delta t > 0$. Supporting technical details are provided in Appendix A. For example, measuring time, as we shall, in years, a current default intensity of 0.04 implies that the conditional probability of default within the next three months is approximately 0.01. Immediately after default, the intensity drops to zero. Stochastic variation in the intensity over time, as new conditioning information becomes available, reflects any changes in perceived credit quality. Correlation across obligors in the changes over time of their credit qualities is reflected by correlation in the changes of those obligors' default intensities. Indeed, our model has the property that *all* correlation in default timing arises in this manner. [See Appendix A for details on this point.] An alternative would be a model in which simultaneous defaults could be caused by certain common credit events, such as a multivariate-exponential model, as explained by Duffie and Singleton [1998]. Another alternative is a contagion model, such as the static "infectious default" model of Davis and Lo [1999]. We will call a stochastic process λ a *pre-intensity* for a stopping time τ if, whenever $t < \tau$: *(i)* the current intensity is λ_t, and *(ii)*

$$P\left(\tau > t + s \;\middle|\; \mathcal{F}_t\right) = E\left[\exp\left(\int_t^{t+s} -\lambda_u \, du\right) \;\middle|\; \mathcal{F}_t\right], \quad s > 0. \tag{2}$$

A pre-intensity need not fall to zero after default. For example, default at a constant pre-intensity of 0.04 means that the intensity itself is 0.04 until default, and zero thereafter. Regularity conditions for pre-intensity processes are found in Appendix A.

We adopt a pre-intensity model that is a special case of the "affine" family of processes that have been used for this purpose and for modeling short-term interest rates. Specifically, we suppose that each obligor's default time has some pre-intensity process λ solving a stochastic differential equation of the form

$$d\lambda(t) = \kappa(\theta - \lambda(t))\, dt + \sigma\sqrt{\lambda(t)}\, dW(t) + \Delta J(t), \tag{3}$$

where W is a standard Brownian motion and $\Delta J(t)$ denotes any jump that occurs at time t of a pure-jump process J, independent of W, whose jump sizes are independent and exponentially distributed with mean μ and whose jump times are those of an independent Poisson process with mean jump arrival rate ℓ. (Jump times and jump sizes are also independent.)[2] We call a process λ of this form (3) a *basic affine process with parameters* $(\kappa, \theta, \sigma, \mu, \ell)$. These parameters can be adjusted in several ways to control the manner in which default risk changes over time. For example, we can vary the mean-reversion rate κ, the long-run mean $\overline{m} = \theta + \ell\mu/\kappa$, or the relative contributions to the total variance of λ_t that are attributed to jump risk and to diffusive volatility. The long-run variance of λ_t is given by

$$\mathrm{var}_\infty = \lim_{t\to\infty} \mathrm{var}(\lambda_i(t)) = \frac{\sigma^2 \overline{m}}{2\kappa} + \frac{\ell\mu^2}{\kappa}, \tag{4}$$

which can be verified by applying Ito's Formula to compute $E[\lambda_i(t)^2]$, subtracting $E[\lambda_i(t)]^2$, and taking a limit in t. We can also vary the relative contributions to jump risk of the mean jump size μ and the mean jump arrival rate ℓ. A special case is the no-jump ($\ell = 0$) model of Feller [1951], which was used by Cox, Ingersoll, and Ross [1985] to model interest rates. From the results of Duffie and Kan [1996], we can calculate that, for any t and any $s \geq 0$,

$$E\left[\exp\left(\int_t^{t+s} -\lambda_u \, du\right) \,\middle|\, \mathcal{F}_t\right] = e^{\alpha(s)+\beta(s)\lambda(t)}, \tag{5}$$

where explicit solutions for the coefficients $\alpha(s)$ and $\beta(s)$ are provided in Appendix B. Together, (2) and (5) give a simple, reasonably rich, and tractable model for the default-time probability distribution, and how it varies at random over time as information arrives into the market.

3.2 Multi-Issuer Default Model

In order to study the implications of changing the correlation in the default times of the various participations (collateralizing bonds or loans) of a CDO, while holding constant the default-risk model of each underlying obligor, we will exploit the following result, stating that a basic affine model can be written as the sum of independent basic affine models, provided the parameters κ, σ, and μ governing, respectively, the mean-reversion rate, diffusive volatility, and mean jump size are common to the underlying pair of independent basic affine processes.

[2] A technical condition that is sufficient for the existence of a strictly positive solution to (3) is that $\kappa\theta \geq \frac{\sigma^2}{2}$. We do not require it, since none of our results depends on strict positivity.

Proposition 1. Suppose X and Y are independent basic affine processes with respective parameters $(\kappa, \theta_X, \sigma, \mu, \ell_X)$ and $(\kappa, \theta_Y, \sigma, \mu, \ell_Y)$. Then $Z = X + Y$ is a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, \ell)$, where $\ell = \ell_X + \ell_Y$ and $\theta = \theta_X + \theta_Y$.

A proof is provided in Appendix A. This result allows us to maintain a fixed parsimonious and tractable 1-factor Markov model for each obligor's default probabilities, while varying the correlation among different obligors' default times, as explained below.

We suppose that there are N participations in the collateral pool, whose default times $\tau_1, \ldots, \tau_N$ have pre-intensity processes $\lambda_1, \ldots, \lambda_N$, respectively, that are basic affine processes. In order to introduce correlation in a simple way, we suppose that

$$\lambda_i \quad = \quad X_c + X_i, \tag{6}$$

where X_i and X_c are basic affine processes with respective parameters $(\kappa, \theta_i, \sigma, \mu, \ell_i)$ and $(\kappa, \theta_c, \sigma, \mu, \ell_c)$, and where $X_1, \ldots, X_N, X_c$ are independent. By Proposition 1, λ_i is itself a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, \ell)$, where $\theta = \theta_i + \theta_c$ and $\ell = \ell_i + \ell_c$. One may view X_c as a state process governing common aspects of economic performance in an industry, sector, or currency region, and X_i as a state variable governing the idiosyncratic default risk specific to obligor i. The parameter

$$\rho = \frac{\ell_c}{\ell} \tag{7}$$

is the long-run fraction of jumps to a given obligor's intensity that are common to all (surviving) obligor's intensities. One can also see that ρ is the probability that λ_j jumps at time t given that λ_i jumps at time t, for any time t and any distinct i and j. If $X_c(0)/\lambda_i(0) = \rho$ then, for any distinct i and j, we may also treat ρ as the initial instantaneous correlation between λ_i and λ_j, that is, the limiting correlation between $\lambda_i(t)$ and $\lambda_j(t)$ as t goes to zero. In fact, if $\sigma = 0$, then, for all t,

$$\lim_{s \downarrow 0} \ \operatorname{corr}_{\mathcal{F}(t)}(\lambda_i(t+s), \lambda_j(t+s)) = \rho,$$

where $\operatorname{corr}_{\mathcal{F}(t)}(\,\cdot\,)$ denotes $\mathcal{F}_t$-conditional correlation. We also suppose that

$$\theta_c = \rho\theta, \tag{8}$$

maintaining the constant ρ of proportionality between $E(X_c(t))$ and $E(\lambda_i(t))$ for all t, provided that $X_c(0)/\lambda_i(0) = \rho$ (or, in any case, in the limit as t goes to ∞).

3.3 Sectoral, Regional, and Global Risk

Extensions to handle multi-factor risk (regional, sectoral, and other sources), could easily be incorporated with repeated use of Proposition 1. For example, one could suppose that the default time τ_i of the i-th obligor has a pre-intensity $\lambda_i = X_i + Y_{c(i)} + Z$, where the sector factor $Y_{c(i)}$ is common to all issuers in the "sector" $c(i) \subset \{1, \ldots, N\}$ for S different sectors, where Z is common to all issuers, and where $\{X_1, \ldots, X_N, Y_1, \ldots, Y_S, Z\}$ are independent basic affine processes.

If one does not restrict the parameters of the underlying basic affine processes, then an individual obligor's pre-intensity need not itself be a basic affine process, but calculations are nevertheless easy. We can use the independence of the underlying state variables to see that

$$E\left[\exp\left(\int_t^{t+s} -\lambda_i(u)\,du\right) \;\middle|\; \mathcal{F}_t\right] = \exp\left[\alpha(s) + \beta_i(s)X_i(t) + \beta_{c(i)}(s)Y_{c(i)}(t) + \beta_Z(s)Z(t)\right], \tag{9}$$

where $\alpha(s) = \alpha_i(s) + \alpha_{c(i)}(s) + \alpha_Z(s)$, and where all of the α and β coefficients are obtained explicitly from Appendix B, from the respective parameters of the underlying basic affine processes X_i, $Y_{c(i)}$, and Z.

Even more generally, one can adopt multi-factor affine models in which the underlying state variables are not independent. Appendix A summarizes some extensions, and also allows for interest rates that are jointly determined by an underlying multi-factor affine jump-diffusion model.

3.4 Risk-Neutral and Actual Intensities

In order to calculate credit spreads, we adopt a standard arbitrage-free defaultable term-structure model in which, under risk-neutral probabilities defined by some equivalent martingale measure Q, the default time of a given issuer has some "risk-neutral" pre-intensity process, a basic affine process λ^Q with some parameters $(\kappa^Q, \theta^Q, \sigma^Q, \mu^Q, \ell^Q)$. The actual pre-intensity process λ and the risk-neutral pre-intensity process λ^Q are different processes.[3] Neither their sample paths nor their parameters are required by the absence of arbitrage alone to have any particular relationship to each other. As the actual stochastic behavior of bond or CDO prices may be of concern, and as bond prices depend on risk-neutral default intensities, it is sometimes useful to describe the stochastic behavior of λ^Q under the actual probability measure P. For example, we could suppose that, under the actual probability measure P, the risk-neutral intensity λ^Q is also a basic affine process with parameters $(\kappa^{QP}, \theta^{QP}, \sigma^{QP}, \mu^{QP}, \ell^{QP})$. Other than purely technical existence conditions, the only parameter restriction is that $\sigma^Q = \sigma^{QP}$, because the diffusion parameter σ^Q is determined by the path of λ^Q itself, and of course this path is the same under both P and Q.

[3] Both are guaranteed to exist if one does, as shown by Artzner and Delbaen [1995].

At this writing, there is little empirical evidence with which to differentiate the parameters guiding the dynamics of risk-neutral and actual default intensities, nor is there much evidence for distinguishing the actual and risk-neutral dynamics of risk-neutral intensities. Jarrow, Lando, and Turnbull [1997] provide some methods for calibrating risk-neutral default intensities from ratings-based transition data and bond-yield spreads. Duffee [1998] provides some empirical estimates of the actual dynamics of risk-neutral default intensities for relatively low-risk corporate-bond issuers, based on time-series data on corporate bonds.

There is, however, some information available for distinguishing the actual and risk-neutral levels of default probabilities. Sources of information on actual default probabilities include the actuarial incidence of default by credit rating; statistically estimated empirical default-probability studies, such as those of Altman [1968], Lennox [1997], Lundstedt and Hillgeist [1998], and Shumway [1996]; and industry "EDF" default-probability estimates provided to commercial users by KMV Corporation. Data sources from which risk-neutral default probabilities might be estimated include market yield spreads for credit risk and credit-derivative prices. (See, for example, Jarrow and Turnbull [1995] Lando [1998] or Duffie and Singleton [1999] for modeling.) Fons [1994] and Fons and Carty [1995] provide some comparison of term structures of market credit spreads with those implied by actuarial default incidence (corresponding, in effect, to expected discounted cash flows, treating actual default probabilities as though they are risk-neutral, and assuming that they do not change over time).

In this study, we use the same parametric default models for both valuation and risk analysis. Where we are addressing initial market valuation, our parametric assumptions are with regard to the risk-neutral behavior of the risk-neutral pre-intensity processes, unless otherwise indicated. Indeed, in order to keep notation simple and avoid referring back and forth to actual and risk-neutral probabilities, we refer exclusively throughout the remainder to risk-neutral behavior, unless otherwise noted. It is intended, however, that the reader will draw some insight into actual risk analysis from the results presented in this form.

3.5 Recovery Risk

We suppose that, at default, any given piece of debt in the collateral pool may be sold for a fraction of its face value whose risk-neutral conditional expectation given all information $\mathcal{F}_t$ available at any time t before default is a constant $\overline{f} \in (0, 1)$ that does not depend on t. The recovery fractions of the underlying participations are assumed to be independently distributed, and independent of default times and interest rates. (Here again, we are referring to risk-neutral behavior.)

For simplicty, we will assume throughout that the recovered fraction of face value is uniformly distributed on $[0, 1]$. The empirical cross-sectional distribution of recovery of face value for various types of debt, as measured by Moody's Investor Services, is illustrated in Figure 3. (For each debt class, a box illustrates the range from the 25-th percentile to the 75-th percentile.) Among the illustrated classes of debt, the recovery distribution of senior unsecured bonds is the most similar to uniform.

