Term Structures of Credit Spreads with Incomplete Accounting Information[1]

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Abstract: We study the implications of imperfect information for term structures of credit spreads on corporate bonds. We suppose that bond investors cannot observe the issuer's assets directly, and receive instead only periodic and imperfect accounting reports. For a setting in which the assets of the firm are a geometric Brownian motion until informed equityholders optimally liquidate, we derive the conditional distribution of the assets, given accounting data and survivorship. Contrary to the perfect-information case, there exists a default-arrival intensity process. That intensity is calculated in terms of the conditional distribution of assets. Credit yield spreads are characterized in terms of accounting information. Generalizations are provided.

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1 Introduction

This paper analyses term structures of credit risk and yield spreads in secondary markets for the corporate debt of firms that are not perfectly transparent to bond investors.

The valuation of risky debt is central to theoretical and empirical work in corporate finance. A leading paradigm in corporate-bond valuation has taken as given the dynamics of the assets of the issuing firm, and priced corporate bonds as contingent claims on the assets, as in Black and Scholes (1973) and Merton (1974). Generalizations treat coupon bonds and the effects of bond indenture provisions [Black and Cox (1976) and Geske (1977)] and stochastic interest rates [Shimko, Tejima and van Deventer (1993) and Longstaff and Schwartz (1995)]. By introducing bankruptcy costs and tax effects, this framework has been extended to treat endogenous capital structure, liquidation policy, re-capitalization, and re-negotiation of debt [Brennan and Schwartz (1984), Fischer, Heinkel, and Zechner (1989), Leland (1994, 1998), Leland and Toft (1996), Uhrig-Homburg (1998), Anderson and Sundaresan (1996), Mella-Baral and Perraudin (1997).] These second-generation models allow for endogenous default, optimally triggered by equity owners when assets fall to a sufficiently low level.

In practice, it is typically difficult for investors in the secondary market for corporate bonds to observe a firm's assets directly, because of noisy or delayed accounting reports, or barriers to monitoring by other means. Investors must instead draw inference from the available accounting data and from other publicly available information, for example business-cycle data, that would bear on the issuer's credit quality. Under informational assumptions, we derive public investors' conditional distribution of the issuer's assets, explicitly accounting for the implications of imperfect information and survivorship. This provides a model for conditional default probabilities at each future maturity. Default occurs at an arrival intensity that bond investors can calculate in terms of observable variables.

We show several significant implications of incomplete information for the level and shape of the term structure of secondary-market yield spreads (the excess over risk-free interest rates at which corporate bond prices are quoted in public markets). With perfect information, yield spreads for surviving firms are zero at zero maturity, and are relatively small for small maturities, regardless of the riskiness of the firm. As illustrated in Figure 1, yield spreads for relatively risky firms would, with perfect information, eventually climb rapidly with maturity. (The assumptions and parameters used for this figure are explained in Section 2.) Such severe variation in the shape of the term structure of yield spreads is uncommon in practice [Fons (1994); Helwege and Turner (1999); Johnson (1967); Jones, Mason, and Rosenfeld (1984); and Sarig and Warga (1989).] With imperfect information, however, yield spreads are strictly positive at zero maturity because investors' are uncertain about the nearness of current assets to the trigger level at which the firm would declare default. This uncertainty causes a more moderate variation in spreads with maturity, as illustrated in Figure 1 for an otherwise identical firm whose accounting reports are noisy. The shape of the term structure of credit spreads may indeed play a useful empirical role in estimating the degree of transparency of a firm as viewed by bond-market participants. The existence of a default intensity, moreover, is consistent with the fact that bond prices often drop precipitously at or around the time of default. (With perfect information, as default approaches bond prices would converge continuously to their default-contingent values.)

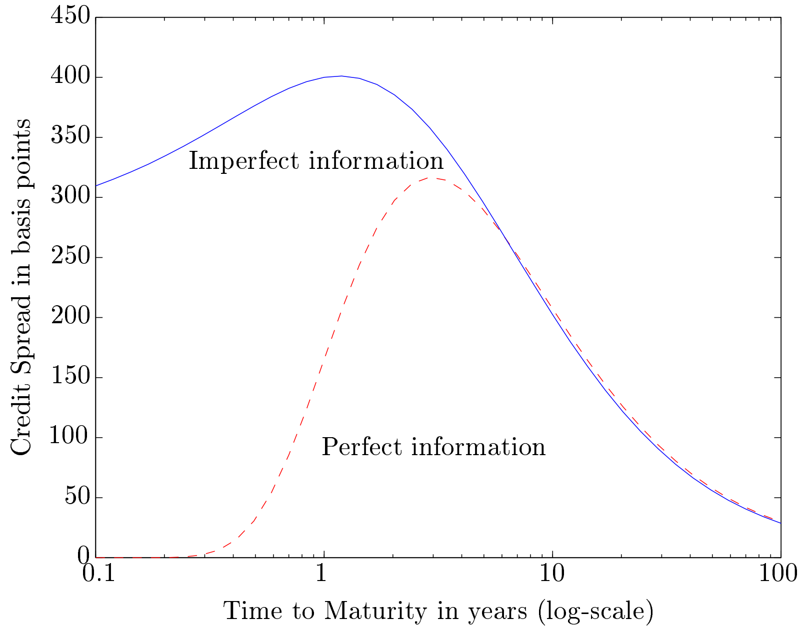

Figure 1: Credit Spreads under perfect and imperfect information.

As opposed to the *structural models* described above, which link default explicitly to the first time that assets fall below a certain level, a more recent literature has adopted a *reduced-form* approach, assuming that the default arrival intensity exists, and formulating it directly as a function of latent state variables or predictors of default.[2] The popularity of this reduced-form approach to modeling defaultable bonds and credit derivatives, particularly in business and econometric applications [Duffee (1999)] arises from its tractability.

This approach allows the application of statistical methods for estimating the incidence of default [Altman (1968), Bijnen and Wijn (1994), Lennox (1999), Lundstedt and Hillgeist (1998), McDonald and Van de Gucht (1999), Shumway (1996)], and the use of convenient methods for risk management and derivative pricing that were originally developed for default-free term structures.

We present here a first example of a structural model that is consistent with such a reduced-form representation.

Bounding short spreads away from zero can also be obtained in a structural model in which the assets of a firm are perfectly observable and given by a jump-diffusion process, as in Zhou (1997). This approach is not, however, consistent[3] with a stochastic intensity for default (unless the only variation in asset levels is through jumps). Hence, this approach does not lay the theoretical groundwork for hazard-rate based estimation of default intensities. With precisely measured assets, moreover, this jump-diffusion model offers no role in the estimation of default risk for such potentially useful explanatory variables as duration of survivorship, industrial performance, or consumer confidence. One could, of course, build a model for a firm with complete information in which various different state variables determine the dynamics of assets or liabilities, and would therefore also be determinants of default risk.

[2]See for example Artzner and Delbaen (1995), Arvanitis, Gregory and Laurent (1999) Duffie and Singleton (1999), Duffie, Schroder, and Skiadas (1996), Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), Lando (1994, 1998), Madan and Unal (1998), and Schönbucher (1998).

[3]For a stopping time τ to have an associated intensity, it must (among other properties) be totally inaccessible, meaning that, for *any* sequence of stopping times, the probability that the sequence approaches τ strictly from below is zero. See, for example, Meyer (1966), page 130, Definition D42. The first time $\tau = \inf\{V_t \leq V_B\}$ that the asset process V defined by Zhou (1997) cross the default boundary V_B could be the time of a jump, or could be via a continuous ("diffusion") crossing. That is, if we let $\tau_n = \inf\{t : V_t \leq V_B + n^{-1}\}$, then there is a strictly positive probability that τ_n converges to τ, so τ is not totally inaccessible, and therefore does not have an intensity.

2 The Basic Model

This section presents our basic model of a firm with incomplete secondary-market information about the credit quality of its debt. For simplicity, we treat a time-homogeneous setting, staying within the tradition of the work of Anderson, Pan, and Sundaresan (1995), Anderson and Sundaresan (1996), Fan and Sundaresan (1997), Fischer, Heinkel, and Zechner (1989), Leland (1994, 1998), Leland and Toft (1996), Mella-Barral (1999), Mella-Barral and Perraudin (1997), and Uhrig-Homburg (1998). We solve for the optimal capital structure and default policy, and then derive the conditional distribution of the firm's assets, given incomplete accounting information, along with the associated default probabilities, default arrival intensity, and credit spreads. In Section 3, we treat extensions of the basic model.

2.1 Setup and Optimal Liquidation

We begin by reviewing a standard model of a firm's assets, capital structure, and optimal liquidation policy. With some exceptions, the results are basically those of Leland and Toft (1996).

The stochastic process V describing the stock of assets of our given firm is modeled as a geometric Brownian motion, which is defined, along with all other random variables, on a fixed probability space $(\Omega, \mathcal{F}, P)$. In particular, $V_t = e^{Z(t)}$, where $Z_t = Z_0 + mt + \sigma W_t$, for a standard Brownian motion W, a volatility parameter $\sigma > 0$, and a parameter $m \in (-\infty, \infty)$ that determines the expected asset growth rate $\mu = t^{-1} \log[E(V_t/V_0)] = m + \sigma^2/2$. The firm generates cash flow at the rate δV_t at time t, for some constant $\delta \in (0, \infty)$.

The firm issues debt so as to take advantage of the tax shields offered for interest expense, at the constant tax rate $\theta \in (0, 1)$. In order to stay in a simple time-homogeneous setting, the debt is modeled as a consol bond, meaning a commitment to pay coupons indefinitely at some constant total coupon rate $C > 0$. Tax benefits for this bond are therefore received at the constant rate θC, until liquidation. (We briefly consider callable debt in Section 3.)

All agents in our model are risk-neutral, and discount cash flows at a fixed market interest rate r.

The debt is sold at time 0 for some amount D to be determined shortly. For contractual purposes, it assumed that the debt is issued at par, determining its face value D and coupon rate C/D. The optimal amount of debt to be issued will be discussed after considering its valuation.

The firm is operated by its equity owners, who are completely informed at all times of the firm's assets. This means that they have the information filtration $(\mathcal{F}_t)$ generated[4] by V. The expected present value at time t of the cash flows to be generated by the assets, excluding the effects of liquidation losses and tax shields, conditional on the current level V_t of assets, is

$$E\left[\int_t^\infty e^{-r(s-t)}\delta V_s\, ds \;\Big|\; V_t\right] = \frac{\delta V_t}{r-\mu}, \tag{1}$$

provided that $r > \mu$. (If $\mu \geq r$, the present value of cash flows is infinite, a case we do not consider.) One could therefore equally well take this value or the firm's cash flow rate δV_t, as the state variable for the firm's opportunities, as these alternative state variables are simply multiples of the stock of assets V_t.

For simplicity, the equity owners' only choice is when to liquidate the firm. A liquidation policy is an $(\mathcal{F}_t)$-stopping time $\tau : \Omega \to [0, \infty]$. Given an asset level at liquidation of V_τ, we choose for simplicity to define the liquidation value of the assets to be $\delta(r-\mu)^{-1}V_\tau$, i.e. the present unlevered value defined in (1). Alternatively, one could consider a liquidation value which reflects the value of assets in an optimally levered, reorganized firm. This would require the solution of an associated fixed-point problem - a problem which is not central to the objectives of this paper. At the chosen liquidation time τ, a fraction $\alpha \in [0,1]$ of the assets are lost as a frictional cost. The value of the remaining assets, $\delta(r-\mu)^{-1}(1-\alpha)V_\tau$, is, by an assumption of strict priority, assigned to debtholders.[5]

Proceeds from the sale of debt are paid at time 0 as a cash distribution to initial equity holders. After this distribution, the initial value of equity to

[4]That is, for each $t > 0$, $\mathcal{F}_t$ is the σ-algebra generated by $\{V_s : 0 \leq s \leq t\}$.

[5]For an alternative formulation, one might suppose that equity holders would receive at liquidation any excess of the recovery value of assets, $\delta(r-\mu)^{-1}(1-\alpha)V_\tau$, over the face value D of debt, with $\min(\delta(r-\mu)^{-1}(1-\alpha)V_\tau, D)$ going to debtholders. For certain parameters, such as those used in our illustrative numerical example to follow, the two formulations imply the same default policy and valuations, as equityholders would in any case optimally declare default only when there are insufficient post-recovery assets to cover the face value of debt. In other parameters cases (with large V_0, r, and C, and with negligible σ, μ, and δ), equityholders given the chance for recovery of $\max(0, \delta(r-\mu)^{-1}(1-\alpha)V_\tau - D)$ would liquidate at time 0.

shareholders, given a liquidation policy τ and coupon rate C, is

$$F(V_0, C, \tau) = E\left[\int_0^\tau e^{-rt}(\delta V_t + (\theta - 1)C)\, dt\right]. \tag{2}$$

Equity shareholders would therefore choose the liquidation policy solving the optimization problem

$$S_0 = \sup_{\tau \in \mathcal{T}} F(V_0, C, \tau), \tag{3}$$

where $\mathcal{T}$ is the set of $(\mathcal{F}_t)$-stopping times.

If at some time t the asset level V_t is less than C/δ, then equity owners have a net negative dividend rate.[6] Equity owners may nevertheless prefer to continue operating the firm, suffering such negative distributions, bearing in mind that assets might eventually rise and create large future dividends. At some sufficiently low level of assets, however, the prospects for such a future recovery are dim enough to warrant immediate liquidation. Indeed, the optimal liquidation time, as shown in a different[7] context by Leland (1994), is the first time $\tau(V_B) = \inf\{t : V_t \leq V_B\}$ that the asset level falls to some sufficiently low boundary $V_B > 0$. The optimality of such a trigger policy is unsurprising, as V_t is a sufficient statistic for the firm's future cash flows, and cash flows are increasing in V_t.

Specifically, one conjectures that the optimal equity value at time t,

$$S_t = \operatorname{ess}\ \sup_{\tau \in \mathcal{T}} E\left[\int_t^\tau e^{-r(s-t)}(\delta V_s + (\theta - 1)C)\, ds \;\middle|\; \mathcal{F}_t\right], \tag{4}$$

is given by $S_t = w(V_t)$, where w solves the Hamilton-Jacobi-Bellman differential equation

$$w'(v)\mu v + \frac{1}{2}w''(v)\sigma^2 v^2 - rw(v) = (1 - \theta)C - \delta v, \qquad v > V_B, \tag{5}$$

with the boundary conditions

$$w(v) = 0, \qquad v \leq V_B, \tag{6}$$

[6]In a model restricted to have non-negative net equity dividends per share, such net negative cash distributions could be funded by dilution, for example through share purchase rights issued to current shareholders at the current valuation.

[7]Leland's 1994 model has $\delta = 0$. With $\delta = 0$, the manner in which equity holders extract value is not modeled. The subsequent results of Leland and Toft (1996) and Leland (1998) have $\delta > 0$, although the candidate solution was not subjected to verification.

and

$$w'(V_B) = 0. \tag{7}$$

Under this conjecture, to be verified shortly, (6) means that it is no longer optimal to operate the firm once the equity value has been reduced to its liquidation value, while (7) is the so-called "smooth-pasting" condition. The differential equation (5) states that, so long as it optimal to continue operating the firm, the expected rate of increase in equity value, net of the rate of opportunity cost $rw(v)$ of equity capital, is equal the net rate at which cash is paid out by equity.

The HJB equation is solved by $w(v) = 0$ for $v \leq V_B = v_B(C)$, where

$$v_B(C) = \frac{(1-\theta)C\gamma(r-\mu)}{r(1+\gamma)\delta}, \tag{8}$$

where

$$\gamma = \frac{m + \sqrt{m^2 + 2r\sigma^2}}{\sigma^2},$$

and, for $v > V_B$,

$$w(v) = \frac{\delta v}{r-\mu} - \frac{v_B(C)\delta}{r-\mu}\left(\frac{v}{v_B(C)}\right)^{-\gamma} + (\theta - 1)\frac{C}{r}\left[1 - \left(\frac{v}{v_B(C)}\right)^{-\gamma}\right]. \tag{9}$$

The first term in (9) represents the present value of all future cash flows generated by assets, assuming no liquidation. The second term represents the present value of the cash flows lost to distress or transferred to debtholders at the time of liquidation. The last term represents the costs of all future debt coupon payments, net of tax shields. The key factor $(v/v_B(C))^{-\gamma}$ is the present value at an asset level $v > v_B(C)$ of receiving one unit of account at the stopping time $\tau(v_B(C))$.

The optimality property

$$S_0 = w(V_0) = F(V_0, C, \tau(V_B)) \tag{10}$$

is verified as follows. For each t, let

$$q_t = e^{-rt}w(V_t) + \int_0^t e^{-rs}(\delta V_s - (1-\theta)C)\, ds.$$

From (5) and Ito's Formula,[8] and noting that for $v \leq v_B(C)$ we have both $w(v) = 0$ and $\delta v - (1-\theta)C \leq 0$, it follows that q is a uniformly integrable super-martingale. Thus, for any stopping time τ, we have $q_0 \geq E(q_\tau)$. This implies that, for any stopping time τ, we have

$$E\left[\int_0^\tau e^{-rs}(\delta V_s - (1-\theta)C)\,ds\right] \leq w(V_0) - E[e^{-r\tau}w(V_\tau)]. \tag{11}$$

As w is non-negative, $w(V_0) \geq F(V_0, C, \tau)$. For the candidate optimal policy $\tau = \tau(V_B)$, we have $w(V_\tau) = 0$ and equality in (11), confirming optimality and (10).

The associated expected present value of the cash flows to the bond at any time t before liquidation, conditional on V_t, is $d(V_t, C)$, where

$$d(v, C) = \frac{(1-\alpha)v_B(C)\delta}{r-\mu}\left(\frac{v}{v_B(C)}\right)^{-\gamma} + \frac{C}{r}\left[1 - \left(\frac{v}{v_B(C)}\right)^{-\gamma}\right]. \tag{12}$$

We summarize as follows:

Proposition 2.1 *Suppose $r > \mu$. Let $v_B(C)$ and $d(V_0, C))$ be defined by (8) and (12), respectively. Then the optimal liquidation problem (3) is solved by the first time $\tau(v_B(C))$ that V is at or below $v_B(C)$. The associated initial values of equity and debt are $w(V_0)$ and $d(V_0, C)$, respectively, where w is given by (9) for $v > v_B(C)$ and $w(v) = 0$ for $v \leq v_B(C)$.*

We suppose that the total coupon rate $C^*(V_0)$ of the bonds to be issued is chosen so as to maximize over C the total initial firm valuation, $F(V_0, C, \tau(v_B(C)) + d(V_0, C)$, which is the initial value of equity plus the sale value of debt.

[8] Although w need not be twice continuously differentiable, it is convex, C^1, and C^2 except at V_B, where $w'(V_B) = 0$. We have $w(v) = 0$ for $v \leq V_B$. Under these conditions,

$$w(V_s) = w(V_0) + \int_0^s 1_{\{V(t) > V_B\}}[w'(V_t)\mu V_t + \frac{1}{2}w''(V_t)\sigma^2 V_t^2]\,dt + \int_0^s w'(V_s)\sigma V_s\,dB_s.$$

(See, for example, Karatzas and Shreve (1988), page 219.) Because $w'(v)$ is bounded, the last term is a martingale.

2.1.1 Example: Optimal Capital Structure and Liquidation

As a numerical illustration, we consider the case

$$\theta = 0.35; \quad \sigma = 0.05; \quad r = 0.06; \quad m = 0.01; \quad \delta = 0.05; \quad \alpha = 0.3. \tag{13}$$

As the solutions for the optimal total coupon rate $C^*(V_0)$ and the optimal liquidation boundary $V_B = v_B(C^*(V_0))$ are linear in V_0, we may without loss of generality take $V_0 = 100$. For these parameters, we have the optimal total coupon rate $C = 8.00$, liquidation boundary $V_B = v_B(C) = 78.0$, and initial par debt level $D = d(V_0, C) = 129.4$. At these parameters, we easily satisfy the zero-equity-at-liquidation condition $\delta(r-\mu)^{-1}(1-\alpha)v_B(C) \leq d(V_0, C)$, so the conclusions of Proposition 1 apply.

The yield of the debt is $C/D = 6.18\%$. Recovery of the debt at default, as a fraction of face value, is $\delta(r-\mu)^{-1}(1-\alpha)V_B/D = 43.3\%$. For comparison, the average recovery, as a fraction of face value, of all defaulted bonds monitored[9] by the rating agency Moody's, for 1920 to 1997, is 41%. Of course, recovery varies by subordination and for other reasons.

2.2 Imperfect Bond Market Information

Now we turn to how the secondary-market assesses the firm's credit risk and values its bonds.

After issuance, bond investors are not kept fully informed of the status of the firm. While they do understand that optimizing equity owners will force liquidation when assets fall to V_B, bond investors cannot observe the asset process V directly. Instead, they receive imperfect information at selected times $t_1, t_2, \ldots$, with $t_i < t_{i+i}$. While extensions to more general observation schemes are provided in the next section, for now we assume that at each observation time t there is a noisy accounting report of assets, given by $\hat{V}_t$, where $\log \hat{V}_t$ and $\log V_t$ are joint normal. Specifically, we suppose that $Y(t) = \log \hat{V}_t = Z(t) + U(t)$, where $U(t)$ is normally distributed and independent of $Z(t)$. (The independence assumption is without loss of generality, given joint normality.)

Also observed at each $t \in [0, \infty)$ is whether the equity owners have liquidated the firm. That is, the information filtration $(\mathcal{H}_t)$ available to the

[9] See "Historical Default Rates of Corporate Issuers," Moody's Investors Service, February, 1998, page 20.

secondary market is defined by

$$\mathcal{H}_t = \sigma(\{Y(t_1), \ldots, Y(t_n), 1_{\{\tau \leq s\}} : 0 \leq s \leq t\}), \tag{14}$$

for the largest n such that $t_n \leq t$, where $\tau = \tau(V_B)$.

For simplicity, we suppose that equity is not traded on the public market, and that equity owner-managers are precluded, say by insider-trading regulation, from trading in public debt markets. This allows us to maintain the simple model $(\mathcal{H}_t)$ for the information reaching the secondary bond market, and avoiding a complex rational-expectations equilibrium problem with asymmetric information.

Our main objective for the remainder of this sub-section is to compute the conditional distribution of V_t given $\mathcal{H}_t$. We will begin with the simple case of having observed a single noisy observation at time $t = t_1$. In Section 3, we extend to multiple observation times.

We will need,[10] as an intermediate calculation, the probability $\psi(z_0, x, \sigma\sqrt{t})$, conditional on Z starting at some given level z_0 at time 0 and ending[11] at some level x at a given time t, that $\min\{Z_s : 0 \leq s \leq t\} > 0$. As indicated by our notation, this probability does not depend on the drift parameter m, and depends on the variance parameter σ^2 and time t only through the term $k = \sigma\sqrt{t}$. From the density of the first-passage time recorded in Chapter 1 of Harrison (1985), and from Bayes' Rule, one obtains after some simplification that

$$\psi(z, x, k) = 1 - \exp\left(-\frac{2zx}{k^2}\right). \tag{15}$$

Next, fixing $z_0 = Z_0$, we calculate the density $b(\,\cdot \mid Y_t, z_0, t)$ of Z_t, "killed" at $\tau = \inf\{t : Z_t \leq \underline{v}\}$, conditional on the observation $Y_t = Z_t + U_t$. That is, using the conventional informal notation,

$$b(x \mid Y_t, z_0, t)\, dx = P(\tau > t \text{ and } Z_t \in dx \mid Y_t), \qquad x > \underline{v}. \tag{16}$$

Using the definition of ψ and Bayes' Rule,

$$b(x \mid Y_t, z_0, t) = \frac{\psi(z_0 - \underline{v}, x - \underline{v}, \sigma\sqrt{t})\phi_U(Y_t - x)\phi_Z(x)}{\phi_Y(Y_t)}, \tag{17}$$

[10] The approach taken here, as well as the specific calculations for a slightly different case, were generously shown to us by Michael Harrison. We are much in his debt for this assistance.