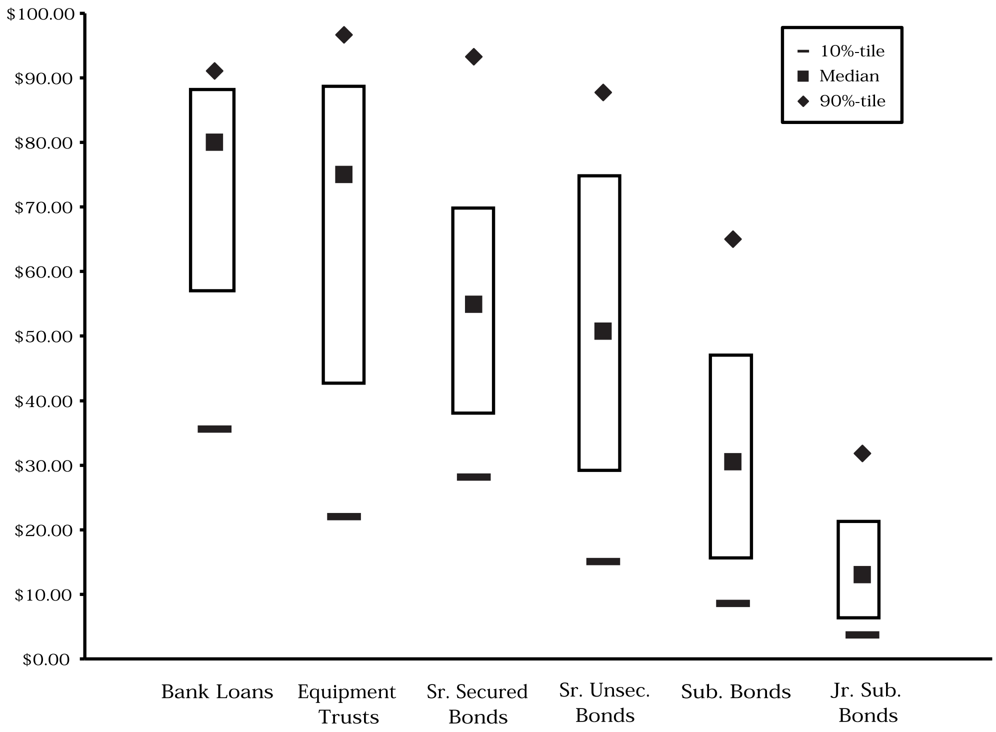

Figure 3: Recovery Distributions. (Source: Moody's Investors Services)

3.6 Collateral Credit Spreads

We suppose for simplicity that changes in default intensities and changes in interest rates are (risk-neutrally) independent. An extension to treat correlated interest rate risk is provided in Appendix A. Combined with the above assumptions, this implies that, for an issuer whose default time τ has a basic affine pre-intensity process λ, a zero-coupon bond maturing at time t has an initial market value of

$$p(t, \lambda(0)) = \delta(t) e^{\alpha(t)+\beta(t)\lambda(0)} + \overline{f} \int_0^t \delta(u)\pi(u)\, du, \tag{10}$$

where $\delta(t)$ denotes the default-free zero-coupon discount to time t and

$$\pi(t) = -\frac{\partial}{\partial t} P(\tau > t) = -e^{\alpha(t)+\beta(t)\lambda(0)}[\alpha'(t) + \beta'(t)\lambda(0)]$$

is the (risk-neutral) probability density at time t of the default time. The first term of (10) is the market value of a claim that pays 1 at maturity in the event of survival. The second term is the market value of a claim to any default recovery between times 0 and t. The integral is computed numerically, using our explicit solutions from Appendix B for $\alpha(t)$ and $\beta(t)$. This pricing approach was developed by Lando [1998] for slightly different default intensity models. Lando also allows for correlation between interest rates and default intensity, as summarized in Appendix A. Lando [1998] assumes deterministic recovery at default; for our case of random recovery, see Duffie [1998b].

Using this defaultable discount function $p(\cdot)$, we can value any straight coupon bond, or determine par coupon rates. For example, for quarterly coupon periods, the (annualized) par coupon rate $c(s)$ for maturity in s years (for an integer $s > 0$) is determined at any time t by the identity

$$1 = p(s, \lambda_t) + \frac{c(s)}{4} \sum_{j=1}^{4s} p\left(\frac{j}{4}, \lambda_t\right), \tag{11}$$

which is trivially solved for $c(s)$.

3.7 Diversity Scores

A key measure of collateral diversity developed by Moody's for CDO risk analysis is the diversity score. The diversity score of a given pool of participations is the number n of bonds in a idealized comparison portfolio that meets the following criteria:

1. The total face value of the comparison portfolio is the same as the total face value of the collateral pool.

2. The bonds of the comparison portfolio have equal face values.

3. The comparison bonds are equally likely to default, and their default is independent.

4. The comparison bonds are, in some sense, of the same average default probability as the participations of the collateral pool.

5. The comparison portfolio has, according to some measure of risk, the same total loss risk as does the collateral pool.

At least in terms of publicly available information, it is not clear how the (equal) default probability p of default of the bonds of the comparison portfolio is determined. One method that has been discussed by Schorin and Weinreich [1998] for this purpose is to assign a default probability corresponding to the weighted average rating score of the collateral pool, using rating scores such as those illustrated in Table 1, and using weights that are proportional to face value. Given the average rating score, one can assign a default probability p to the resulting "average" rating. For the choice of p, Schorin and Weinreich [1998] discuss the use of the historical default frequency for that rating.

A diversity score of n and a comparison-bond default probability of p imply, using the independence assumption for the comparison portfolio, that the probability of k defaults out of the n bonds of the comparison portfolio is

$$q(k,n) = \frac{n!}{(n-k)!k!}\ p^k(1-p)^{n-k}. \tag{12}$$

From this "binomial-expansion" formula, a risk analysis of the CDO can be conducted by assuming that the performance of the collateral pool is sufficiently well approximated by the performance of the comparison portfolio. Moody's would not rely exclusively on the diversity score in rating the CDO tranches.

Table 2 shows the diversity score that Moody's would apply to a collateral pool of equally-sized bonds of different firms within the same industry. It also lists the implied probability of default of one participation given the default of another, as well as the correlations of the 0-1, survival-default, random variables associated with any two participations, for two levels of individual default probability ($p = 0.5$ and $p = 0.05$). Figure 4 presents the conditional probability information graphically.

4 Pricing Examples

This section applies a standard risk-neutral derivative valuation approach to the pricing of CDO tranches. In the absence of any tractable alternative, we use Monte-Carlo simulation of the default times. Essentially any intensity model could be subsituted for the basic affine model that we have adopted here. The advantage of the affine model is the ability to quickly calibrate the model to the underlying participations, in terms of given correlations, default probabilities, yield spreads, and so on, and in obtaining an understanding of the role of diffusion, jumps, mean reversion, and diversification for both valuation and (when working under the actual probabilities) various risk measures.

We will study various alternative CDO cash-flow structures and default-risk parameters. The basic CDO structure consists of a special purpose vehicle (SPV) that acquires a collateral portfolio of participations (debt instruments of various obligors), and allocates interest, principal, and default-recovery cash flows from the collateral pool to the CDO tranches, and perhaps to a manager, as described below.

Table 1: Rating Scores Used to Derive Weighted Average Ratings

	Moody's	Fitch	DCR
Aaa/AAA	1	1	0.001
Aa1/AA+	10	8	0.010
Aa2/AA	20	10	0.030
Aa3/AA-	40	14	0.050
A1/A+	70	18	0.100
A2/A	120	23	0.150
A3/A-	180	36	0.200
Baa1/BBB+	260	48	0.250
Baa2/BBB	360	61	0.350
Baa3/BBB-	610	94	0.500
Ba1/BB+	940	129	0.750
Ba2/BB	1,350	165	1.000
Ba3/BB-	1,780	210	1.250
B1/B+	2,220	260	1.600
B2/B	2,720	308	2.000
B3/B-	3,490	356	2.700
CCC+	NA	463	NA
Caa/CCC	6,500	603	3.750
CCC-	NA	782	NA
<Ca/<CCC-	10,000	1,555	NA

Source: Schorin and Weinreich [1998], from Moody's Investors Service, Fitch Investors Service, and Duff and Phelps Credit Rating

4.1 Collateral

There are N participations in the collateral pool. Each participation pays quarterly cash flows to the SPV at its coupon rate, until maturity or default. At default, a participation is sold for its recovery value, and the proceeds from sale are also made available to the SPV. In order to be precise, let $A(k) \subset \{1, \ldots, N\}$ denote the set of surviving participations at the k-th coupon period. The total interest income in coupon period k is then

$$W(k) = \sum_{i \in A(k)} M_i \frac{C_i}{n}, \tag{13}$$

where M_i is the face value of participation i and C_i is the coupon rate on participation i. Letting $B(k) = A(k-1) - A(k)$ denote the set of participations defaulting between

Table 2: Moody's Diversity Scores for Firms within an Industry

Number of Firms in Same Industry	Diversity Score	Conditional Default Probability ($p = 0.5$)	($p = 0.05$)	Default Correlation ($p = 0.5$)	($p = 0.05$)
1	1.00				
2	1.50	0.78	0.48	0.56	0.45
3	2.00	0.71	0.37	0.42	0.34
4	2.33	0.70	0.36	0.40	0.32
5	2.67	0.68	0.33	0.36	0.30
6	3.00	0.67	0.31	0.33	0.27
7	3.25	0.66	0.30	0.32	0.26
8	3.50	0.65	0.29	0.31	0.25
9	3.75	0.65	0.27	0.29	0.24
10	4.00	0.64	0.26	0.28	0.23
>10	Evaluated on a case-by-case basis				

Source: Moody's Investors Service, from Schorin and Weinreich [1998]

coupon periods $k-1$ and k, the total total cash flow in period k is

$$Z(k) = W(k) + \sum_{i \in B(k)} (M_i - L_i), \tag{14}$$

where L_i is the loss of face value at the default of participation i.

For our example, the initial pool of collateral available to the CDO structure consists of $N = 100$ participations that are straight quarterly-coupon 10-year par bonds of equal face value. Without loss of generality, we take the face value of each bond to be 1.

Table 3 shows four alternative sets of parameters for the default pre-intensity $\lambda_i = X_c + X_i$ of each individual participation. We initiate X_c and X_i at their long-run means, $\theta_c + \ell_c\mu/\kappa$ and $\theta_i + \ell_i\mu/\kappa$, respectively. This implies an initial condition (and long-run mean) for each obligor's (risk-neutral) default pre-intensity of 5.33%. Our base-case default-risk model is defined by Parameter Set Number 1, and by letting $\rho = 0.5$ determine the degree of diversification.

The three other parameter sets shown in Table 3 are designed to illustrate the effects of replacing some or all of the diffusive volatility with jump volatility, or the effects of reducing the mean jump size and increasing the mean jump arrival frequency ℓ. All parameter sets have the same long-run mean $\theta + \mu\ell/\kappa$. The parameters θ, σ, ℓ, and μ are adjusted so as to maintain essentially the same term structure of survival probabilities, illustrated in Figure 5. (As all parameter sets have the same κ parameter, this is a rather straightforward numerical exercise, using (5) for default probabilities.) This in turn implies essentially the same term structure of zero-coupon yields, as illustrated in

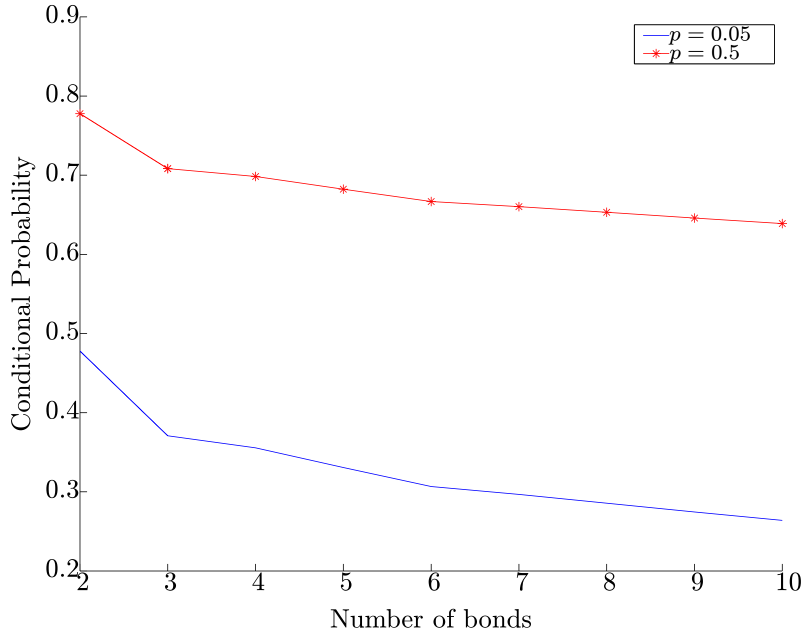

Figure 4: Probability of default of one bond given the default of another, as implied by Moody's diversity scores.