[11] To be more precise, if we take Z to be a pinned standard Brownian motion, with $Z_0 = z > 0$ and $Z_t = x > 0$, we want the probability $\psi(z, x, t)$ that $\min\{Z_s : 0 \leq s \leq t\} > 0$.

where ϕ_U denotes the density of U_t, and likewise for ϕ_Z and ϕ_Y. These densities are normal, with respective means $\overline{u} = E(U_t), mt + z_0$, and $mt + z_0 + \overline{u}$, and with respective variances $a^2 = \mathrm{var}(U_t), \sigma^2 t$, and $a^2 + \sigma^2 t$. The standard deviation a of U_t may be thought of as a measure of the degree of accounting noise.

We have

$$P(\tau > t \,|\, Y_t) = \int_{\underline{v}}^{+\infty} b(z \,|\, Y_t, z_0, t)\, dz. \tag{18}$$

Finally, we compute the density $g(\,\cdot\,|\, Y_t, z_0, t)$ of Z_t, conditional on the noisy observation Y_t and on $\tau > t$. Using (16) and (18), and another application of Bayes' Rule,

$$g(x \,|\, y, z_0, t) = \frac{b(x \,|\, y, z_0, t)}{\int_{\underline{v}}^{+\infty} b(z \,|\, y, z_0, t)\, dz}. \tag{19}$$

Letting $\tilde{y} = y - \underline{v} - \overline{u}$, $\tilde{x} = x - \underline{v}$, and $\tilde{z}_0 = z_0 - \underline{v}$, a calculation of the integral in (19) leaves us with

$$g(x \,|\, y, z_0, t) = \frac{\sqrt{\frac{\beta_0}{\pi}}\; e^{-J(\tilde{y}, \tilde{x}, \tilde{z}_0)} \left[1 - \exp\left(\frac{-2\tilde{z}_0 \tilde{x}}{\sigma^2 t}\right)\right]}{\exp\left(\frac{\beta_1^2}{4\beta_0} - \beta_3\right) \Phi\left(\frac{\beta_1}{\sqrt{2\beta_0}}\right) - \exp\left(\frac{\beta_2^2}{4\beta_0} - \beta_3\right) \Phi\left(-\frac{\beta_2}{\sqrt{2\beta_0}}\right)}, \tag{20}$$

where

$$J(\tilde{y}, \tilde{x}, \tilde{z}_0) = \frac{(\tilde{y} - \tilde{x})^2}{2a^2} + \frac{(\tilde{z}_0 + mt - \tilde{x})^2}{2\sigma^2 t}, \tag{21}$$

for

$$\beta_0 = \frac{a^2 + \sigma^2 t}{2a^2 \sigma^2 t} \tag{22}$$

$$\beta_1 = \frac{\tilde{y}}{a^2} + \frac{\tilde{z}_0 + mt}{\sigma^2 t} \tag{23}$$

$$\beta_2 = -\beta_1 + 2\frac{\tilde{z}_0}{\sigma^2 t} \tag{24}$$

$$\beta_3 = \frac{1}{2}\left(\frac{\tilde{y}^2}{a^2} + \frac{(\tilde{z}_0 + mt)^2}{\sigma^2 t}\right), \tag{25}$$

and where Φ is the standard-normal cumulative distribution function. Given survival to t, this gives us the conditional distribution of assets, because the conditional density of V_t at some level v is easily obtained from the conditional density of Z_t at $\log v$.

2.2.1 Numerical Illustration - Continued

We extend the numerical illustration begun in Section 2.1.1 by considering the conditional distribution of assets for the same firm at some current date $t > 1$. We suppose that a noise-free asset report of $\hat{V}(t-1) = V(t-1) = 86.3$ was provided one year ago. Figure 2 shows the conditional density of the current asset level V_t, given $\mathcal{H}_t$, that would be realized in the event that the bond has not yet defaulted ($\tau(V_B) > t$) and the current asset report $\hat{V}_t$ has an outcome equal to the previous year's report, 86.3. As the default boundary is $V_B = 78$, the firm has become rather risky in this scenario. We indicate various cases for the standard deviation a of the accounting noise U_t. Our base case is $a = 10\%$. We have no empirical evidence at this writing of a reasonable level of accounting noise, which in any case would presumably vary with the nature of a firm, so we consider the effect of variation in a. We suppose in all cases that U_t has expectation $\overline{u} = -a^2/2$, so that $E(e^{U(t)}) = 1$, implying an unbiased accounting report.[12] Figure 3 shows how this conditional density is affected by the lagged asset report $\hat{V}(t-1)$, for all other parameters at their base cases.

We also compute the $\mathcal{H}_t$-conditional probability $p(t,s)$ of survival to some future time $s > t$. That is, $p(t,s) = P(\tau > s \,|\, \mathcal{H}_t)$. For $t < \tau$, we have

$$p(t,s) = \int_{\underline{v}}^{\infty} \left(1 - \pi\left(s - t, x - \underline{v}\right)\right) g(x \,|\, Y_t, z_0, t)\, dx, \tag{26}$$

where $\pi(t,x)$ denotes the probability of first passage of a Brownian motion with drift m and volatility parameter σ from an initial condition $x > 0$ to a level below 0 before time t, which is known explicitly [Harrison (1985)]. Figure 4 illustrates outcomes of the conditional default probability $1 - p(t,s)$, for our base-case example, for various time horizons $s - t$ and various levels a of accounting noise. For example, with perfect information, the conditional probability of default within one year is approximately 2.9%, while this conditional probability is approximately 6.7% if the accounting assets are reported with a 10% level of accounting noise.

[12] This does *not* imply that the accounting report is conditionally unbiased given survivorship. One may think, for example, of an accounting report based on a physical measurement of the stock of assets, which is imperfect but unbiased. An alternative would be an accounting report in the form of a conditional distribution of assets, given survivorship and the results of noisy measurements. The mean of this conditional distribution would, by definition, be conditionally unbiased.

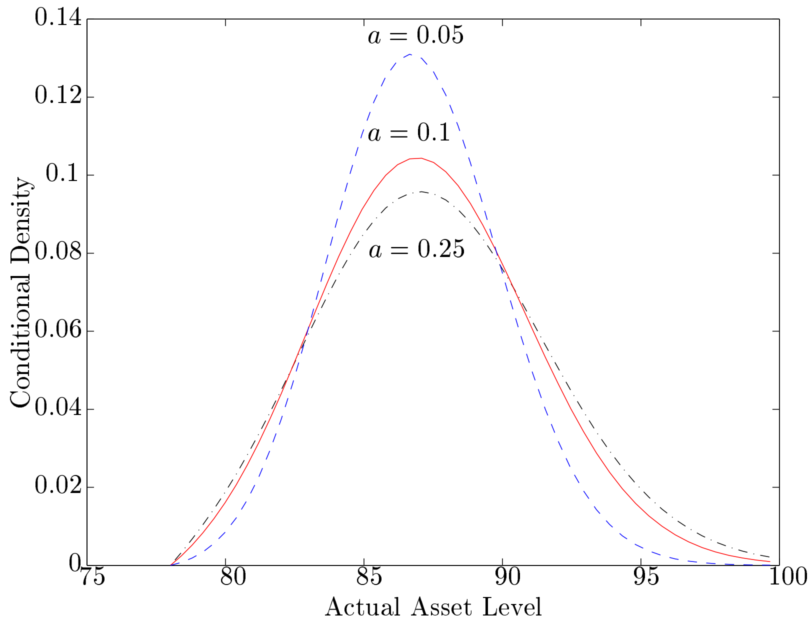

Figure 2: Conditional density for varying accounting precision.

2.3 Default Intensity

The conditional probability that default occurs within h units of time goes to zero as h goes to zero, regardless of the informational assumptions. What distinguishes perfect from imperfect information is the rate of convergence. In the case of perfect information, that is, for a filtration such as $(\mathcal{F}_t)$ to which Z is adapted, on the event that $\tau > t$ we have

$$\lim_{h \downarrow 0} \frac{P(\tau \leq t + h \,|\, \mathcal{F}_t)}{h} = \frac{\pi(h, Z_t - \underline{v})}{h} = 0 \qquad a.s. \tag{27}$$

In structural models with default defined as the first hitting time of an observable diffusion, it is this fact which forces credit spreads on zero-coupon bonds to go to zero as time to maturity goes to zero, as in Figure 1. It is tempting to conclude that the same is true with imperfect information, because, as we see from (26),

$$\frac{1 - p(t, t+h)}{h} = \int_{\underline{v}}^{\infty} \frac{\pi(h, x - \underline{v})}{h} g(x \,|\, Y_t, z_0, t)\, dx,$$

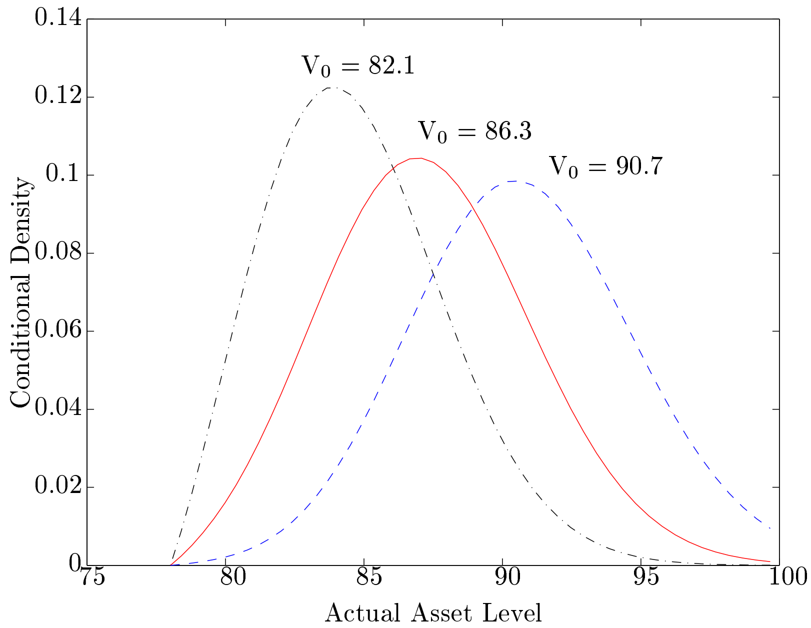

Figure 3: Conditional asset density, varying previous year asset level.

and the first factor of the integrand converges to 0 for all x. The integral itself, however, does *not* converge to 0 as h goes to 0. In this section, we prove that (in general) a non-zero limit exists, we give an explicit expression for the limit, and we show that this limit is in fact the *intensity* of τ, defined as follows. The default stopping time τ has an intensity process λ with respect to the filtration $(\mathcal{H}_t)$ if λ is a non-negative progressively measurable process satisfying $\int_0^t \lambda_s \, ds < \infty$ a.s. for all t, such that $\{1_{\{\tau \leq t\}} - \int_0^t \lambda_s \, ds : t \geq 0\}$ is an $(\mathcal{H}_t)$-martingale. For details, see, for example, Brémaud (1981).

The intuitive meaning of the intensity is that it gives a local default rate, in that

$$P\left(\tau \in (t, t + dt] \mid \mathcal{H}_t\right) = \lambda_t \, dt.$$

From the results in Section 2.2, at any (ω, t) such that $0 < t < \tau(\omega)$, the $\mathcal{H}_t$-conditional distribution of Z_t has a continuously differentiable conditional density $f(t, \cdot, \omega)$. This density is zero at the boundary $\underline{v}$, and has a derivative (from the right) $f_x(t, \underline{v}, \omega)$ that exists and is non-zero. We are ready to state the key result of this section.

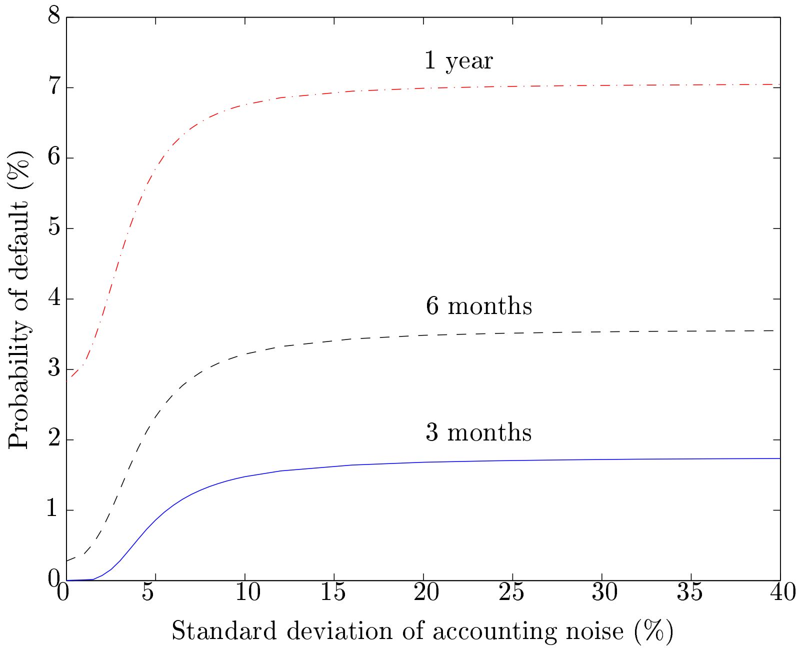

Figure 4: Default probability, varying accounting precision.

Proposition 2.2 *Define a process λ by $\lambda(t) = 0$ for $t > \tau$ and*

$$\lambda_t(\omega) = \frac{1}{2}\sigma^2 f_x(t, \underline{v}, \omega), \quad 0 < t \leq \tau. \tag{28}$$

Then λ is an[13] *$(\mathcal{H}_t)$-intensity process of τ.*

A proof is found in Appendix A. In order to gain intuition for the result, we consider a standard binomial "random-walk" approximation for Z, supposing at first that $m = 0$. We can assume without loss of generality that the default-triggering boundary $\underline{v}$ for Z is 0. For notational simplicity, we let $f(\,\cdot\,)$ also denote the $\mathcal{H}_t$-conditional density of Z_t on the event that $\tau > t$. The conditional probability that $Z_t = 0$ is zero. The conditional probability that Z_t is one step above 0 is, to first order, approximated by $f(\sigma\sqrt{h})\sigma\sqrt{h}$, as illustrated in Figure 5.

Because the only level from which Z_t can reach 0 within one time step of length h is $\sigma\sqrt{h}$, the conditional probability of hitting 0 by time h is equal

[13]While there may exist other $(\mathcal{H}_t)$-intensity processes for τ, they are equivalent for our purposes. For the sense in which uniqueness applies, see Brémaud (1981), page 30.

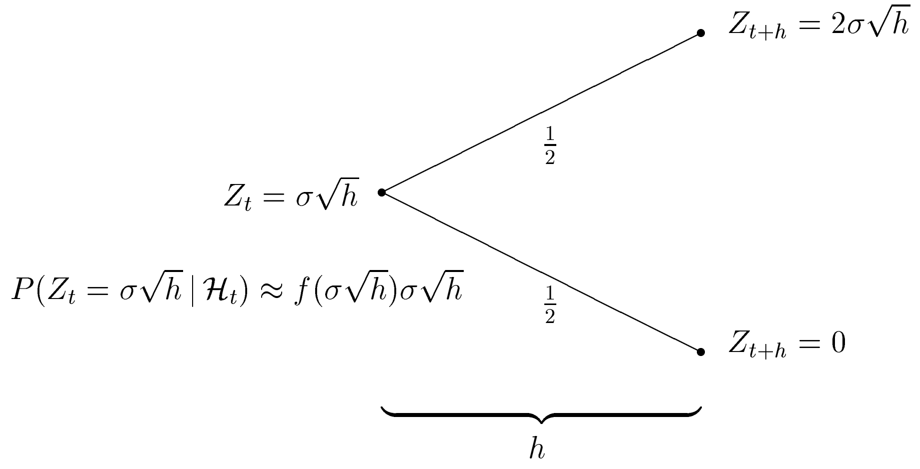

Figure 5: Binomial Approximation of Hitting Intensity.

to $\frac{1}{2}f(\sigma\sqrt{h})\sigma\sqrt{h}$. Thus,

$$\begin{aligned} \lim_{h\to 0}\frac{1}{h}P(\tau \le t+h \mid \mathcal{H}_t) &= \lim_{h\to 0}\frac{\frac{1}{2}f(\sigma\sqrt{h})\sigma\sqrt{h}}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{2}f(\sigma\sqrt{h})\sigma^2}{\sigma\sqrt{h}} \\ &= \frac{1}{2}f'(0)\sigma^2, \end{aligned}$$

where we have used the fact that $f(0) = 0$ to calculate the derivative. This binomial approximation can easily be extended to handle a drift term m by changing the probability of up and down moves to $\frac{1}{2} + \frac{m\sqrt{h}}{2\sigma}$ and $\frac{1}{2} - \frac{m\sqrt{h}}{2\sigma}$, respectively. As h goes to zero these probabilities go to $\frac{1}{2}$, and we have the same limit result for the intensity.[14]

[14] This is a reflection of the fact that, locally, the volatility dominates the evolution of the Brownian motion. This can be seen from the law of the iterated logarithm [for example, Karatzas and Shreve (1988)].

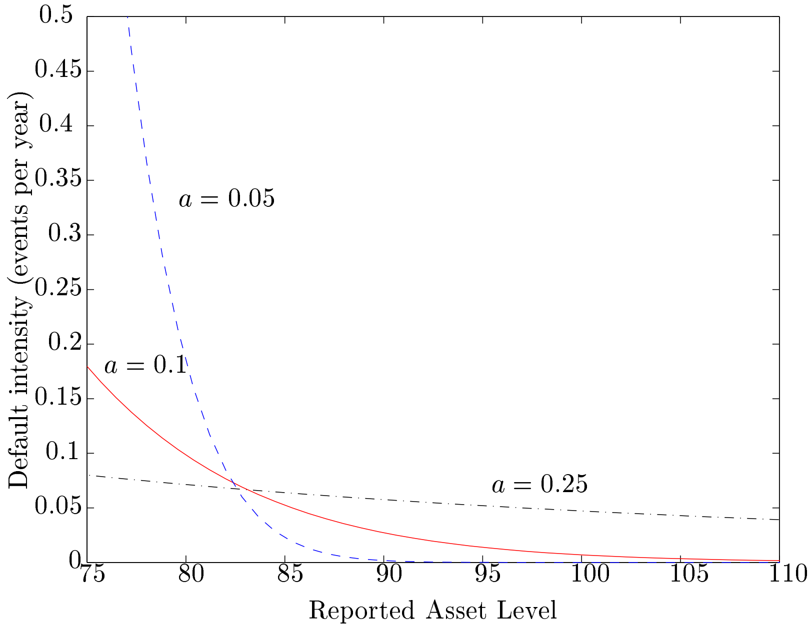

Figure 6: Default intensity, varying accounting precision.

2.4 Credit Spreads

We turn now to the implications of incomplete information for the term structures of credit yield spreads of the modeled firm.

For a given time T to maturity, the yield spread on a given zero-coupon bond selling at a price $\varphi > 0$ is the real number η such that $\varphi = e^{-(r+\eta)T}$. If we assume that a bond with maturity date $s > t$ issued by our modeled firm recovers some fraction $R(s) \in [0, 1]$ of its face value at default, then the secondary market price $\varphi(t, s)$ at time t of such a bond, in the event that the firm has yet to default by t, is given by

$$\varphi(t, s) = e^{-r(s-t)}p(t, s) - R(s) \int_t^s e^{-r(u-t)}\, p(t, du), \tag{29}$$

where we recall that $p(t, u)$ is the probability of survival to time u. The first term in (29) is the value of the survival-contingent contractual payoff of the bond at maturity, while the second is the value of any recovery at default, contingent on default before maturity. Using (26), our calculation of $\varphi(t, s)$ is reduced to a single numerical integration over the current log-asset

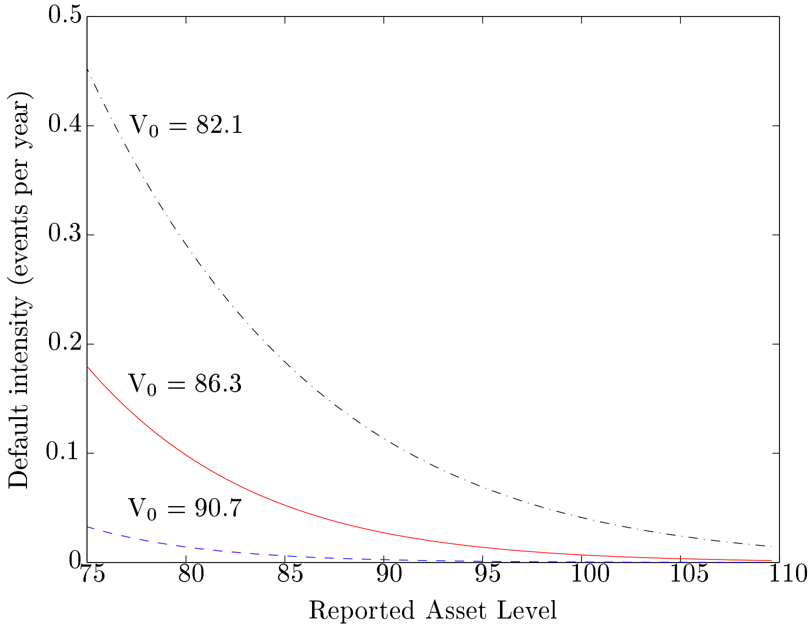

Figure 7: Default intensity, varying previous year asset level.

level x, weighted by its conditional density, applying Fubini's Theorem to the integral in (29). As our model was based for simplicity on a single consol bond, and as our calculation of the survival probability $p(t, s)$ is based on that capital structure, some interpretation of (29) is appropriate. One possibility is that the consol is stripped in the secondary market into a continuum of zero-coupon bonds. While various recovery assumptions could be made, it is convenient for our purposes to assume that, given default at time τ, the recovery $R(s)$ for a bond with maturity date $s > \tau$ is proportional to the default-free discount $e^{-r(s-\tau)}$ for that maturity. (This could, for instance, be a contractual provision.) Later, we consider default swap spreads, which do not call for such recovery assumptions.

Figure 1 compares the term structure of credit spreads in our base-case numerical illustration with the term structure that would apply with perfect accounting data (dashed curve, for the case with $a = 0$). Figure 8 compares the base case (solid curve) with the term structure that would apply with various qualities of accounting information. Figure 9 compares the base case (solid curve) against the term structure that would apply with various lagged asset reports. With perfect accounting information, the previous accounting report would be irrelevant, given the current report.