Figure 6. Table 3 provides, for each parameter set, the 10-year par-coupon spreads and the long-run variance of $\lambda_i(t)$. In order to illustrate the qualitative differences between parameter sets, Figure 7 shows sample paths of new 10-year par spreads for two issuers, one with the base-case parameters (Set 1), the other with pure-jump intensity (Set 2), calibrated to the same initial spread curve.

Letting d_i denote the event of default by the i-th participation, Table 4 shows, for each parameter set and each of 3 levels of the correlation parameter ρ, the unconditional probability of default and the conditional probability of default by one participation given default by another. The table also shows the diversity score of the collateral pool that is implied by matching the variance of the total loss of principal of the collateral portfolio to that of a comparison portfolio of bonds of the same individual default probabilities. This calculation is based on the analytical methods described in Section 5.

For our basic examples, we suppose first that any cash in the SPV reserve account is invested at the default-free short rate. We later consider investment of SPV free cash flows in additional risky participations.

Table 3: Risk-Neutral Default Parameter Sets

Set	κ	θ	σ	ℓ	μ	Spread	var_∞
1	0.6	0.0200	0.141	0.2000	0.1000	254 bp	0.42%
2	0.6	0.0156	0.000	0.2000	0.1132	254 bp	0.43%
3	0.6	0.0373	0.141	0.0384	0.2500	253 bp	0.49%
4	0.6	0.0005	0.141	0.5280	0.0600	254 bp	0.41%

Table 4: Conditional probabilities of default and diversity scores

		$\rho = 0.1$		$\rho = 0.5$		$\rho = 0.9$	
Set	$P(d_i)$	$P(d_i \mid d_j)$	divers.	$P(d_i \mid d_j)$	divers.	$P(d_i \mid d_j)$	divers.
1	0.386	0.393	58.5	0.420	21.8	0.449	13.2
2	0.386	0.393	59.1	0.420	22.2	0.447	13.5
3	0.386	0.392	63.3	0.414	25.2	0.437	15.8
4	0.386	0.393	56.7	0.423	20.5	0.454	12.4

4.2 Sinking-Fund Tranches

We will consider a CDO structure that pays SPV cash flows to a prioritized sequence of sinking-fund bonds, as well as a junior subordinate residual, as follows.

In general, a sinking-fund bond with n coupon periods per year has some remaining principal $F(k)$ at coupon period k, some annualized coupon rate c, and a scheduled interest payment at coupon period k of $F(k)c/n$. In the event that the actual interest paid $Y(k)$ is less than the scheduled interest payment, any difference $F(k)c/n - Y(k)$ is accrued at the bond's own coupon rate c so as to generate an accrued un-paid interest at period k of $U(k)$, where $U(0) = 0$ and

$$U(k) = \left(1 + \frac{c}{n}\right) U(k-1) + \frac{c}{n} F(k) - Y(k).$$

There may also be some pre-payment of principal, $D(k)$ in period k, and some contractual unpaid reduction in principal, $J(k)$ in period k, in order to prioritize payments in light of the default and recovery history of the collateral pool. By contract, we have $D(k) + J(k) \leq F(k-1)$, so that the remaining principal at quarter k is

$$F(k) = F(k-1) - D(k) - J(k). \tag{15}$$

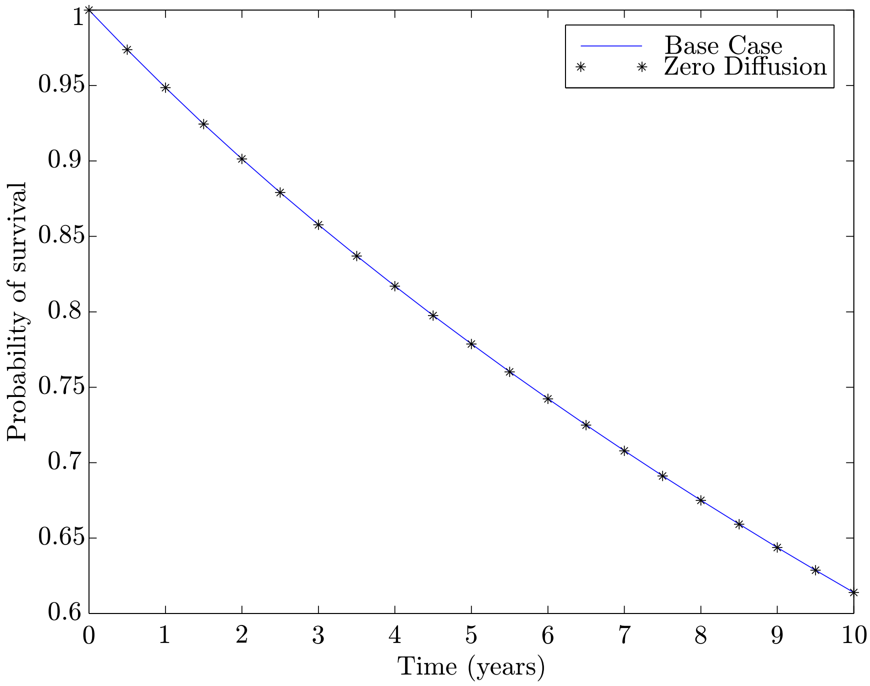

Figure 5: Term Structures of Survival Probabilities, With and Without Diffusion

At maturity, coupon period number K, any unpaid accrued interest and unpaid principal, $U(K)$ and $F(K)$ respectively, are paid to the extent provided in the CDO contract. (A shortfall does not constitute default[4] so long as the contractual prioritization scheme is maintained.) The total actual payment in any coupon period k is $Y(k)+D(k)$.

The par coupon rate on a given sinking-find bond is the scheduled coupon rate c with the property that the initial market value of the bond is equal to its initial face value, $F(0)$. If the default-free short rate $r(k)$ is constant, as in our initial set of results, any sinking-fund bond that pays all remaining principal and all accrued unpaid interest by or at its maturity date has a par-coupon rate equal to the default-free coupon rate, no matter the timing of the interest and principal payments. We will illustrate our initial valuation results in terms of the par-coupon spreads of the respective tranches, which are the excess of the par-coupon rates of the tranches over the default-free par coupon rate.

[4]As a practical matter, Moody's may assign a "default" to a CDO tranche even if it meets its contractual payments, if the investors' losses from default in the underlying collateral pool are sufficiently severe.

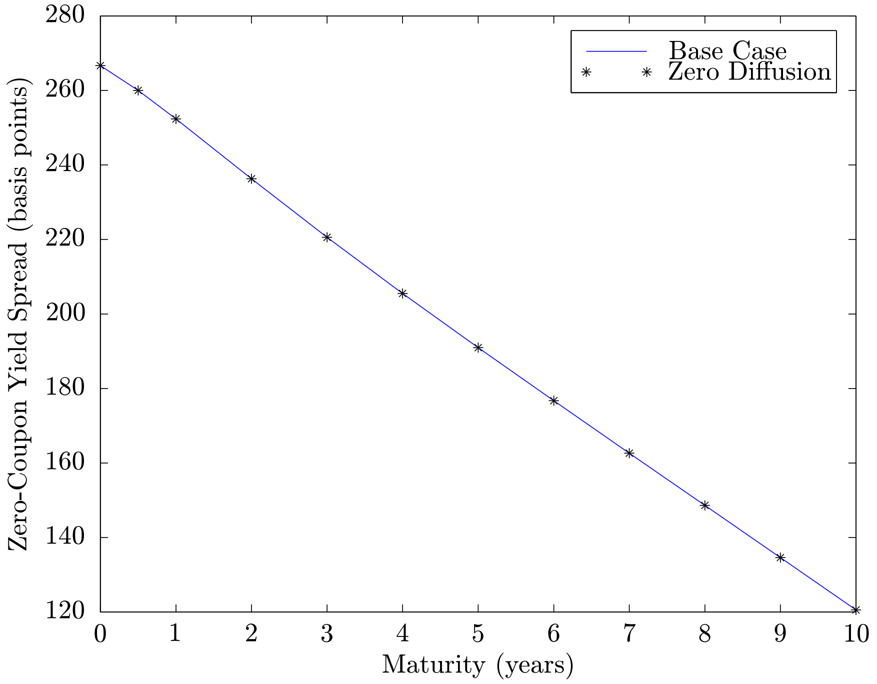

Figure 6: Zero-Coupon Yield Spreads, With and Without Diffusion

4.3 Prioritization Schemes

We will experiment with the relative sizes and prioritization of two CDO bond tranches, one 10-year senior sinking-fund bond with some initial principal $F_1(0) = P_1$, and one 10-year mezzanine sinking-fund bond with initial principal $F_2(0) = P_2$. The residual junior tranche receives any cash-flow remaining at the end of the 10-year structure. As the base-case coupon rates on the senior and mezzanine CDO tranches are, by design, par rates, the base-case initial market value of the residual tranche is $P_3 = 100 - P_1 - P_2$.

At the k-th coupon period, Tranche j has a face value of $F_j(k)$ and an accrued unpaid interest $U_j(k)$ calculated at its own coupon rate, c_j. Any excess cash flows from the collateral pool (interest income and default recoveries), are deposited in a reserve account. To begin, we suppose that the reserve account earns interest at the default-free one-period interest rate, denoted $r(k)$ at the k-th coupon date. At maturity, coupon period K, any remaining funds in the reserve account, after payments at quarter K to the two tranches, are paid to the subordinated residual tranche. (Later, we investigate the effects of investing the reserve account in additional participations that are added to the collateral pool.) We neglect any management fees.

We will investigate valuation for two prioritization schemes, which we now describe. Given the definition of the sinking funds in the previous sub-section, in order to com-

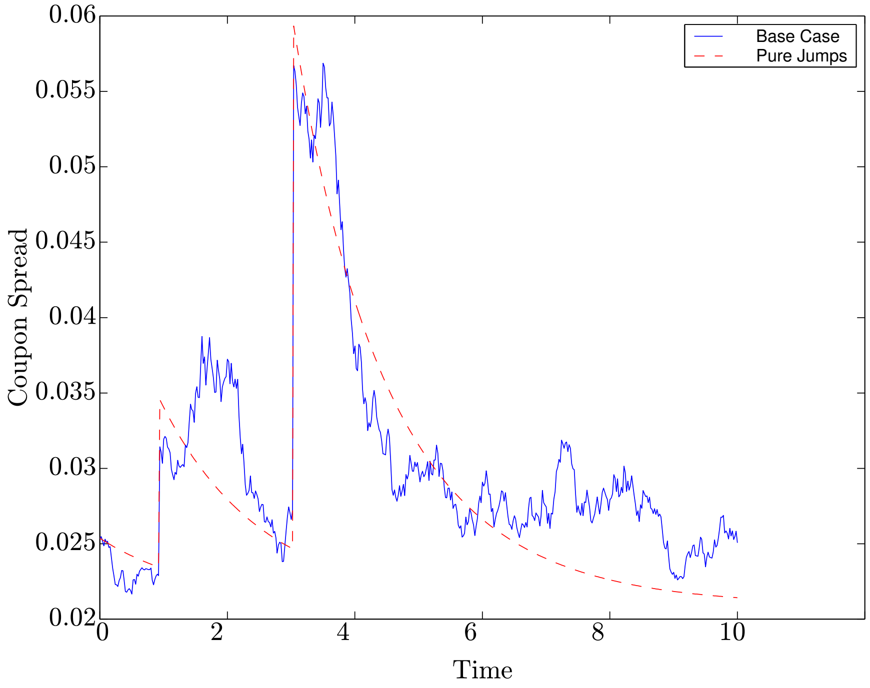

Figure 7: New 10-year par-coupon spreads for the base-case parameters and for the pure-jump intensity parameters.

pletely specify cash flows to all tranches, it is enough to define the actual interest payments $Y_1(k)$ and $Y_2(k)$ for the senior and mezzanine sinking funds, respectively, any payments of principal, $D_1(k)$ and $D_2(k)$, and any contractual reductions in principal, $J_1(k)$ and $J_2(k)$.

Under our *uniform prioritization* scheme, the interest $W(k)$ collected from the surviving participations is allocated in priority order, with the senior tranche getting

$$Y_1(k) = \min\left[U_1(k), W(k)\right]$$

and the mezzanine getting

$$Y_2(k) = \min\left[U_2(k), W(k) - Y_1(k)\right].$$

The available reserve $R(k)$, before payments at period k, is thus defined by

$$R(k) = \left(1 + \frac{r(k)}{4}\right)\left[R(k-1) - Y_1(k-1) - Y_2(k-1)\right] + Z(k), \tag{16}$$

recalling that $Z(k)$ is the total cash flow from the participations in period k.