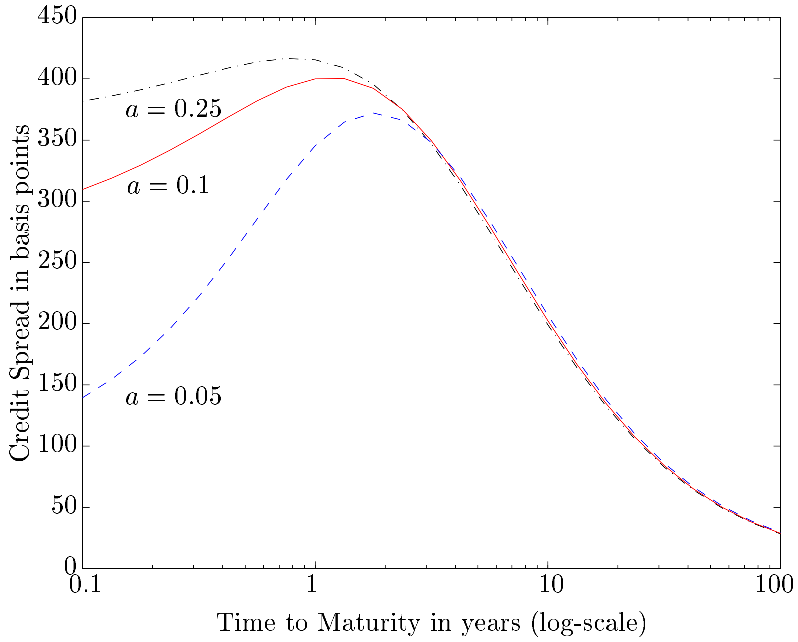

Figure 8: Credit spreads for varying accounting precision.

In a setting of perfect information ($a = 0$), we have $\varphi(t, s) = J_{t,s}(V_t)$, for $J_{t,s} : [0, \infty) \to (0, \infty)$ determined by (29). If $m > 0$, it can be shown by calculation of its second derivative that $J_{t,s}(\,\cdot\,)$ is concave.[15] It follows by Jensen's Inequality that yield spreads are larger at an outcome for $\hat{V}_t$ that is conditionally unbiased for V_t than they would be in the case of perfect information. This is analogous to the fact that, in a Black-Scholes setting, the equity price as a function of asset level is increasing in the volatility of assets, and therefore the debt price is decreasing in asset volatility. Here, however, the asset volatility is fixed, and it is a question of precision of the accounting observation. One might extrapolate to practical settings and anticipate that, other things equal, secondary-market yield spreads are decreasing in the degree of transparency of a firm. We emphasize this as separate from the adverse-selection effect on new-issue prices, which may suffer from a lemon's premium associated with the issuer's superior information regarding its own

[15] A proof, due to Nicolae Gârleanu, is provided in a working-paper version of this paper. If $m < 0$, counterexamples can be constructed.

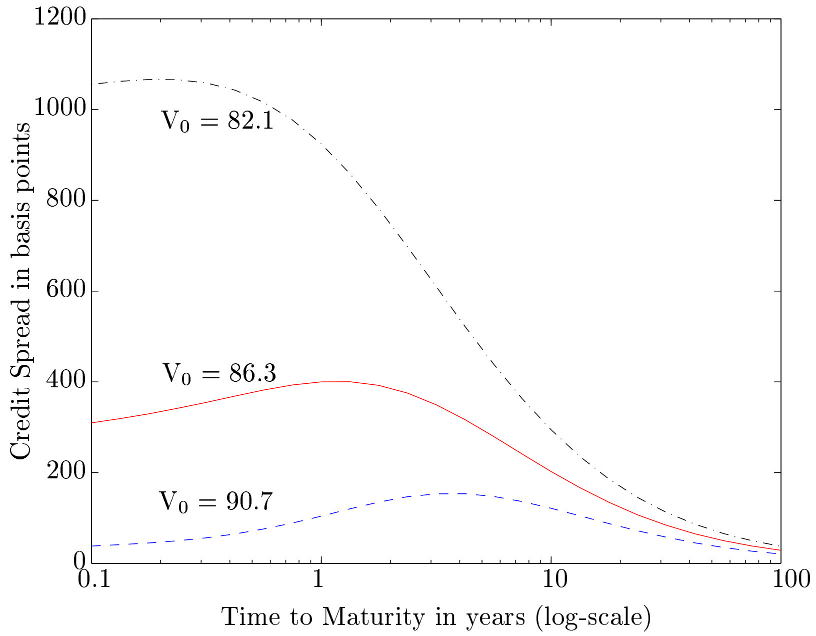

Figure 9: Credit spreads, varying previous year's asset level.

credit quality. In our setting, all participants in the secondary market for bonds are equally well informed; they simply adjust their views regarding credit risk based on the precision of their information.

2.4.1 Default-Swap Spreads

Default swaps are the most common form of credit derivative. With a given maturity T, a default swap is an exchange of an annuity stream at a constant coupon rate until maturity or default, whichever is first, in return for a payment of X at default, if default is before T, where X is the difference between the face value and the recovery value on the stipulated underlying bond. A default swap can thus be thought of as a default insurance contract for bond holders that expires at a given date T, and makes up the difference between face and recovery values in the event of default.

By considering the market valuation of default swaps, we can effectively uncover (as explained below) the term structure of credit spreads for par coupon bonds. This is a measure of credit spreads that is more standard than the zero-coupon bond term structure considered above. Moreover, our results regarding the shape of the zero-coupon term structure $\varphi(t,\,\cdot)$ are sensitive to our assumption regarding how these strips share in the assets remaining after default. Default-swap spreads depend only on the total recovery value to the underlying consol bond.

We assume, as typical in practice, that the default-swap annuity payments are made semiannually, and that the default swap's maturity date T is a coupon date. We take the underlying bond to be the consol bond issued by the firm in our example. This fixes the default-contingent payment, for each default swap, of

$$X = 1 - \frac{(1-\alpha)\delta V_B}{(r-\mu)D}.$$

We can solve for the at-market default-swap spread, which is the annualized coupon rate $c(t,T)$ that makes the default swap sell at time t for a market value of 0. With $T = t + n/2$, for a given positive integer n, we have

$$c(t,T) = \frac{2XE[e^{-r(\tau-t)}1_{\{\tau<T\}}]}{\sum_{i=1}^{n} e^{-0.5ri}E[1_{\{\tau>t+0.5i\}}]}.$$

Default-swap spreads are a standard for price quotation and credit information in bond markets. In our setting, they have the additional virtue of providing implicitly the term structure of credit spreads for par floating-rate bonds of the same credit quality as the underlying consol bond, in terms of default time and recovery at default.[16]

2.5 Summary of Empirical Implications

We summarize some of the empirical implications of our model for corporate debt valuation as follows.

Even if default is triggered by an insufficient level of assets relative to liabilities, observable variables that are correlated with current asset-liability

[16]This follows from the fact that, without transactions costs, a default swap can be viewed as an exchange of a par floating-rate default-free bond of maturity T for a par defaultable floating-rate bond of the same maturity. This is the case because this portfolio has an initial market value of 0 (as an exchange of par for par), provides an annuity until $\min(\tau, T)$ at a coupon rate which is the credit spread of the defaultable bond over the default-free bond, and at default, if before T, is worth $X - 100$. This follows from the fact that a default-free floating rate bond is always worth its face value. This argument ignores the impact of default between coupon dates, which is negligible except in extreme cases. See Duffie (1999).

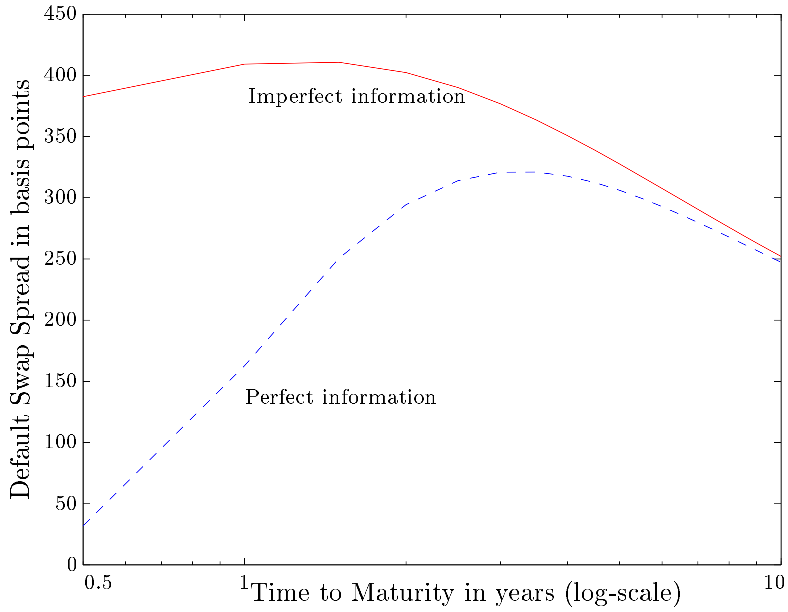

Figure 10: Default swap spreads with perfect and imperfect information. Base case.

ratios have explanatory power for credit spreads and for default probabilities. For example, as with Bijnen and Wijn (1994), Lennox (1999), Lundstedt and Hillgeist (1998), McDonald and Van de Gucht (1999), and Shumway (1996), firm-specific ratios (such as interest expense to income), or macroeconomic variables related to the business cycle, may have predictive power for default probabilities or yield spreads, even after exploiting asset and liability data. (We offer some specific modeling in Section 3.) We have also shown the contributing role of lagged accounting variables.

Perfect observation in standard diffusion models of corporate debt, such as those Leland (1994) and Longstaff and Schwartz (1995), implies that credit spreads also go to zero as maturity goes to zero, regardless of the credit quality of the issuer.[17] For poorer-quality firms, credit spreads would widen sharply with maturity, and then typically decline. With imperfect information about the firm's value, however, credit spreads remain bounded away

[17] In a model such as that of Merton (1974), when the "solvency ratio" of asset value to the face value of debt is less than one, the default probability goes to 1, and spreads therefore to infinity, as time to maturity goes to 0.

from zero as maturity goes to zero. The shape of the term structure of credit spreads is moderated, and may provide some indication of the quality of accounting information assumed by investors.[18]

A "Jensen effect" implies a lower price of debt, fixing the reported level of assets, for an issuer with a conditionally unbiased but imperfect asset report than for a perfectly observable issuer that is otherwise identical. This follows from the fact that, under certain conditions, including examples provided in this paper, corporate bond prices in a setting of perfect information are concave in the issuer's asset level.

Our model is consistent with the fact that bond or equity prices often drop precipitously at or around the time of default. Empirically, given the imperfect manner in which the "surprise" may be revealed around the time of default, we would expect to see a marked increase in the volatility of bond returns during the time window bracketing default. Some evidence in this direction is already available in Beneish and Press (1995) and Slovin, Sushka, and Waller (1997).[19]

3 Extensions

This section outlines some extensions of the basic model. First, we allow for inference regarding the distribution of assets from several variables correlated with asset value, or from more than one period of accounting reports. We briefly discuss how to model re-capitalization, or decisions by the firm that may be triggered by more than one state variable, such as a stochastic liquidation boundary. We characterize the default intensity for cases in which the asset process V is a general diffusion process, with a general observation scheme.

[18] For theoretical or empirical evidence of how the shape of the term structure of credit spreads depends on credit quality, see Fons (1994); He, Hu, and Lang (1999); Helwege and Turner (1999); Johnson (1967); Jones, Mason, and Rosenfeld (1984); Pitts and Selby (1983); and Sarig and Warga (1989). For an empirical study linking the quality of accounting information, as measured by a rating of disclosure quality provided by the Financial Analysts Federation, to the price and rating of corporate debt, see Sengupta (1998).

[19] Some of the evidence presented in Slovin, Sushka, and Waller (1997) is through the stock price reaction of creditors of the defaulting firm.