Unpaid reductions in principal from default losses occur in reverse priority order, so that the junior residual tranche suffers the reduction

$$J_3(k) = \min(F_3(k-1), H(k)),$$

where

$$H(k) = \max\left(\sum_{i \in B(k)} L_i - (W(k) - Y_1(k) - Y_2(k)), 0\right)$$

is the total of default losses since the previous coupon date, less collected and undistributed interest income. Then the mezzanine and senior tranches are successively reduced in principal by

$$\begin{aligned} J_2(k) &= \min(F_2(k-1), H(k) - J_3(k)) \\ J_1(k) &= \min(F_1(k-1), H(k) - J_3(k) - J_2(k)), \end{aligned}$$

respectively. Under uniform prioritization, there are no early payments of principal, so $D_1(k) = D_2(k) = 0$ for $k < K$. At maturity, the remaining reserve is paid in priority order, and principal and accrued interest are treated identically, so that without loss of generality for purposes of valuation we take $Y_1(K) = Y_2(K) = 0$,

$$D_1(K) = \min[F_1(K) + U_1(K), R(K)],$$

and

$$D_2(K) = \min[F_2(K) + U_2(K), R(K) - D_1(K)].$$

The residual tranche finally collects $D_3(K) = R(K) - Y_1(K) - D_1(K) - Y_2(K) - D_2(K)$.

For our alternative *fast-prioritization* scheme, the senior tranche is allocated interest and principal payments as quickly as possible, until maturity or until its principal remaining is reduced to zero, whichever is first. Until the senior tranche is paid in full, the mezzanine tranche accrues unpaid interest at its coupon rate. Then the mezzanine tranche is paid interest and principal as quickly as possible until maturity or until retired. Finally, any remaining cash flows are allocated to the residual tranche. Specifically, in coupon period k, the senior tranche is allocated the interest payment

$$Y_1(k) = \min\left[U_1(k), Z(k)\right]$$

and the principal payment

$$D_1(k) = \min[F_1(k-1), Z(k) - Y_1(k)],$$

where the total cash $Z(k)$ generated by the collateral pool is again defined by (14). The mezzanine receives the interest payments

$$Y_2(k) = \min\left[U_2(k), Z(k) - Y_1(k) - D_1(k)\right],$$

and principal payments

$$D_2(k) = \min[F_2(k-1), Z(k) - Y_1(k) - D_1(k) - Y_2(k)].$$

Finally, any residual cash flows are paid to the junior subordinated tranche. For this scheme, there are no contractual reductions in principal ($J_i(k) = 0$).

In practice, there are many other types of prioritization schemes. For example, during the life of a CDO, failure to meet certain contractual over-collateralization ratios in many cases triggers a shift to some version of fast prioritization. For our examples, the CDO yield spreads for uniform and fast prioritization would provide upper and lower bounds, respectively, on the senior spreads that would apply if one were to add such a feature to the uniform prioritization scheme that we have illustrated.

4.4 Simulation Methodology

Our computational approach consists of simulating piece-wise linear approximations of the paths of X_c and $X_1, \ldots, X_N$, for time intervals of some relatively small fixed length Δt. (We have taken an interval Δt of one week.) Defaults during one of these intervals are simulated at the corresponding discretization of the total arrival intensity $\Lambda(t) = \sum_i \lambda_i(t) 1_{A(i,t)}$, where $A(i,t)$ is the event that issuer i has not defaulted by t. With the arrival of some default, the identity of the defaulter is drawn at random, with the probability that i is selected as the defaulter given by the discretization approximation of $\lambda_i(t) 1_{A(i,t)} / \Lambda(t)$. Based on experimentation, we chose to simulate $10,000$ pseudo-independent scenarios. The basis for this and other multi-obligor default-time simulation approaches is discussed by Duffie and Singleton [1998].

Table 5: Par spreads ($\rho = 0.5$).

	Principal		Spread (Uniform)		Spread (Fast)	
Set	P_1	P_2	s_1 (bp)	s_2 (bp)	s_1 (bp)	s_2 (bp)
1	92.5	5	18.7 (1)	636 (16)	13.5 (0.4)	292 (1.6)
2	92.5	5	17.9 (1)	589 (15)	13.5 (0.5)	270 (1.6)
3	92.5	5	15.3 (1)	574 (14)	11.2 (0.5)	220 (1.5)
4	92.5	5	19.1 (1)	681 (17)	12.7 (0.4)	329 (1.6)
1	80	10	1.64 (0.1)	67.4 (2.2)	0.92 (0.1)	38.9 (0.6)
2	80	10	1.69 (0.1)	66.3 (2.2)	0.94 (0.1)	39.5 (0.6)
3	80	10	2.08 (0.2)	51.6 (2.0)	1.70 (0.2)	32.4 (0.6)
4	80	10	1.15 (0.1)	68.1 (2.0)	1.70 (0.2)	32.4 (0.6)

4.5 Results for Par CDO Spreads

Table 5 shows the estimated par spreads, in basis points, of the senior (s_1) and mezzanine (s_2) CDO tranches for the 4 parameter sets, for various levels of over-collateralization, and for our two prioritization schemes. In order to illustrate the accuracy of the simulation methodology, estimates of the standard deviation of these estimated spreads that are due to "Monte Carlo noise" are shown in parentheses. Tables 6 and 7 show estimated par spreads for the case of "low" ($\rho = 0.1$) and "high" ($\rho = 0.9$) default correlation. In all of these examples, the risk-free rate is 0.06, and there are no management fees.

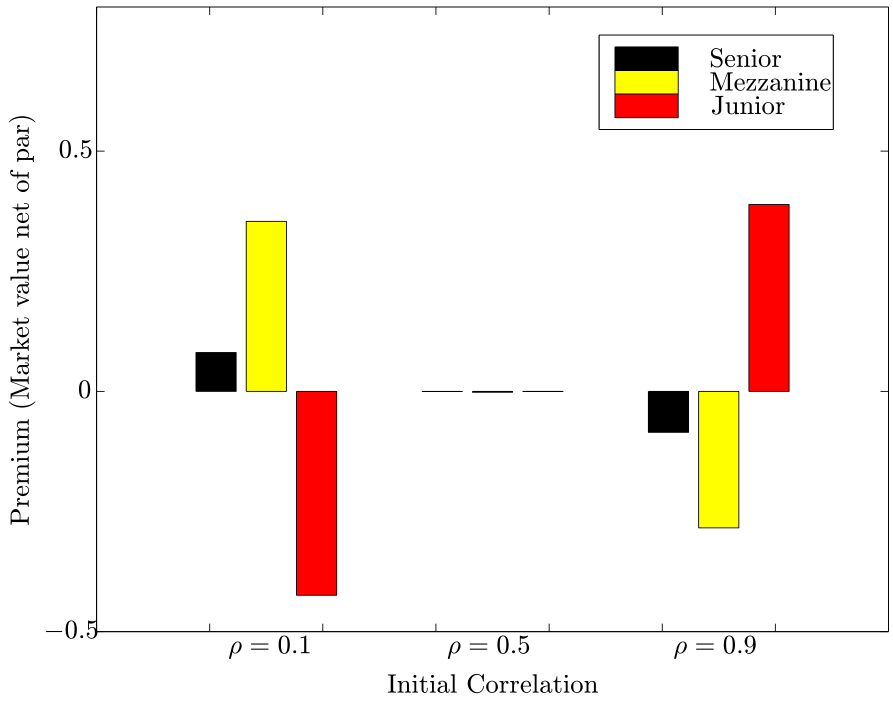

Figure 8: Impact on market values of correlation, uniform prioritization, Parameter Set 1, high overcollateralization ($P_1 = 80$).

Figures 8 and 9 illustrate the impacts on the market values of the 3 tranches of a given CDO structure of changing the correlation parameter ρ. The base-case CDO structures used for this illustration are determined by uniform prioritization of senior and mezzanine tranches whose coupon rates are at par for the base-case Parameter Set 1 and correlation $\rho = 0.5$. For example, suppose this correlation parameter is moved from the base case of 0.5 to 0.9. Figure 9, which treats the case ($P_1 = 92.5$) of relatively little subordination available to the senior tranche, shows that this loss in diversification

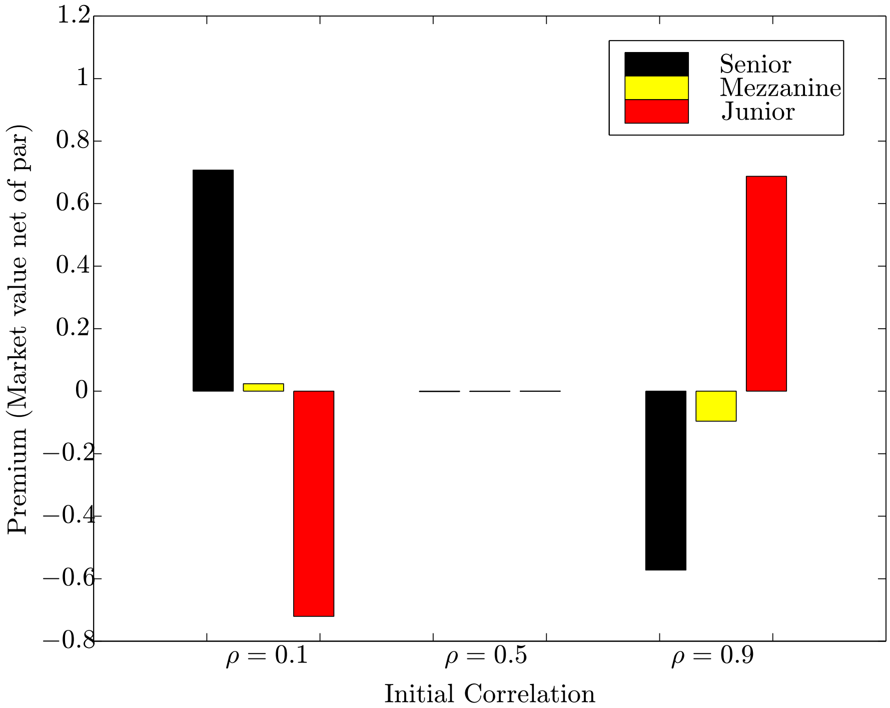

Figure 9: Impact on market values of correlation, uniform prioritization, Parameter Set 1, low overcollateralization ($P_1 = 92.5$).

reduces the market value of the senior tranche from 92.5 to about 91.9. The market value of the residual tranche, which benefits from volatility in the manner of a call option, increases in market value from 2.5 to approximately 3.2, a dramatic relative change. While a precise statement of convexity is complicated by the timing of the prioritization effects, this effect is along the lines of Jensen's Inequality, as an increase in correlation also increases the (risk-neutral) variance of the total loss of principal. These opposing reactions to diversification of the senior and junior tranches also show that the residual tranche may offer some benefits to certain investors as a default-risk-volatility hedge for the senior tranche.

The mezzanine tranche absorbs the net effect of the impacts of correlation changes on the market values of the senior and junior residual tranches (in this example resulting a decline in market value of the mezzanine from 5.0 to approximately 4.9), as it must given that the total market value of the collateral portfolio is not affected by the correlation of default risk. We can compare these effects with the impact of correlation on the par spreads of the senior and mezzanine tranches that are shown in Tables 5, 6, and 7. As we see, the mezzanine par spreads can be dramatically influenced by correlation, given the relatively small size of the mezzanine principal. Moreover, experimenting with various mezzanine over-collateralization values shows that the effect is ambiguous: Increasing default correlation may *raise or lower* mezzanine spreads.

Table 6: Par spreads ($\rho = 0.1$).

	Principal		Spread (Uniform)		Spread (Fast)	
Set	P_1	P_2	s_1 (bp)	s_2 (bp)	s_1 (bp)	s_2 (bp)
1	92.5	5	6.7	487	2.7	122
2	92.5	5	6.7	492	2.9	117
3	92.5	5	6.3	473	2.5	102
4	92.5	5	7.0	507	2.7	137
1	80	10	0.27	17.6	0.13	7.28
2	80	10	0.31	17.5	0.15	7.92
3	80	10	0.45	14.9	0.40	6.89
4	80	10	0.16	19.0	0.05	6.69

Table 7: Par spreads ($\rho = 0.9$).

	Principal		Spread (Uniform)		Spread (Fast)	
Set	P_1	P_2	s_1 (bp)	s_2 (bp)	s_1 (bp)	s_2 (bp)
1	92.5	5	30.7	778	23.9	420
2	92.5	5	29.5	687	23.7	397
3	92.5	5	25.3	684	20.6	325
4	92.5	5	32.1	896	23.1	479
1	80	10	3.17	113	1.87	68.8
2	80	10	3.28	112	1.95	70.0
3	80	10	4.03	90	3.27	60.4
4	80	10	2.52	117	1.06	65.4

4.6 Risky Reinvestment

We also illustrate how one can implement a contractual proviso that recoveries on defaulted participations and excess collected interest are to be invested in collateral of comparable quality to that of the original pool. This method can also be used to allow for collateral assets that mature before the termination of the CDO.

The default intensity of each new collateral asset is of the type given by equation (6), where X_i is initialized at the time of the purchase at the initial base-case level (long-run mean of λ_i). Par spreads are computed for these bonds. Figures 10 and 11 show the effect of changing from safe to risky reinvestment. Given the "short-option" aspect of the senior tranche, it becomes less valuable when reinvestment becomes risky. It is interesting to note that the mezzanine tranche benefits from the increased variance in this case.

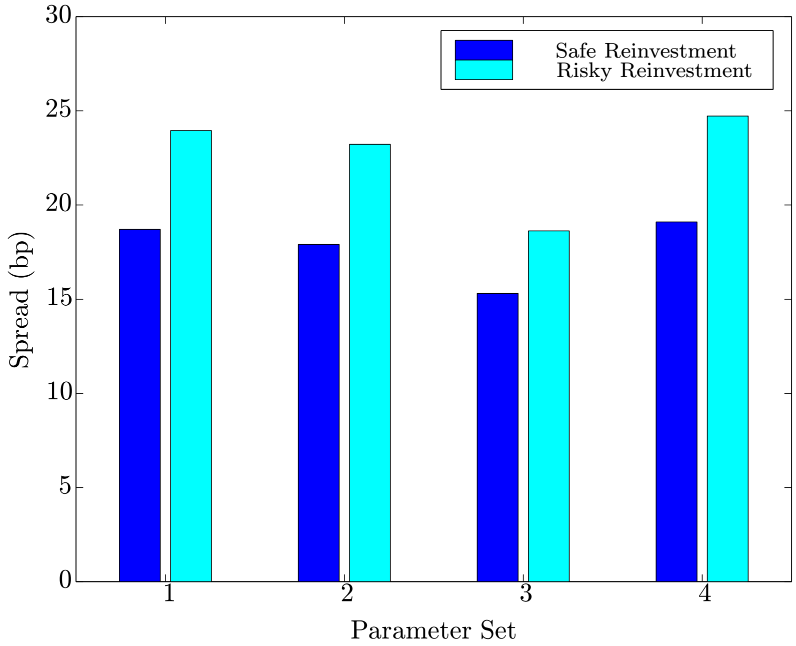

Figure 10: Senior Tranche spreads in the uniform prioritization and risky reinvestment schemes ($P_1 = 92.5, P_2 = 5$).

5 Analytical Results

Here, exploiting the symmetry assumptions of our special example, analytical results are provided for the probability distribution for the number of defaulting participations, and the total of default losses of principal, including the effects of random recovery.

The key is an ability to compute explicitly the probability of survival of all participations in any chosen sub-group of obligors. These explicit probabilities, however, must be evaluated with extremely high numerical accuracy given the large number of combinations of sub-groups to be considered.

For a given time horizon T, let d_j denote the event that obligor j defaults by T. That is, $d_j = \{\tau_j < T\}$. We let M denote the number of defaults. Assuming symmetry (invariance under permutation) in the unconditional joint distribution of default times,

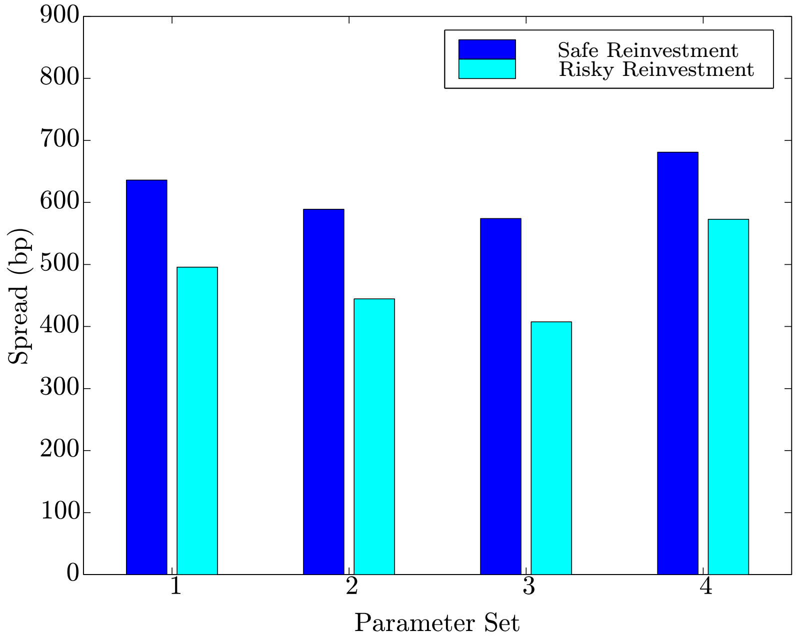

Figure 11: Mezzanine Tranche spreads in the uniform prioritization and risky reinvestment schemes ($P_1 = 92.5, P_2 = 5$).

$$P(M = k) = \binom{N}{k} P(d_1 \cap \cdots \cap d_k \cap d^c_{k+1} \cap \cdots \cap d^c_N),$$

where $\binom{N}{k} = N!/[(N-k)!k!]$. Let $q(k,N) = P(d_1 \cap \cdots \cap d_k \cap d^c_{k+1} \cap \cdots \cap d^c_N)$. The probability $p_j = P(d_1 \cup \cdots \cup d_j)$ that at least one of the first j names defaults by T is computed later; for now, we take this calculation as given.

Proposition 2: $q(k,N) = \sum_{j=1}^{N} (-1)^{(j+k+N+1)} \binom{k}{N-j} p_j$, using the convention that $\binom{m}{l} = 0$ if $l < 0$ or $m < l$.

A proof is provided in Appendix A. Using the fact that the pre-intensity of the first-to-arrive $\tau^{(j)} = \min(\tau_1, \ldots, \tau_j)$ of the stopping times $\tau_1, \ldots, \tau_j$ is $\lambda_1 + \cdots + \lambda_j$ (Duffie [1998a]), and using the independence of $X_1, \ldots, X_N, X_c$, we have

$$\begin{aligned} p_j = 1 - P(\tau^{(j)} > T) &= 1 - E\left[\exp\left(-\int_0^T \sum_{i=1}^{j} \lambda_i(t))\, dt\right)\right] && (17) \\ &= 1 - e^{\alpha_c(T) + \beta_c(T) X_c(0) + j\alpha_i(T) + j\beta_i(T) X_i(0)}, && (18) \end{aligned}$$

where $\alpha_c(T)$ and $\beta_c(T)$ are given explicitly in Appendix B as the solutions of the ODEs (B.1)-(B.2) for the case $n = -\kappa$, $p = \sigma^2$, $q = -j$, $\ell = \ell_c$, and $m = \kappa\theta_c$; while $\alpha_i(T)$

and $\beta_i(T)$ are the explicitly-given solutions of (B.1)-(B.2) for the case $n = -\kappa$, $p = \sigma^2$, $q = -1$, $\ell = \ell_i$, and $m = \kappa\theta_i$.

It is not hard to see how one would generalize Proposition 2 so as to accommodate more than one type of intensity — that is, how to treat a case with several internally symmetric pools. Introducing each such group, however, increases by one the dimensions of the array p and the summation. Given the relatively lengthy computation required to obtain adequate accuracy for even 2 sub-groups of issuers, one may prefer simulation to this analytical approach for multiple types of issuers.

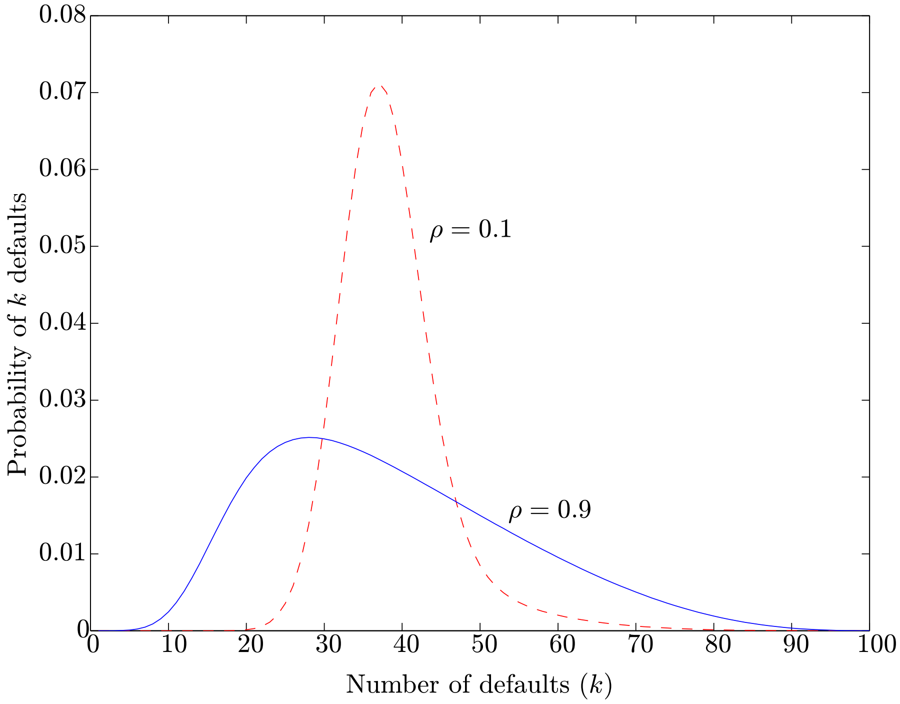

Figure 12: The probability of k defaults, for high and low correlation (base case).

Based on this analytical method, Figure 12 shows the probability $q(k, 100)$ of k defaults within 10 years, out of the original group of 100 issuers, for Parameter Set 1, for a correlation-determining parameter ρ that is high (0.9) or low (0.1). Figure 13 shows the associated cumulative probability functions, including the base case of $\rho = 0.5$. For example, the likelihood of at least 60 defaults out of the original 100 participations in 10 years is on the order of 1 percent for the low-correlation case, while roughly 12 percent for the high-correlation case.

Figure 14 shows the corresponding low-correlation and high-correlation probability $q(k, 100)$ of k defaults out of 100 for the no-diffusion case, Parameter Set 2. Figures 15, 16, and 17 compare the probability $q(k, 100)$ of k defaults for all 4 parameter sets. The

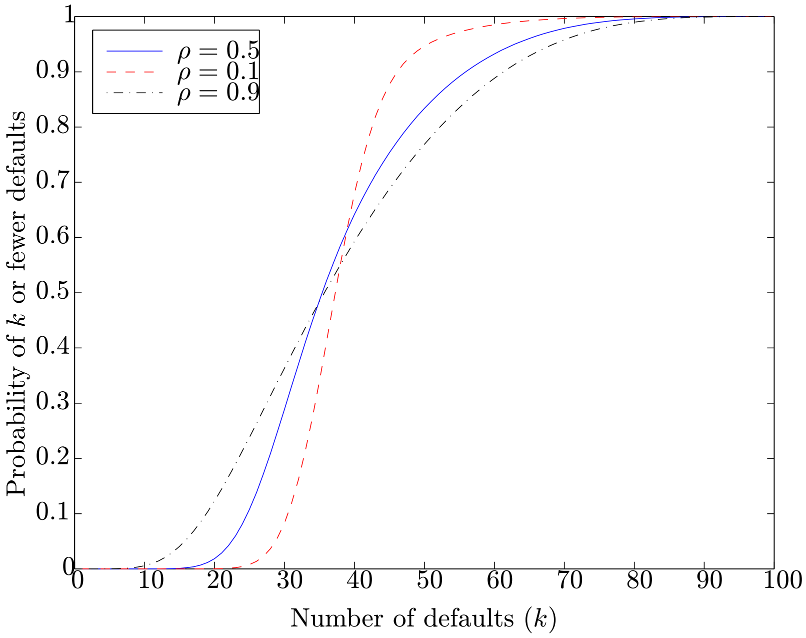

Figure 13: The cumulative probability of the number of defaults, high and low correlation (base case).

low-correlation distributions are rather more similar across the various parameter sets than are the high-correlation distributions.

One can also compute analytically the likelihood of a total loss of principal of a given amount x. This is done by adding up, over k, the probabilities $q(k, 100)$ of k defaults multiplied by the probabilities that the total loss of principal from k defaults is at least x. For the latter, we do not use the actual distribution of the total fractional loss of principal of a given number k of defaulting participations. For ease of computation, we substitute with a central-limit (normal) approximation for the distribution of the sum of k *iid* uniform-$[0, 1]$ random variables, which is merely the distribution of a normally distributed variable with the same mean and variance. We are interested in this calculation for moderate to large levels of x, corresponding for example to estimating the probability of failure to meet an over-collateralization target. We have verified that, even for relatively few defaults, the central-limit approximation is adequate for our purposes. A sample of the resulting loss distributions is illustrated in Figure 18.