3.1 Conditioning on Several Signals

Suppose, extending from Section 2.2, that the observation Y_t made at time t is valued in $\mathbb{R}^k$ for some $k \geq 1$, and is joint normal with Z_t. This may accommodate observations, beyond merely noisy observation of assets, that have explanatory power for credit risk and yield spreads, such as various accounting ratios, peer performance measures, or even macro-economic variables, such as consumer sentiment panel data, stock-market indices, or economic growth-rate measures.

For convenience, we let $G_Y(Z_t)$ denote the conditional expectation of Y_t given Z_t. That is, $G_Y : \mathbb{R} \to \mathbb{R}^k$ is affine, and determined as usual by the means and covariances of (Y_t, Z_t). We note that, by joint normality, if we define U_t to be the "residual vector" $U_t = Y_t - G_Y(Z_t)$, then U_t is independent of Z_t. We let $\phi_U(\,\cdot\,)$ denote the (joint-normal) density of U_t, and $\phi_Y(\,\cdot\,)$ denote the joint density of Y_t. Now, b in (17) is re-defined by

$$b(x \mid Y_t, z_0, t) = \frac{\psi(z_0 - \underline{v}, x - \underline{v}, \sigma\sqrt{t})\phi_U(Y_t - G_Y(x))\phi_Z(x)}{\phi_Y(Y_t)}, \tag{30}$$

and the solution for g given by (19) continues to apply.

3.2 Conditioning with Several Periods of Reports

We now suppose that noisy reports of assets arrive at successive integer dates. Because it is reasonable to allow for persistence in accounting noise, we suppose that $U_1, U_2, \ldots$ may be serially correlated. Specifically, we suppose that $Y_i = Z_i + U_i$, for

$$U_i = \kappa U_{i-1} + \epsilon_i, \tag{31}$$

where κ is a fixed coefficient and $\epsilon_1, \epsilon_2, \ldots$ are independent and identically distributed normal random variables, independent of Z.

Let

- $z^{(n)} = (z_1, z_2, \ldots, z_n)$ denote an outcome of $Z^{(n)} = (Z_1, Z_2, \ldots, Z_n)$.
- $y^{(n)} = (y_1, y_2, \ldots, y_n)$ denote an outcome of $Y^{(n)} = (Y_1, Y_2, \ldots, Y_n)$.
- $u^{(n)} = y^{(n)} - z^{(n)}$ denote an outcome of $U^{(n)} = Y^{(n)} - Z^{(n)}$.

We let $b_n(\,\cdot\,|\,Y^{(n)})$ denote the improper conditional density of $Z^{(n)}$, killed with first exit before time n, given $Y^{(n)}$. As with (16), this is the density defined by $P(Z^{(n)} \in dz^{(n)}$ and $\tau > n\,|\,Y^{(n)})$. For notational simplicity, at the outcomes $z^{(n)}, y^{(n)}$, and $u^{(n)}$, we let $p_U(u_n\,|\,u_{n-1})$ denote the transition density of $U_1, U_2, \ldots$; $p_Z(z_n\,|\,z_{n-1})$ denote the transition density of $Z_1, Z_2, \ldots$; and $p_Y(y_n\,|\,y^{(n-1)})$ denote the conditional density of Y_n given $Y^{(n-1)}$. For the case of $\kappa = 0$ (serially uncorrelated noise), it may be seen that $Y_1, Y_2, \ldots$ is auto-regressive of order 1, and p_Y takes a simple form.

By Bayes' Rule,

$$b_n\left(z^{(n)}\,|\,y^{(n)}\right) \;=\; \frac{P(\tau > n, Z^{(n)} \in dz^{(n)}, Y^{(n)} \in dy^{(n)})}{P(Y^{(n)} \in dy^{(n)})}.$$

As the event $\{\tau > n, Z^{(n)} \in dz^{(n)}, Y^{(n)} \in dy^{(n)}\}$ is the same as the event $A \cap B$, where $A = \{\tau > n, Z_n \in dz_n, Y_n \in dy_n\}$ and

$$B = \{\tau > n-1, Z^{(n-1)} \in dz^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)}\},$$

we have

$$b_n\left(z^{(n)}\,|\,y^{(n)}\right) \;=\; \frac{P(A\,|\,B)P(B)}{p_Y(y_n\,|\,y^{(n-1)})P(Y^{(n-1)} \in dy^{(n-1)})}. \tag{32}$$

After some straightforward manipulation of this expression, we find that $b_n\left(z^{(n)}\,|\,y^{(n)}\right)$ is given by

$$\frac{\psi(z_{n-1}-\underline{v}, z_n-\underline{v}, \sigma)p_Z(z_n\,|\,z_{n-1})p_U(y_n - z_n\,|\,y_{n-1}-z_{n-1})b(z^{(n-1)}\,|\,y^{(n-1)})}{p_Y(y_n\,|\,y^{(n-1)})}. \tag{33}$$

After conditioning on $Y^{(n)}$ and also on survival to time n (that is, on the event $\tau > n$), we have the conditional density $g_n(\,\cdot\,|\,Y^{(n)})$ of $Z^{(n)}$, given by

$$g_n(z^{(n)}\,|\,y^{(n)}) = \frac{b_n(z^{(n)}\,|\,y^{(n)})}{\int_{A(n)} b_n(z\,|\,y^{(n)})\,dz}, \tag{34}$$

where the region of integration for the denominator (survival) probability is

$$A(n) = \{z \in I\!\!R^n : z_1 \geq \underline{v}, \ldots, z_n \geq \underline{v}\}.$$

We do not have an explicit solution for the survival probability, $\int_{A(n)} b_n(z\,|\,y^{(n)})\,dz$, but numerical integration can be done recursively, one dimension at a time, from the explicit recursive solution (33) for b_n. Given the joint density $g_n\left(z^{(n)} \mid y^{(n)}\right)$ for $Z^{(n)}$, we can integrate with respect to $z^{(n-1)}$ to obtain the marginal density of z_n, and from this the intensity[20] of τ on date n, for $n < \tau$.

3.3 General Default or Re-Structuring Intensity

Our results in Section 2 on default intensity can be significantly extended.

As with Fischer, Heinkel, and Zechner (1989) and Leland (1998) one can consider cases in which the firm will re-capitalize when the level of assets reaches some sufficiently high level $\overline{v}$. For example, the firm could issue callable debt, giving it the right to pay down the existing debt at face value. For a given fixed frictional cost for calling the debt, it would be optimal to call when V hits a sufficiently high level $\overline{v}$. In order to compute the density f of Z_t given a noisy signal Y_t, and neither liquidation nor re-capitalization by t, we would simply replace the probability $\psi(z, x, \sqrt{t})$ used in (17) with the analogous term based on the probability $\psi^*(z, y, \overline{z}, \sqrt{t})$ that a pinned standard Brownian motion starting at $z \in (0, \overline{z})$ at time 0, and finishing at time t at level y, does not leave the interval $(0, \overline{z})$ during $[0, t]$. This probability can be obtained explicitly from results found, for example, in Harrison (1985). As for the intensity, we let $\tau^* = \inf\{t : V_t \notin (\underline{v}, \overline{v})\}$, $\overline{\tau} = \inf\{t \leq \tau^* : V_t \geq \overline{v}\}$, and $\tau = \inf\{t \leq \tau^* : V_t \leq \underline{v}\}$. At $t < \tau^*$, the intensity $\overline{\lambda}_t$ of $\overline{\tau}$ is given by

$$\overline{\lambda}_t = -\frac{1}{2}\sigma^2 f_x(t, \overline{v}), \quad t < \tau. \tag{35}$$

The $(\mathcal{H}_t)$-intensity λ^* for τ^* is given at $t < \tau^*$ by $\lambda + \overline{\lambda}$.

Another useful generalization replaces the constant default-triggering boundary $\underline{v}$ with a process $\underline{Z}$, whose value $\underline{Z}(t)$ at t may not be observable. Suppose, for example, that $\underline{Z}$ is a Brownian motion (with some drift and volatility parameters). The difference $Z^* = Z - \underline{Z}$ is then itself a Brownian motion, and $\tau = \inf\{t : Z^*_t \leq 0\}$ is a first-passage time that may be treated by the same methodology just developed. An extension of our result by Song (1998) treats multi-dimensional cases, for example for a model in which the difference $Z^* = Z - \underline{Z}$ is not a one-dimensional diffusion, and the hitting problem

[20] In order to compute the intensity, it is of course numerically easier to obtain the derivative of b_n with respect to z_n explicitly, and then to integrate this with respect to $z^{(n-1)}$, reversing the order of differentiation and integration to save one numerical step.

therefore must be formulated as that of a two-dimensional process crossing a smooth surface.

Indeed, even if the boundary $\underline{Z}$ is constant in t, but uncertain, we can apply our results to Z^*, defined by $Z_t^* = Z_t - \underline{Z}$, provided Z is itself incompletely observed. If Z is perfectly observed, however, then our theory does not apply, as the conditional density of Z_t^* is not both zero and smooth at zero[21]. Lambrecht and Perraudin (1996) have presented a model of default based on first passage of an observed diffusion for assets to a trigger boundary that is unobserved by certain debtholders, who are each uncertain of the other debtholder's valuation of the firm in default.

We can also accommodate an underlying asset process V that need not be a geometric Brownian motion. For example, suppose V satisfies a stochastic differential equation of the form

$$dV_t = \mu(V_t, t)\, dt + \sigma(V_t, t)\, dW_t.$$

Consider, moreover, some general observation scheme, perhaps continual in time as in Kusuoka (1999), for which, at $t < \tau$ there exists a conditional density $f(t, \cdot)$ for V_t. We show in Appendix B, under technical conditions on μ, σ, and f, that the intensity process λ for first hitting of V at $\underline{v}$ is given by

$$\lambda_t = \frac{1}{2}\sigma^2(\underline{v}, t) f_x(t, \underline{v}), \quad t < \tau. \tag{36}$$

[21] For an argument, see our working-paper version.

Appendices

A Proof of Proposition 2.2

Let $Z_t = Z_0 + mt + \sigma W_t$. Without loss of generality, let $\underline{v} = 0$ and define $\tau_0 = \inf\{s > 0 : Z_s = 0\}$. To begin with, we consider a fixed time $t > 0$. For now, we suppose that the only information available is survivorship; that is, $\mathcal{H}_t$ is generated by $\{1_{\{\tau_0 > s\}} : s \leq t\}$. On the event $\tau_0 > t$, it follows directly from results in Harrison (1985) that Z_t has a conditional density $f(t, \cdot)$, denoted $f(\cdot)$ for notational simplicity. This density is bounded, satisfies $f(0) = 0$, and has a bounded derivative, with $f'(0)$ defined from the right. We let

$$\overline{\lambda} = \lim_{h \to 0} \frac{1}{h} \int_{(t,\infty)} \pi(h, x, m, \sigma) f(x)\, dx,$$

where $\pi(h, x, m, \sigma)$ denotes the probability of first passage to zero before time h for an (m, σ)-Brownian motion with initial condition x. Our first objective is to show that this limit exists and that

$$\overline{\lambda} = \frac{1}{2}\sigma^2 f'(0). \tag{A.1}$$

From expression (11) on page 14 of Harrison (1985) (noting that the probability of hitting 0 before h starting from $x > 0$ with drift m is the probability of hitting the level x starting from 0 before h with drift $-m$), we have, by substituting $z = x/(\sigma\sqrt{h})$,