One can also analytically compute the variance of total loss of principal, whence diversity scores, as tabulated for our example in Table 4. A description of the computation of divesity score on a general pool of collateral, not necessarily with symmetric default risk, can be found in Appendix C. Given a (risk-neutral) diversity score of n, one can then estimate CDO yield spreads by a much simpler algorithm, which approximates by

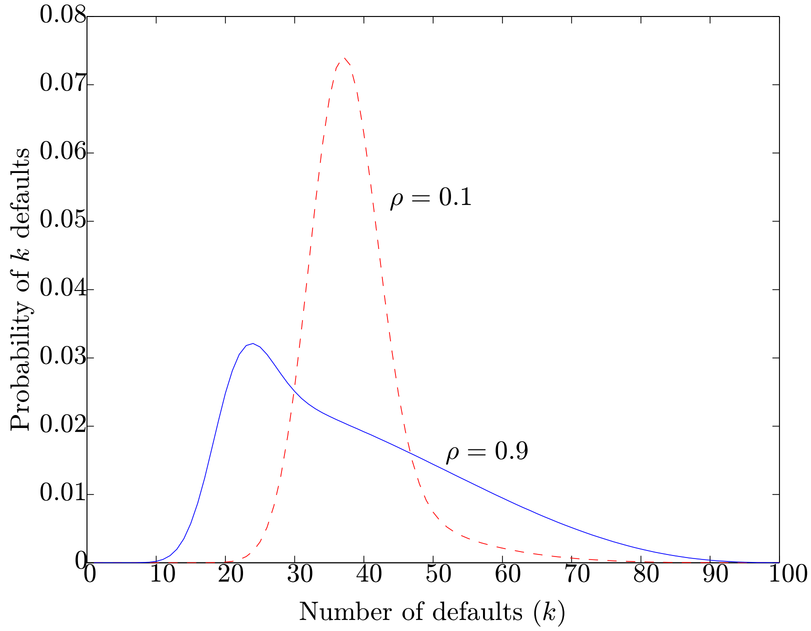

Figure 14: The probability of k defaults, for high and low correlation (Parameter Set 2).

substituting the comparison portfolio of n independently defaulting participations for the actual collateral portfolio. The default times can be simulated independently, directly from an explicit unconditional distribution, rather than using a much more arduous simulation of the pre-intensity processes. The algorithm is roughly as follows:

1. Simulate a draw from the explicit distribution of the number M of defaults of the comparison portfolio of n bonds, with probability $q(k, n)$ of k defaults.
2. For each of the M comparison defaults, simulate, independently, the corresponding default times, using the explicit default-time distribution of each particpation.
3. Simulate M fractional losses of principal.
4. Allocate cash flows to the CDO tranches, period by period, according the desired prioritization scheme.
5. Discount the cash flows of each CDO tranche to a present value at risk-free rates.
6. For each tranche, average the discounted cash flows over independently generated scenarios.

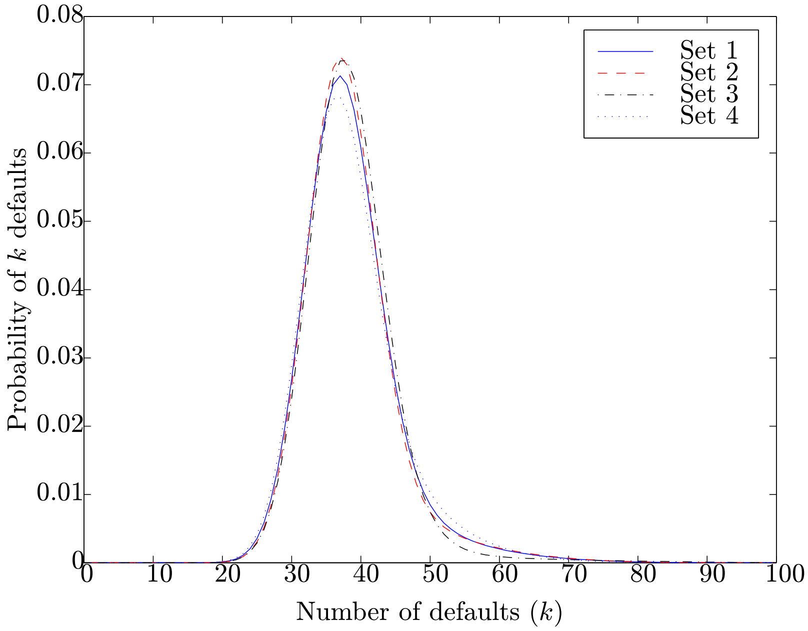

Figure 15: Probability of k defaults, all parameter sets, low correlation ($\rho = 0.1$).

A comparison of the resulting approximation of CDO spreads with those computed earlier are provided, for certain cases, in Figures 19 and 21. For well-collateralized tranches, the diversity-based estimates of spreads are reasonably accurate, at least relative to the uncertainty that one would, in any case, have regarding the actual degree of diversification in the collateral pool. For highly subordinated tranches and with moderate or large default correlation, the diversity-based spreads can be rather inaccurate, as can be seen from Figure 20 and Table 8.

Figure 22 shows the likelihood of a total loss of principal of at least 24.3 percent of the original face value, as we vary the correlation-determining parameter ρ. This is also illustrative of a calculation of the probability of failing to meet an over-collateralization target.

Appendices

A Default Probabilities and Pricing for the Affine Model

This appendix provides some technical background and extensions for affine default probability and bond valuation models. A probability space $(\Omega, \mathcal{F}, P)$ and filtration

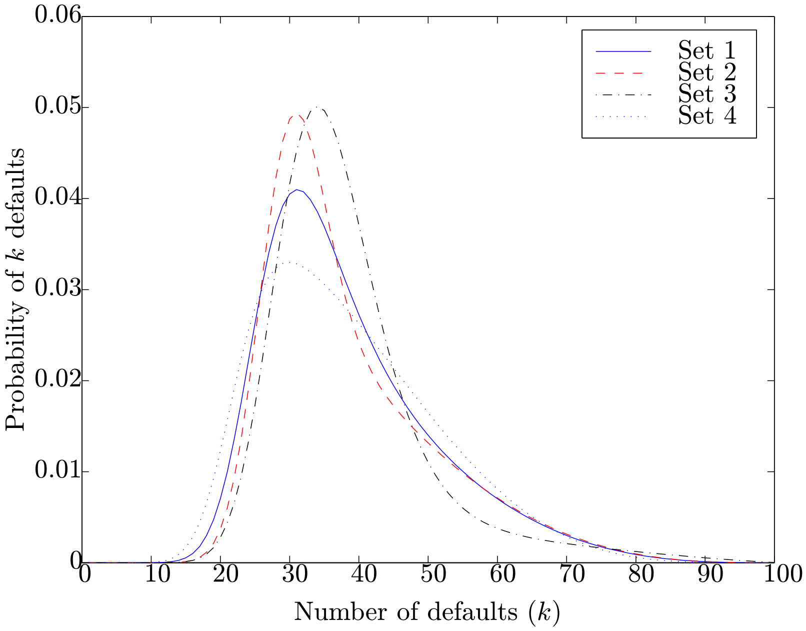

Figure 16: Probability of k defaults, all parameter sets, moderate correlation ($\rho = 0.5$).

$\{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions are fixed. For details, see Protter [1990].

A stopping time τ has an intensity process η (non-negative, predictable, with $\int_0^t \eta_s\, ds < \infty$ for all t) if a martingale M is defined by $M_t = Y_t - \int_0^t \eta_s\, ds$, where $Y_t = 0$ for $t < \tau$ and $Y_t = 1$ for $t \geq \tau$. For a given Markov process X valued in some state space $\mathcal{X}$, the point process Y associated with τ is said to be doubly-stochastic, driven by X with pre-intensity λ, if, conditional on X, τ is the first jump time of a Poisson process with deterministic intensity $\{\lambda(t) : t \geq 0\}$, for $\lambda_t = \Lambda(X_t)$, where $\Lambda : \mathcal{X} \to [0, \infty)$. This in turn implies, by the law of iterated expectations that, on the event that $\tau > t$, for any $s > 0$ we have

$$P\left(\tau > t + s \;\middle|\; \mathcal{F}_t\right) = E\left[\exp\left(\int_t^{t+s} -\lambda_u\, du\right) \;\middle|\; \mathcal{F}_t\right]. \tag{A.1}$$

For our case of stopping times $\tau_1, \ldots, \tau_N$ with respective intensities $\lambda_1, \ldots, \lambda_N$, we suppose that the corresponding multivariate point process $Y = (Y_1, \ldots, Y_N)$ is doubly stochastic driven by X, in the sense that Y_i is doubly stochastic driven by X, and that, conditional on X, the processes $Y_1, \ldots, Y_N$ are independent.

Our state process $X = (X_1, \ldots, X_N, X_c)$, consisting of independent basic affine processes, is a special case of a general multivariate affine jump-diffusion process X valued in a subset $\mathcal{X}$ of $\Re^d$, formulated by Duffie and Kan [1996]. We can suppose that $\lambda_i(t) = a_i + b_i \cdot X_t$ and that the short rate process r is of the form $r_t = a_r + b_r \cdot X_t$.

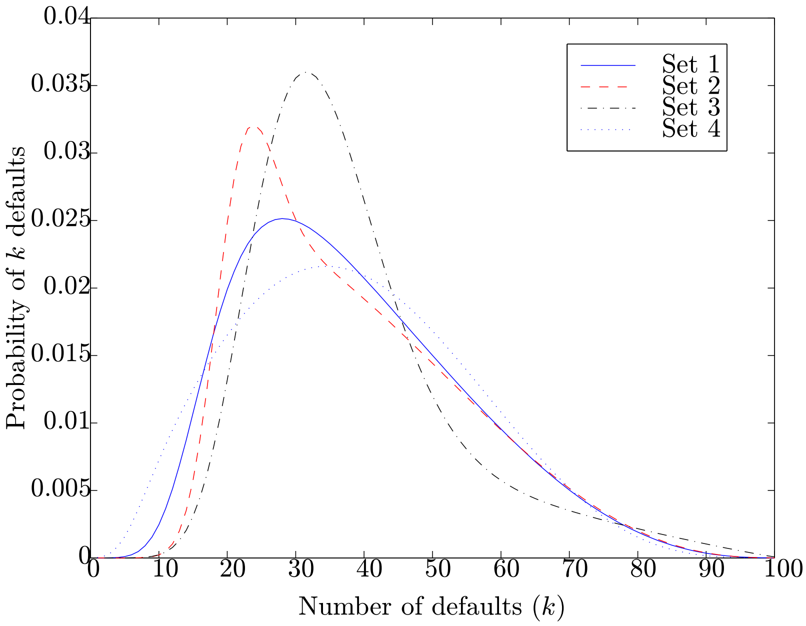

Figure 17: Probability of k defaults, all parameter sets, high correlation ($\rho = 0.9$).

Under regularity conditions, and working under an equivalent martingale measure, this implies a default-free zero-coupon discount at time t, for maturity date T, of the form $e^{A(t,T)+B(t,T)\cdot X(t)}$, for coefficients $A(t,T)$ and $B(t,T)$ in $\Re^d$ that satisfy ordinary differential equations (ODEs). As for the defaultable discount, the pricing formula (10) extends to obtain a zero-coupon price of the form

$$p(t, X(0)) = e^{\alpha(t)+\beta(t)\cdot X(0)} + \overline{f} \int_0^t e^{\alpha(u)+\beta(u)\cdot X(0)} [a(u) + b(u) \cdot X(0)] \, ds,$$

where α, β, a, and b, solve given ordinary differential equations. See Lando [1998] and (for computation of the coefficients in the general affine jump-diffusion case) Duffie, Pan, and Singleton [1998].

B Solution for The Basic Affine Model

For the basic affine model, the coefficients $\alpha(s)$ and $\beta(s)$ determining the solution (5) of the survival-time distribution are given as solutions of the ordinary differential equations

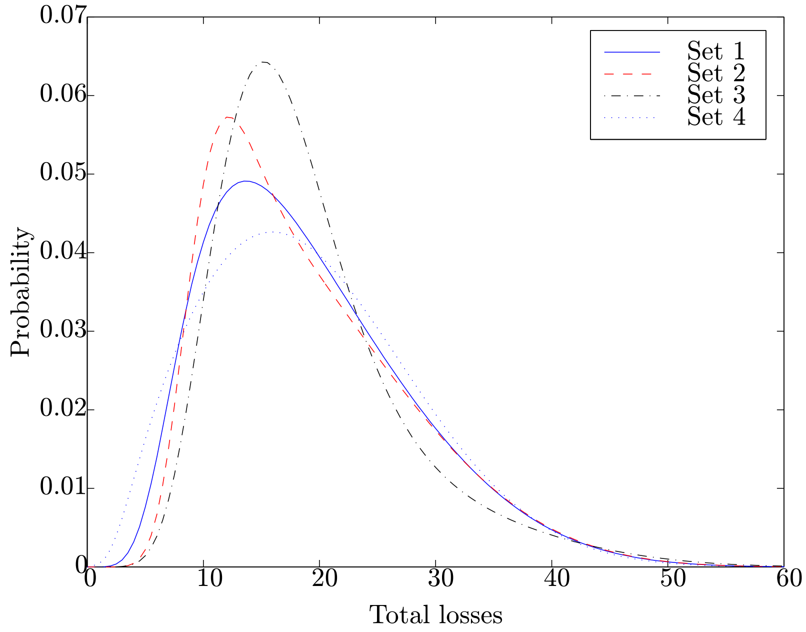

Figure 18: Probability Density of total losses of principal through default. (Parameter Set 1), high correlation.

(ODEs)

$$\beta'_s = n\beta_s + \frac{1}{2}p\beta_s^2 + q, \tag{B.1}$$

$$\alpha'_s = m\beta_s + \ell\frac{\mu\beta_s}{1-\mu\beta_s}, \tag{B.2}$$

with boundary conditons $\alpha(0) = \beta(0) = 0$, for $n = -\kappa$, $p = \sigma^2$, $q = -1$, and $m = \kappa\theta$. The solutions are given by:

$$\beta_s = \frac{1-e^{b_1 s}}{c_1 + d_1 e^{b_1 s}} \tag{B.3}$$

$$\begin{aligned} \alpha_s = \; & \frac{m(-c_1-d_1)}{b_1 c_1 d_1}\log\frac{c_1+d_1e^{b_1 s}}{c_1+d_1} + \frac{m}{c_1}s + \\ & \frac{\ell(a_2c_2-d_2)}{b_2c_2d_2}\log\frac{c_2+d_2e^{b_2 s}}{c_2+d_2} + \left(\frac{\ell}{c_2}-\ell\right)s, \end{aligned} \tag{B.4}$$

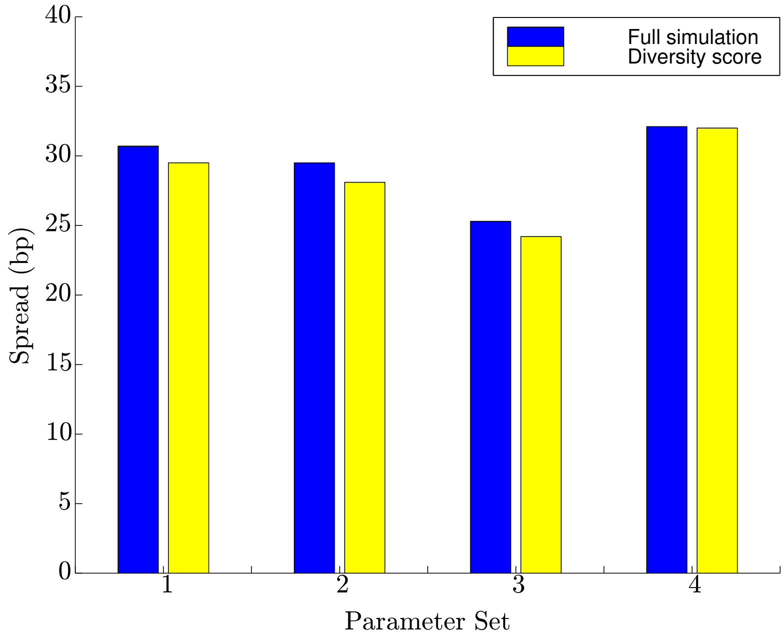

Figure 19: Senior Tranche spreads and diversity-based approximate spreads, with high correlation ($\rho = 0.9$) and low subordination ($P_1 = 92.5$).

where

$$
\begin{aligned}
c_1 &= \frac{-n + \sqrt{n^2 - 2pq}}{2q} \\
d_1 &= \frac{n + \sqrt{n^2 - 2pq}}{2q} \\
b_1 &= -\frac{n(d_1 - c_1) + 2qc_1d_1 - p}{c_1 + d_1} \\
a_2 &= \frac{d_1}{c_1} && \text{(B.5)} \\
b_2 &= b_1 && \text{(B.6)} \\
c_2 &= 1 - \frac{\mu}{c_1} && \text{(B.7)} \\
d_2 &= \frac{d_1 + \mu}{c_1}. && \text{(B.8)}
\end{aligned}
$$

More generally, for constants q, u, and v, under technical integrability conditions we

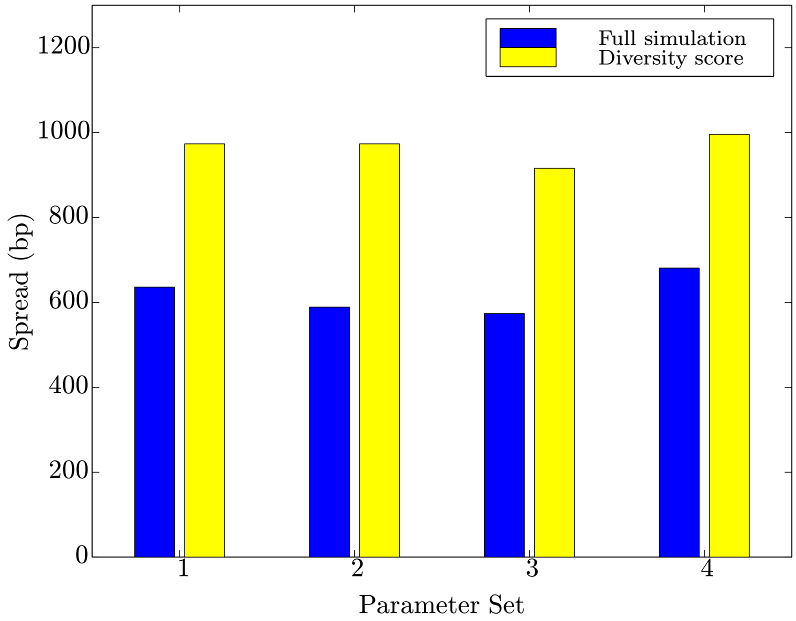

Figure 20: Mezzanine Tranche spreads and diversity-based approximate spreads, with moderate correlation ($\rho = 0.5$) and low subordination ($P_1 = 92.5, P_2 = 5$).

have

$$E\left[\exp\left(\int_0^s q\lambda(t)\,dt\right)e^{v+u\lambda(s)}\right] = e^{\alpha(s)+\beta(s)\lambda(s)}, \tag{B.9}$$

where α and β solve the same ODEs (B.1)-(B.2) with the more general boundary conditions $\alpha(0) = v$ and $\beta(0) = u$. For this general case, the solutions (barring degenerate cases, which may require separate treatment) are given by

$$\beta_s = \frac{1 + a_1 e^{b_1 s}}{c_1 + d_1 e^{b_1 s}} \tag{B.10}$$

$$\begin{aligned} \alpha_s = \; & v + \frac{m(a_1 c_1 - d_1)}{b_1 c_1 d_1} \log \frac{c_1 + d_1 e^{b_1 s}}{c_1 + d_1} + \frac{m}{c_1} s + \\ & \frac{\ell(a_2 c_2 - d_2)}{b_2 c_2 d_2} \log \frac{c_2 + d_2 e^{b_2 s}}{c_2 + d_2} + \left(\frac{\ell}{c_2} - \ell\right) s, \end{aligned} \tag{B.11}$$

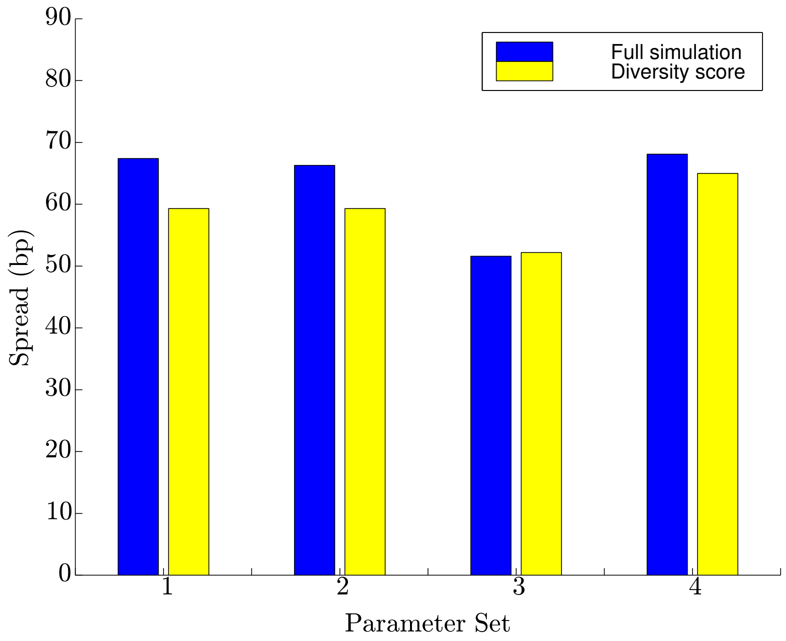

Figure 21: Mezzanine spreads and diversity-based approximate spreads, with moderate correlation ($\rho = 0.5$) and high subordination ($P_1 = 80$, $P_2 = 10$).

where

$$
\begin{aligned}
c_1 &= \frac{-n + \sqrt{n^2 - 2pq}}{2q} \\
d_1 &= (1 - c_1 u)\frac{n + pu + \sqrt{n^2 - 2pq}}{2nu + pu^2 + 2q} \\
a_1 &= (d_1 + c_1)u - 1 \\
b_1 &= \frac{d_1(n + 2qc_1) + a_1(nc_1 + p)}{a_1 c_1 - d_1} \\
a_2 &= \frac{d_1}{c_1} \\
b_2 &= b_1 \\
c_2 &= 1 - \frac{\mu}{c_1} \\
d_2 &= \frac{d_1 - \mu a_1}{c_1}.
\end{aligned}
$$

With this, we can compute the characteristic function $\varphi(\,\cdot\,)$ of $\lambda(s)$ given $\lambda(0)$, defined

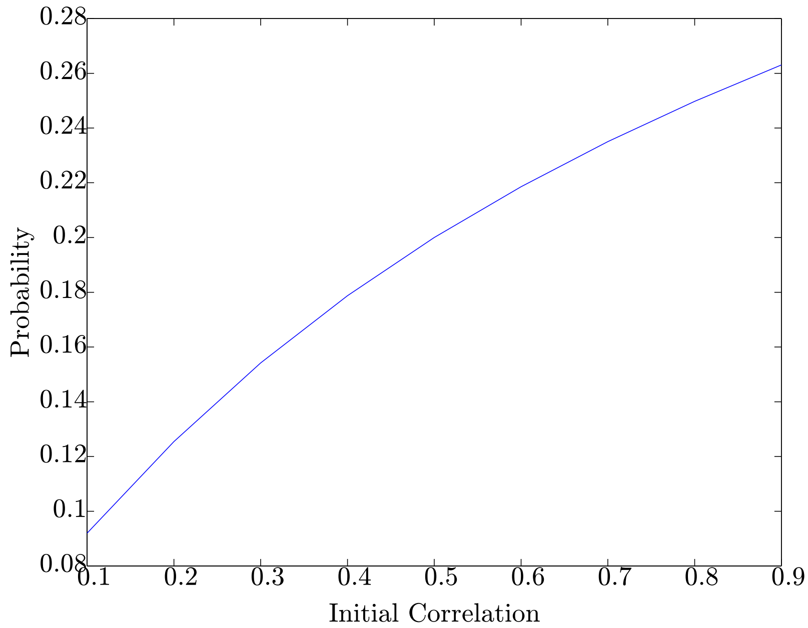

Figure 22: Likelihood of total default losses of at least 24.3% of principal within 10 years.

by

$$\varphi(z) = E\left(e^{iz\lambda(s)}\right) = e^{\alpha(s)+\beta(s)\lambda(s)}, \tag{B.12}$$

by taking $q = 0$ and by taking $u = iz$, treating the coefficients $\alpha(s)$ and $\beta(s)$ as complex. The Laplace transform $L(\,\cdot\,)$ of $\lambda(s)$ is computed analogously:

$$L(z) = E\left(e^{-z\lambda(s)}\right) = e^{\alpha(s)+\beta(s)\lambda(s)}, \tag{B.13}$$

this time taking $u = -z$. For details and extensions to a general affine jump-diffusion setting, see Duffie, Pan, and Singleton [1998].