$$\begin{aligned}
\overline{\lambda} &= \lim_{h \to 0} \frac{1}{h} \int_{(0,\infty)} \left(1 - \Phi\left(\frac{x + mh}{\sigma\sqrt{h}}\right) + \exp\left(-\frac{2mx}{\sigma^2}\right) \Phi\left(\frac{-x + mh}{\sigma\sqrt{h}}\right)\right) \\
&\qquad \times f(x)\, dx \\
&= \lim_{h \to 0} \int_{(0,\infty)} \left(1 - \Phi\left(z + \frac{m\sqrt{h}}{\sigma}\right) + \exp\left(-\frac{2m\sqrt{h}z}{\sigma}\right) \Phi\left(-z + \frac{m\sqrt{h}}{\sigma}\right)\right) \\
&\qquad \times \frac{1}{h} f\left(\sigma\sqrt{h}z\right) \sigma\sqrt{h}\, dz, \\
&= \lim_{h \to 0} \int_{(0,\infty)} G_1(z,h) G_2(z,h) z\, dz,
\end{aligned}$$

where

$$G_1(z,h) = 1 - \Phi\left(z + \frac{m\sqrt{h}}{\sigma}\right) + \exp\left(-\frac{2m\sqrt{h}z}{\sigma}\right)\Phi\left(-z + \frac{m\sqrt{h}}{\sigma}\right)$$

and

$$G_2(z,h) = \frac{f\left(\sigma\sqrt{h}z\right)}{\sigma\sqrt{h}z}\sigma^2.$$

Thus, if the dominated convergence theorem can be applied, we have

$$\begin{aligned}
\overline{\lambda} &= \int_{(0,\infty)} G_1(z,0)G_2(z,0)z\,dz \\
&= \int_{(0,\infty)} \left(1 - \Phi\left(z\right) + \Phi(-z)\right) f'(0)\sigma^2 z\,dz \qquad \text{(A.2)} \\
&= \sigma^2 f'(0)\int_{(0,\infty)} 2\Phi(-z)z\,dz = \frac{1}{2}\sigma^2 f'(0),
\end{aligned}$$

as desired. To justify dominated convergence, first note that since the derivative of f is bounded, there exists a constant K such that

$$\left|\frac{f(\sigma\sqrt{h}z)}{\sigma\sqrt{h}z}\right| \leq K. \qquad \text{(A.3)}$$

Now note that (for $h < 1$)

$$|G_1(z,h)| \leq G(z) \equiv 1 - \Phi\left(z - \frac{|m|}{\sigma}\right) + \exp\left(\frac{2|m|z}{\sigma}\right)\Phi\left(-z\right).$$

We use the fact that $1 - \Phi(x)$ behaves like $\frac{\phi(x)}{x}$ as $x \to \infty$ to see that $G(z)$ goes to zero exponentially fast as $z \to 0$. Hence, we have shown that for $h < 1$,

$$|G_1(z,h)G_2(z,h)z| < K\sigma^2 zG(z),$$

and this provides the integrable upper bound (for all $h < 1$) justifying the application of dominated convergence.

In our model there is perfect information at time Z_0 and we can therefore only establish the existence of an intensity for $t > 0$ and we do this by proving existence on compact intervals of the form $[\underline{t}, T]$ for all $\underline{t} > 0$. Having established (A.1), we need to check (see Aven (1985)) that there exists a positive measurable process γ satisfying for every T

$$\int_{\underline{t}}^{T} \gamma(s,\omega)\, ds < \infty \qquad \text{a.s,}$$

and which dominates the error of approximation in the following sense. For each T, there exists for almost every ω an $n_0 = n(T,\omega)$ such that, for $n > n_0$,

$$|Y_n(s,\omega) - \lambda(s,\omega)| \leq \gamma(s,\omega) \qquad \text{for all } s \leq T,$$

where Y_n is the process defined by

$$Y_n(s) = \frac{1}{h_n} P(\tau_0 \leq s + h_n \,|\, \mathcal{H}_s) 1_{\{\tau > s\}},$$

for some fixed sequence $\{h_n\}$ of positive reals converging to zero. Note that n_0 can depend on ω.

We first verify that the process λ itself satisfies this integrability condition. Assume $Z_0 = z_0$. Between time $\underline{t}$ and the first accounting report at time t_1, assuming no default, the density of Z_t can be written as $\tilde{f}(t,\cdot,z_0)$, which does not depend on ω and has an analytical expression. [See, for example, Harrison (1985).] This density satisfies $\tilde{f}_x(t,0,z_0) = 0$ and is differentiable as a function of (t,x), with the derivative $f_x(t,\cdot,z_0)$ bounded uniformly on $[\underline{t}, t_1)$ as can easily be seen by including t_1 in the domain. The density at the first accounting report is derived in Section 2.3 as $g(x \,|\, Y(t_1,\omega), z_0, t_1)$, which for each given outcome $Y(t_1,\omega)$ is zero for $x = 0$ and has a bounded derivative with respect to x. If we let s be a given time after the first accounting report but before the next, the density of Z_{t+s} is given by

$$\int_0^\infty \tilde{f}(s,x,u) g(u|Y(t_1,\omega), z_0, t_1)\, du.$$

The derivative with respect to x can be taken inside the integral, so the derivative of the density at time $t+s$ with respect to the point x is

$$f_x(t_1 + s, x, \omega) = \int_0^\infty \frac{d}{dx} \tilde{f}(s,x,u) g(u \,|\, Y(t_1,\omega), z_0, t_1)\, du.$$

We see from these two expressions that the density inherits from $\tilde{f}$ the properties of being 0 at $x = 0$ and having a uniformly bounded derivative with respect to x over the finite interval $[t_1, t_2)$. Since there are only finitely many accounting reports over a finite period of time, it follows that the bound of the derivative can be obtained by taking the maximum of the bounds needed for each interval between accounting reports[22]. This proves integrability of a sample path of λ by proving the stronger property that for each ω, $f_x(t, \cdot, \omega)$ is bounded as a function of x, with one bound holding uniformly over $[\underline{t}, T]$.

It now follows from the triangle inequality that we only need to verify that for each t, there exists for almost every ω an $n_0 = n(t, \omega)$ such that

$$\int_{\underline{t}}^{T} |Y_n(s, \omega)| \, ds < \infty, \quad n > n_0. \tag{A.4}$$

To see this, we first note that, by the strong Markov property of Z, on the event $\{\tau > t\}$,

$$P\left(\tau_0 \leq h_n + t \mid Z(t)\right) = \pi(h_n, Z(t), m, \sigma).$$

Because $\pi(h_n, z\sqrt{h_n}, m, \sigma)$ declines exponentially fast in z,

$$|Y_n(s, \omega)| = \int_0^\infty \pi(h_n, \sqrt{h_n} z, m, \sigma) \left| \frac{f(s, \sqrt{h_n} z, \omega)}{\sqrt{h_n} z} \right| z \, dz$$

is bounded, using the fact shown above that, for almost every ω (and n such that $h_n < 1$), the derivative $f_x(s, x, \omega)$ is bounded uniformly in s. Hence $\int_0^t |Y_n(s, \omega)| < \infty$ whenever n_0 is chosen so that $h_n < 1$ for $n > n_0$. This concludes the proof.

B General Diffusion Results

We now provide results regarding the default intensity and conditional density for cases in which the asset process V solves a stochastic differential equation of the form

$$dV_t = \mu(V_t, t) \, dt + \sigma(V_t, t) \, dW_t. \tag{B.1}$$

Without loss of generality we study the first hitting time of the level 0 starting from an unknown positive level. We will use the following regularity for the coefficient functions μ and σ.

Condition A.

[22] We use the fact that there are no exact accounting reports after $\underline{t}$. At such times the intensity would not exist, just as it does not exist at time 0 when Z_0 is known. To have exact reports but still maintain an intensity, one could introduce random report times, where the report times themselves have an intensity process.

A1. The coefficients μ and σ satisfy global Lipschitz and linear growth conditions, in that there exists a fixed constant K such that, for any x, y, and t,

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq K\,|x-y| \tag{B.2}$$
$$\mu(x,t)^2 + \sigma(x,t)^2 \leq K^2(1+x^2). \tag{B.3}$$

A2. There exist constants l and r, with $-\infty \leq l < 0 < r \leq \infty$, such that $\sigma(\,\cdot\,,t)$ is bounded away from 0 on every compact subset of (l,r), uniformly in t, and such that $P(V_t \in (l,r) \text{ for all } t) = 1$.

A3. $\sigma(\,\cdot\,,\,\cdot\,)$ is C^1 on $(l,r) \times [0,\infty)$.

Condition (A1) is standard, ensuring that a unique, strong solution to the SDE exists. (A2) allows for cases (such as geometric Brownian Motion less a constant) in which the diffusion lives in an open (proper) subset of $I\!R$, containing the boundary, and the volatility function is positive on this subset. (A3) facilitates the proof, but can be relaxed. We consider a case with no instances of perfect information and assume that for every $t \geq 0$, V_t admits a regular conditional distribution function $F(\,\cdot\,|\,\mathcal{H}_t)$ given $\mathcal{H}_t$). That is,

1. For each $x > 0$, $F(x\,|\,\mathcal{H}_t)$ is a version of $P(V_t < x\,|\,\mathcal{H}_t)$.

2. For every ω, $F(\,\cdot\,|\,\mathcal{H}_t)(\omega)$ is a distribution function.

On (ω,t) for which $\tau(\omega) > t$, we assume that $F(\,\cdot\,|\,\mathcal{H}_t)$ has a density denoted $f(t,\,\cdot\,,\omega)$ (or sometimes simply "$f(t,\,\cdot\,)$," notationally suppressing dependence on ω). Whenever $\tau(\omega) \leq t$, we arbitrarily set $f(t,x,\omega) = 0$. Condition B forms technical assumptions regarding this conditional density.

Condition B.

B1. For each (ω,t), we have $f(t,0,\omega) = 0$ and $f(t,\,\cdot\,,\omega)$ is continuously differentiable on $(0,\infty)$ and differentiable from the right at $x = 0$. Furthermore, for almost every ω, $|f_x(s,x,\omega)|$ is bounded on sets of the form $\{(s,x):\ 0 \leq s \leq t, 0 \leq x < \infty\}$.

B2. For each fixed $x > 0$, the process $\{f_x(t,x) : t \geq 0\}$ is $(\mathcal{H}_t)$-progressively measurable.

We have seen that Condition B holds in our basic model. Smoothness of f also holds for more general diffusions under purely technical smoothness conditions on the drift coefficients m and the volatility σ, with a filtration $(\mathcal{H}_t)$ defined by "noisy observation," as shown below.

The general result is as follows.

Proposition B.1 *Let V be given by (B.1), with μ and σ satisfying Condition A. Suppose, for $t < \tau = \inf\{t : V_t \leq 0\}$, that the distribution of V_t given $\mathcal{H}_t$ has a density $f(t, \cdot)$, where $f : [0, \infty) \times I\!R \times \Omega \to [0, \infty)$ satisfies Condition B. Let $\lambda(t) = 0$ for $t > \tau$ and let*

$$\lambda_t(\omega) = \frac{1}{2}\sigma^2(0, t) f_x(t, 0, \omega), \quad 0 \leq t \leq \tau. \tag{B.4}$$

Then λ is an[23] *$(\mathcal{H}_t)$-intensity process of τ.*

An outline of the proof[24] is as follows: First one shows, by deterministic scaling and time transformation of a Brownian motion, that the limit result (A.1) holds for an Ornstein-Uhlenbeck process, that is, for V satisfying

$$dV_t = (a + bV_t)\, dt + \sigma\, dW_t.$$

The fact that the limit does not depend on the drift generalizes to SDEs of the form

$$dV_t = \mu(V_t, t)\, dt + \sigma\, dW_t, \tag{B.5}$$

in which μ satisfies a linear growth condition, since the solution to the SDE (B.5) can be kept pathwise between two OU processes on a fixed interval, and for these processes we have just seen that the hitting intensity is independent of the drift coefficients. In the general case (B.1), we define the function H by

$$H(x, t) = \int_0^x \frac{1}{\sigma(y, t)}\, dy, \qquad x \in (l, r).$$

[23] While there may exist other $(\mathcal{H}_t)$-intensity processes for τ, they are equivalent for our purposes. For the sense in which uniqueness applies, see Brémaud (1981), page 30.