Proof of Proposition 1:

Since both X and Y are (weakly) positive processes (see Feller [1951]), Z is, and hence it suffices to verify that its Laplace transform coincides with that of a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, \ell)$ and initial condition $Z(0) = X(0) + Y(0)$. To this end, one needs to compute the Laplace transform of a basic affine process at an arbitrary time t, which is easily done according to equation (B.13) and the explanation following it. In order for equations (B.1)-(B.2) to characterize the transform, it is sufficient that, when applying Ito's Formula to $U_t = e^{\alpha(T-t)+\beta(T-t)\cdot X(t)}$, the Brownian integral term is a martingale (not merely a local martingale). This martingale property holds if

$$E\left[\int_0^t e^{2\alpha(s)+2\beta(s)X(s)}\beta(s)^2 X(s)ds\right] < \infty.$$

This condition is clearly satisfied from Fubini's Theorem, since $\alpha(s)$, $\beta(s)$, and $E\left(X(s)\right)$ are continuous in s, while β is negative, and X is positive. Now it is a simple matter to verifiy that the sum of the Laplace transform of X and Y equals the transform of the basic affine process with parameters $(\kappa, \theta, \sigma, \mu, \ell)$ and initial condition $Z(0) = X(0) + Y(0)$, using the explicit solutions in (B.10)-(B.11).

Proof of Proposition 2:

The argument proceeds by induction in k and N. Assume, therefore, that the claim holds for all positive pairs (k, n) such that $n < N$ and $k \leq n$. Now let $n = N$. First of all, as an immediate consequence of the inclusion-exclusion principle (see, for instance, M. Hall [1986]),

$$q(n, n) = \textstyle\sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} p_j,$$

which is clearly consistent with the claim. Assume now that the claim holds for $(k+1, n), \ldots, (n, n)$. We have

$$\begin{aligned} q(k+1, n) &= P(d_1 \cap \cdots \cap d_{k+1} \cap d_{k+2}^c \cap \cdots \cap d_n^c) \\ q(k, n) &= P(d_1 \cap \cdots \cap d_k \cap d_{k+1}^c \cap \cdots \cap d_n^c). \end{aligned}$$

Adding these two equations,

$$q(k+1, n) + q(k, n) = P(d_1 \cap \cdots \cap d_k \cap d_{k+2}^c \cap \cdots \cap d_n^c) = q(k, n-1), \tag{B.14}$$

whence $q(k, n) = q(k, n-1) - q(k+1, n)$. It is now a simple matter to verify the proposed formula for $q(k, n)$, using the fact that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. This completes the proof.

C Computation of Diversity Scores

We define the diversity score S associated with a "target" portfolio of bonds of total principal F to be the number of identically and independently defaulting bonds, each with principal $\frac{F}{S}$, whose total default losses have the same variance as the target-portfolio default losses. The computation of S entails computation of the variance of losses on the target portfolio of N bonds, which we address in this appendix.

Perhaps because of the goal of simplicity, any interest-rate effects on losses have been ignored in the calculation of diversity, and we shall do so here as well, although such effects could be tractably incorporated, for example by the discounting of losses. (With correlation between interest rates and default losses, the meaningfulness of direct discounting is questionable.) It is also problematic that lost-coupon effects are not separately considered in diversity scores. These effects, which could be particularly important for high-premium bonds, can also be captured along the lines of the following calculations. Finally, diversity scores do not account directly for the replacement of defaulted collateral, or new investment in defaultable securities during the life of a product, except insofar as covenants or ratings requirements stipulate a minimum diversity score that is to be maintained for the current collateral portfolio, for the life of the CDO structure. This might argue for a short-horizon, say 1-year, diversity score, even for long-maturity collateral. That can also be accomodated in our calculations.

Letting "d_i" denote the indicator of the event that participation i defaults by a given time T, assumed to be after its maturity, and letting L_i denote the random loss of principal when this event occurs, we have

$$\begin{aligned}\operatorname{var}\left(\sum_{i=1}^{N} L_i d_i\right) &= E\left[\left(\sum_{i=1}^{N} L_i d_i\right)^2\right] - \left(E\left[\sum_{i=1}^{N} L_i d_i\right]\right)^2 \\ &= \sum_{i=1}^{S} E\left(L_i^2\right) E\left(d_i^2\right) + \sum_{i\neq j} E\left(L_i L_j\right) E\left(d_i d_j\right) - \sum_{i=1}^{N} \left(E\left(L_i\right)\right)^2 \left(E\left(d_i\right)\right)^2 . \quad \text{(C.1)}\end{aligned}$$

Given an affine intensity model, one can compute all terms in (C.1). In the symmetric case, letting $p_{(1)}$ denote the marginal probability of default of a bond, and $p_{(2)}$ the joint probability of default of any two bonds, the above reduces to:

$$\operatorname{var}\left(\sum_{i=1}^{N} L_i d_i\right) = N p_{(1)} E\left(L_i^2\right) + N(N-1) p_{(2)} \left(E\left(L_i\right)\right)^2 - N^2 p_{(1)}^2 \left(E\left[L_i\right]\right)^2 .$$

Equating the variance of the original pool to that of the comparison pool yields

$$\begin{aligned}&\frac{N}{S}\left(p_{(1)} E\left[L_i^2\right] - p_{(1)}^2 \left(E\left[L_i\right]\right)^2\right) \\ &\quad = p_{(1)} E\left[L_i^2\right] + (N-1) p_{(2)} \left(E\left[L_i\right]\right)^2 - N p_{(1)}^2 \left(E\left[L_i\right]\right)^2 . \quad \text{(C.2)}\end{aligned}$$

Solving this equation for the diversity score, S, one gets

$$S = \frac{N\left(p_{(1)} E\left[L_i^2\right] - p_{(1)}^2 \left(E\left[L_i\right]\right)^2\right)}{p_{(1)} E\left[L_i^2\right] + (N-1) p_{(2)} \left(E\left[L_i\right]\right)^2 - N p_{(1)}^2 \left(E\left[L_i\right]\right)^2}.$$

To end the computation, one uses the identities:

$$\begin{aligned}p_{(1)} &= p_1 \\ p_{(2)} &= 2p_1 - p_2,\end{aligned}$$

where p_1 and p_2 are computed according to equation (17), as well as the fact that $E\left[L_i^2\right] = \frac{1}{3}$ and $\left(E\left[L_i\right]\right)^2 = \frac{1}{4}$, assuming losses are uniformly distributed on [0,1]. (Other assumptions on the distribution of L_i, even allowing for correlation here, can be accomodated.)

More generally, suppose $\lambda_i(t) = b_i \cdot X(t)$ and $\lambda_j(t) = b_j \cdot X(t)$, where X is a multivariate affine process of the general type considered in Appendix A, and b_i and b_j are coefficient vectors. Even in the absence of symmetry, for any times $t(i)$ and $t(j)$, assuming without loss of generality that $t(i) \leq t(j)$, the probability of default by i before $t(i)$ and of j before $t(j)$ is

$$E\left(d_i d_j\right) = 1 - E\left[e^{-\int_0^{t(i)} \lambda_i(u)\,du}\right] - E\left[e^{-\int_0^{t(j)} \lambda_j(u)\,du}\right] + E\left[e^{-\int_0^{t(j)} b(t)\cdot X(t)\,du}\right], \qquad \text{(C.3)}$$

where $b(t) = b_i + b_j$ for $t < t(i)$ and $b(t) = b_j$ for $t(i) \leq t \leq t(j)$. Each of the terms in (C.3) is analytically explicit in an affine setting, as can be gathered from Appendices A and B. Beginning with (C.3), the covariance of default losses between any pair of participations, during any pair of respective time windows, may be calculated, and from that the total variance of default losses on a portfolio can be calculated, and finally the diversity score S can be calculated. With lack of symmetry, however, one must take a stand on the definition of an “average” default to be applied to each of the S independently defaulting issues of the comparison portfolio. We do not address that issue here. A pragmatic decision could be based on further investigation, perhaps accompanied by additional empirical work.

Table 8: Par spreads obtained from simulation based on diversity scores

	Corr.	Divers.	Principal		Spread (Uniform)		Spread (Fast)	
Set	ρ	Score	P_1	P_2	s_1 (bp)	s_2 (bp)	s_1 (bp)	s_2 (bp)
1	0.1	59	92.5	5	6.58	604.0	0.99	150
1	0.5	22	92.5	5	17.2	973.6	8.14	340
1	0.9	13	92.5	5	29.5	1172	19.1	452
1	0.1	59	80	10	0.01	14.5	0.00	1.05
1	0.5	22	80	10	0.57	59.3	0.04	18.0
1	0.9	13	80	10	2.15	116	0.77	51.2
2	0.1	59	92.5	5	6.58	604.0	0.99	150
2	0.5	22	92.5	5	17.2	973.6	8.14	340
2	0.9	14	92.5	5	28.1	1148	17.7	442
2	0.1	59	80	10	0.01	14.5	0.00	1.05
2	0.5	22	80	10	0.57	59.3	0.04	18.0
2	0.9	14	80	10	1.93	108.8	0.68	47.5
3	0.1	63	92.5	5	6.19	585	0.78	140
3	0.5	25	92.5	5	15.6	916	7.04	313
3	0.9	16	92.5	5	24.2	1107	14.2	405
3	0.1	63	80	10	0.02	12.5	0.00	0.81
3	0.5	25	80	10	0.43	52.2	0.05	15.5
3	0.9	16	80	10	1.34	90.3	0.44	36.4
4	0.1	57	92.5	5	6.76	605	1.04	153
4	0.5	21	92.5	5	18.6	996	9.46	367
4	0.9	12	92.5	5	32.0	1203	21.3	474
4	0.1	57	80	10	0.02	14.5	0.00	1.14
4	0.5	21	80	10	0.68	65.0	0.11	22.0
4	0.9	12	80	10	2.71	126.8	1.25	56.5

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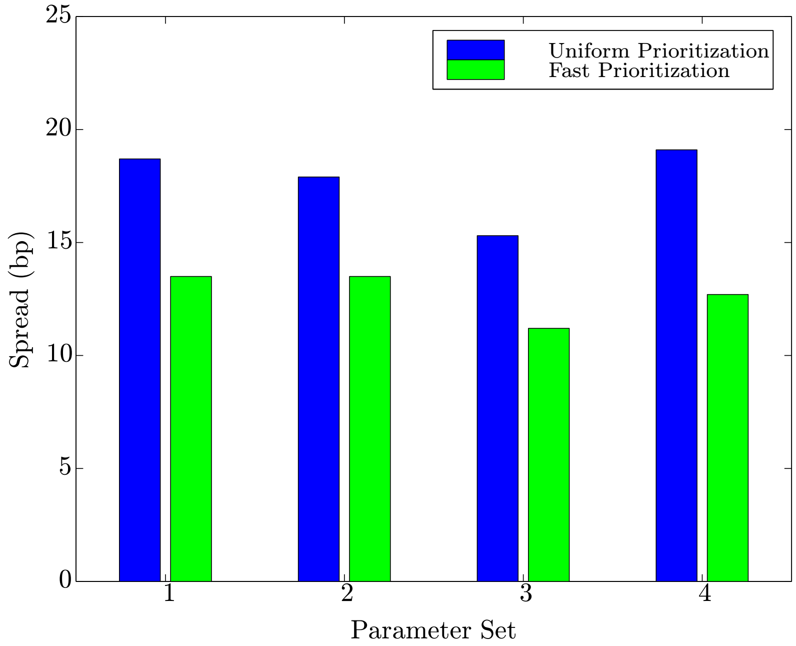

Figure 23: Senior Tranche spreads in the uniform and the fast prioritization schemes ($P_1 = 92.5, P_2 = 5$).

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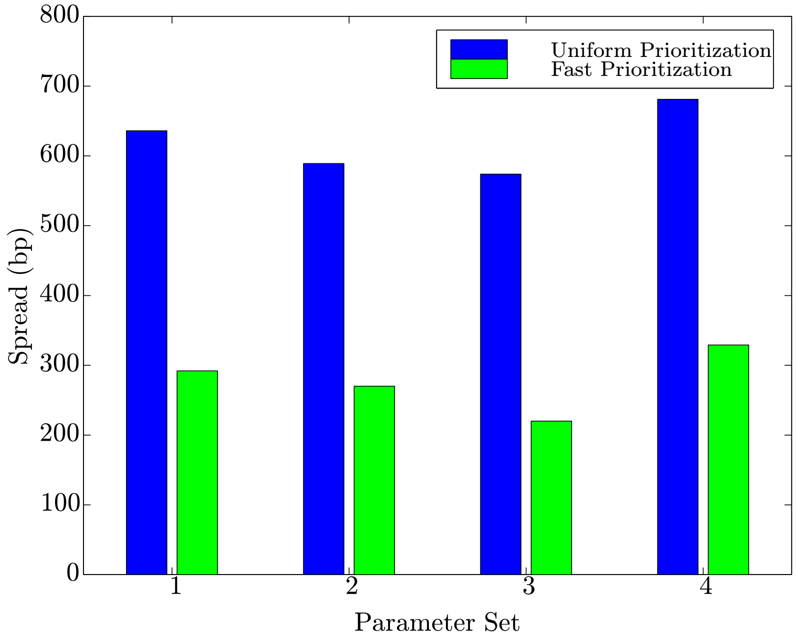

Figure 24: Mezzanine Tranche spreads in the uniform and the fast prioritization schemes ($P_1 = 92.5, P_2 = 5$).

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