[24] For details, see the working-paper version. A new proof of our result is due to Elliott, Jeanblanc and Yor (1999) and a multivariate extension of our intensity formula (B.4) has been developed by Song (1998).

Then, with $Y_t = H(V_t, t)$, we have by Ito's Formula that

$$dY_t = \mu_Y(Y_t, t)\, dt + dW_t,$$

where $\mu_Y(Y_t, t)$ can be computed using Ito's Formula. The first passage time of V to the level 0 is the same as the first passage time of Y to the level 0. Our regularity conditions ensure that the drift coefficient μ_Y of Y satisfies Condition A and that the density $g(\,\cdot\,)$ of the initial level of the process Y satisfies Condition B. Therefore, since Y has a constant volatility of 1, we know from above that the limit $(A.1)$ when considering Y hitting 0 is equal to $g'(0)/2$ which can be shown to equal $\frac{1}{2}f'(0)\sigma^2(0,0)$, proving the limit result for the general case. Given the boundedness condition on f', the verification of Aven's condition is analogous to the argument given in Appendix A.

Finally, we show that existence of a smooth conditional state density that is zero on the hitting boundary, as assumed in Condition B, is natural in our setting. We begin with the following result, which conditions only on survivorship, and follows from Corollary 3.43, page 99, of Cattiaux (1991).

Proposition B.2 *Suppose V satisfies $dV_t = \mu(V_t)\, dt + \sigma(V_t)\, dW_t$, where $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ and all of their derivatives exist and are bounded. Suppose, moreover, that $\sigma(\,\cdot\,)$ is bounded away from zero in an open interval $D \subset \mathbb{R}$, possibly unbounded. Let $\overline{D}$ denote the closure of D and $\tau = \inf\{t : V_t \notin D\}$ denote the time of first exit from D. Then, for any initial condition V_0 in D and any time $t > 0$, the distribution of V_t conditional on $\{\tau > t\}$ has a C^∞ density on $\overline{D}$ that is zero on the boundary of D.*

Now, for a given $V_0 \in D$ and time $t > 0$, we consider the distribution of V_t conditional on no exit from D and some "noisy observations" $Y_1, \ldots, Y_n$ that may be correlated with the process V. To pick a concrete example that generalizes the case studied in Section 2.2, we suppose that $Y_i = V(t_i) + U_i$, where $0 < t_1 < \cdots < t_i < t$, and where $U_1, \ldots, U_n$ are independent, and independent of V, and where the distribution of U_i has a density that is strictly positive on the real line. We let $F(\,\cdot\,; Y_1, \ldots, Y_n)$ denote the conditional distribution of V_t given $\{1_{\{t<\tau\}}, Y_1, \ldots, Y_n\}$.

Corollary B.3 *On the event $\tau > t$, $F(\,\cdot\,; Y_1, \ldots, Y_n)$ has, almost surely, a C^∞ density on $\overline{D}$ that is zero on the boundary of D.*

A proof is provided in the working-paper version of this report, based on Proposition B.2 and Tjur (1974, pp. 260-261), as well as induction in n.

References

[1] E. Altman (1968) "Financial Ratios, Discriminant Analysis, and the Prediction of Corporate Bankruptcy," *Journal of Finance* **23:** 589-609.

[2] R. Anderson and S. Sundaresan (1996) "Design and Valuation of Debt Contracts," *Review of Financial Studies* **9:** 37-68

[3] R. Anderson, Y. Pan, and S. Sundaresan (1995) "Corporate Bond Yield Spreads and the Term Structure," Working Paper, CORE and Columbia Business School.

[4] A. Arvanitis, J. Gregory, and J.-P. Laurent (1999) "Building Models for Credit Spreads," *Journal of Derivatives* (Spring) 27-43.

[5] T. Aven (1985) "A Theorem for Determining the Compensator of a Counting Process," *Scandinavian Journal of Statistics* **12:** 69-72.

[6] P. Artzner and F. Delbaen (1995), "Default Risk Insurance and Incomplete Markets," *Mathematical Finance* **5:** 187-195.

[7] M. Beneish and E. Press (1995) "Interrelation Among Events of Default," *Contemporary Accounting Research* **12:** 57-84

[8] E. J. Bijnen and M.F.C.M. Wijn (1994), "Corporate Prediction Models, Ratios or Regression Analysis?" Working Paper, Faculty of Social Sciences, Netherlands.

[9] F. Black and J. Cox (1976), "Valuing Corporate Securities: Liabilities: Some Effects of Bond Indenture Provisions," *Journal of Finance* **31:** 351-367.

[10] F. Black and M. Scholes (1973), "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* **81:** 637-654.

[11] P. Brémaud (1981) *Point Processes and Queues – Martingale Dynamics*, New York: Springer-Verlag.

[12] M. Brennan and E. Schwartz (1984), "Valuation of Corporate Claims," *Journal of Finance* **39:** 907-929.

[13] P. Cattiaux, (1991) "Calcul Stochastique et Opérateurs Dégénérés du Second Ordre, II. Problème de Dirichlet," *Bulletin des Sciences Mathématiques, Deuxieme Série*, **115:** 81-122.

[14] G. Duffee (1999) "Estimating the Price of Default Risk," *Review of Financial Studies* **12:** 197-226

[15] D. Duffie (1999), "Credit Swap Valuation," *Financial Analysts Journal* **55:** (January-February) pp. 73-87.

[16] D. Duffie, M. Schroder, and C. Skiadas (1996), "Recursive Valuation of Defaultable Securities and the Timing of Resolution of Uncertainty," *Annals of Applied Probability*, **6:** 1075-1090.

[17] D. Duffie and K. Singleton (1999), "Modeling Term Structures of Defaultable Bonds," *Review of Financial Studies* **12:** 687-720.

[18] R. Elliott, M. Jeanblanc, and M. Yor (1999) "Some Models on Default Risk," Working Paper, Department of Mathematics, University of Alberta, forthcoming, *Mathematical Finance*.

[19] H. Fan and S. Sundaresan (1997) "Debt Valuation, Strategic Debt Service and Optimal Dividend Policy," Working Paper, Columbia University.

[20] E. Fischer, R. Heinkel, and J. Zechner (1989) "Dynamic Capital Structure Choice: Theory and Tests," *Journal of Finance* **44:** 19-40.

[21] J. Fons (1994), "Using Default Rates to Model the Term Structure of Credit Risk," *Financial Analysts Journal*, September-October, pp. 25-32.

[22] R. Geske (1977), "The Valuation of Corporate Liabilities as Compound Options," *Journal of Financial Economics* **7:** 63-81.

[23] M. Harrison (1985) *Brownian Motion and Stochastic Flow Systems*, New York, John Wiley and Sons.

[24] J. He, W. Hu, and L. Lang (1999) "The Curvature and Slope in the Credit Spread Term Structure Curve," Working Paper, Chinese University of Hong Kong.

[25] J. Helwege and C. Turner (1999) "The Slope of the Credit Yield Curve for Speculative-Grade Issuers," *Journal of Finance* **54:** 1869-1884.

[26] R. Jarrow, D. Lando, and S. Turnbull (1997), "A Markov Model for the Term Structure of Credit Spreads," *Review of Financial Studies* **10:** 481-523

[27] R. Jarrow and S. Turnbull (1995), "Pricing Options on Financial Securities Subject to Default Risk," *Journal of Finance* **50:** 53-86.

[28] R. Johnson (1967), "Term Structures of Corporate Bond Yields as a Function of Risk of Default," *The Journal of Finance* **22:** 313-345.

[29] E. Jones, S. Mason, and E. Rosenfeld (1984), "Contingent Claims Analysis of Corporate Capital Structures: An Empirical Investigation," *The Journal of Finance* **39:** 611-625.

[30] I. Karatzas and S. Shreve (1988) *Brownian Motion and Stochastic Calculus*, New York: Springer-Verlag.

[31] S. Kusuoka (1999) "A Remark on Default Risk Models," *Advances in Mathematical Economics* **1:** 69-82.

[32] B. Lambrecht and W. Perraudin (1996) "Creditor Races and Contingent Claims," *European Economic Review* **40:** 897-907.

[33] D. Lando (1994), "Three essays on contingent claims pricing", Ph.D. Thesis, Cornell University.

[34] D. Lando (1998), "On Cox Processes and Credit Risky Securities," *Review of Derivatives Research* **2:** 99-120.

[35] H. Leland (1994), "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure," *Journal of Finance* **49:** 1213-1252.

[36] H. Leland (1998), "Agency Costs, Risk Management, and Capital Structure," *Journal of Finance* **53:** 1213-1242.

[37] H. Leland and K. Toft (1996), "Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads," *Journal of Finance* **51:** 987-1019.

[38] C. Lennox (1999) "Indentifying Failing Companies: A Re-evaluation of the Logit, Probit and DA Approaches," *Journal of Economics and Business* **51:** 347-364

[39] F. A. Longstaff and E. S. Schwartz (1995), "A Simple Approach to Valuing Risky Debt," *Journal of Finance* **50:** 789-821.

[40] K. G. Lundstedt and S. A. Hillgeist (1998), "Testing Corporate Default Theories," Working Paper, UC Berkeley and Northwestern University

[41] D. Madan and H. Unal (1998), "Pricing the Risks of Default," *Review of Derivatives Research* **2:** 121-160

[42] P.-A. Meyer (1966) *Probability and Potentials*, Waltham, Massachusetts: Blaisdell Publishing Company.

[43] C. G. McDonald and L.M. Van de Gucht (1999), "High-Yield Bond Default and Call Risks,"*Review of Economics and Statistics* **81:** 409-419.

[44] P. Mella-Barral (1999), "Dynamics of Default and Debt Reorganization," *Review of Financial Studies* **12:** 535-578.

[45] P. Mella-Barral and W. Perraudin (1997), "Strategic Debt Service," *The Journal of Finance* **52:** 531-556.

[46] R. Merton (1974), "On The Pricing of Corporate Debt: The Risk Structure of Interest Rates," *The Journal of Finance* **29:** 449-470.

[47] C. Pitts and M. Selby (1983), "The Pricing of Corporate Debt: A Further Note," *Journal of Finance* **38:** 1311-1313.

[48] P. Protter (1990), *Stochastic Integration and Differential Equations,* New York: Springer-Verlag.

[49] O. Sarig and A. Warga (1989), "Some Empirical Estimates of the Risk Structure of Interest Rates," *The Journal of Finance*, **46:** 1351-1360.

[50] P. Schönbucher (1998), "The Term Structure of Defaultable Bond Prices," *Review of Derivatives Research* **2:** 161-192

[51] D. Shimko, Tejima, N. and D. van Deventer (1993), " The Pricing of Risky Debt when Interest Rates are Stochastic," *The Journal of Fixed Income*, pp. 58-65.

[52] T. Shumway (1996) "Forecasting Bankruptcy More Efficiently: A Simple Hazard Model," Forthcoming in *Journal of Business*.

[53] M. Slovin, M. Sushka, and E. Waller (1997) "The Role of Banks and Trade Creditors in Client-Firm Bankruptcy," Working Paper, Louisiana State University, Baton Rouge.

[54] S. Song (1998) "A Remark on a Result of Duffie and Lando," Working Paper, Department of Mathematics, Université d'Evry, Val d'Essonne, France.

[55] P. Sengupta (1998) "Corporate Disclosure Quality and the Cost of Debt", *The Accounting Review* **73:** 459-474.

[56] T. Tjur (1974) *Conditional Probability Distributions*, Lecture Notes, Institute of Mathematical Statistics, University of Copenhagen.

[57] M. Uhrig-Homburg (1998) "Endogenous Bankruptcy when Issuance is Costly," Working Paper, Department of Finance, University of Mannheim.

[58] C. Zhou (1997) "A Jump Diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities," Working Paper, Federal Reserve Board.