First-to-Default Valuation

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PRELIMINARY AND INCOMPLETE

Abstract: This paper[1] provides simple models and applications for the valuation and simulation of contingent claims that depend on the time and identity of the first to occur of a given list of credit events, such as defaults. Examples include credit derivatives with a first-to-default feature, credit derivatives signed with a defaultable counterparty, credit-enhancement or guarantees, and other related financial positions.

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1 Introduction

This paper provides simple models and applications for the valuation and simulation of contingent claims that depend on the time and identity of the first to occur of a given list of credit events, such as defaults. Examples include credit derivatives with a first-to-default feature, credit derivatives signed with a defaultable counterparty, credit-enhancement or guarantees, and other related financial positions.

2 Model Setting

We suppose there are n credit events to be considered. A credit event is typically default by a particular entity, such as a counterparty, borrower, or guarantor. There are interesting applications, however, in which credit events may be defined instead in terms of downgrades, events that may instigate (with some uncertainty perhaps) the default of one or more counterparties, or other credit-related occurrences.[2] The key to modeling the joint timing of these credit events is a collection $h = (h_1, \ldots, h_n)$ of stochastic intensity processes, where h_i is the intensity process of credit event i. In general, an intensity process λ for a stopping time τ is characterized by the property that, for $N(t) = 1_{\tau \geq t}$, a martingale is defined by

$$N_t - \int_0^t (1 - N_s)\lambda_s \, ds, \qquad t \geq 0.$$

For details, see Brémaud (1980).[3] With constant intensity λ, the event has a Poisson arrival at intensity λ. More generally, for t before a stopping time

[2] At a presentation at the March, 1998 ISDA conference in Rome, Daniel Cunnigham of Cravath, Swaine, and Moore reviwed the documentation of credit swaps, including the specification of credit event types such as "bankruptcy, credit event upon merger, cross acceleration, cross default, downgrade, failure to pay, repudiation, or restructuring." The credit event is to be documented with a notice, suported with evidence of public announcement of the event, for example in the international press. The amount to be paid at the time of the credit event is determined by one or more third parties, and based on physical or cash settlement, as indicated in the confirmation form of the OTC credit swap transaction, a standard contract form with alternatives to be indicated.

[3] All random variables are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. A filtration $\{\mathcal{F}_t : t \geq 0\}$ of σ-algebras, satisfying the usual conditions, is fixed and defines the information available at each time. An intensity process λ is assumed to be non-negative and predictable (a natural measurability restriction) and to satisfy, for each $t > 0$, $\int_0^t \lambda_s \, ds < \infty$

τ with intensity process λ, we may view λ_t as the conditional rate of arrival of the event at time t, given all information available up to that time. In other words, for a small time interval of length Δ, the conditional probability at time t that the event occurs between t and $t + \Delta$, given survival to t, is approximately[4] $\lambda_t \Delta$.

Intensity processes $h_1, \ldots, h_n$ for credit event times $\tau_1, \ldots, \tau_n$ may in some cases be derived from information concerning the incentives and abilities of counterparties to meet their obligations, as in Duffie and Lando (1997), or might be fitted to market yield spreads, firm-specific financial ratios, sovereign risk indicators, or macroeconomic variables, as in Altman (199?), Bijnen and Wijn (1994), McDonald and Van de Gucht (1996), and Shumway (1996). We simply take a joint intensity process $h = (h_1, \ldots, h_n)$ for $(\tau_1, \ldots, \tau_n)$ as given, as in the "reduced-form" defaultable term-structure literature.[5]

3 First-Arrival Intensity

A simplifying and natural assumption is that there is zero probability that more than one credit event occurs at the same time. Under this assumption, the intensity associated with the first[6] credit event is easily obtained.

Lemma 1. *Suppose, for each $i \in \{1, \ldots, n\}$, that event time τ_i has intensity process h_i. Suppose that, $P(\tau_i = \tau_j) = 0$ for $i \neq j$. Then $h_1 + \cdots + h_n$ is an intensity process for $\tau = \min(\tau_1, \ldots, \tau_n)$.*

Proof: Let N_i denote the point process associated with τ_i, and N denote

almost surely. Some authors prefer to to describe "the" intensity as $(1 - N_s)\lambda_s$ rather than λ_s, and this indeed, under technical continuity conditions, allows for a uniqueness-of-intensity property. Our definition is weaker and does not suggest uniqueness, but has other definitional advantages, as in Lemma 1 and Proposition 1.

[4] This is true in the usual sense of derivatives if, for example, λ is a bounded continuous process, and otherwise can be interpreted in an almost-everywhere sense.

[5] Reduced-form models include those of Artzner and Delbaen (1995), Duffie and Singleton (1995), Duffie, Schroder, and Skiadas (1996), Jarrow and Turnbull (1996), Jarrow, Lando, and Turnbull (1997), Lando (1993, 1997, 1998), Madan and Unal (1995), Martin (1997), Nielsen and Ronn (1995), Pye (1974), Schönbucher (1997), and others.

[6] We may have $\tau_i = \infty$ with positive probability, and we take the definition $\tau = \infty$ on the event that $\tau_i = \infty$ for all i.

the point process associated with τ. Then, because $P(\tau_i = \tau_j) = 0$,

$$N(t) = N_1(t) + \cdots + N_n(t), \qquad t \leq \tau.$$

By the definition of the intensities $h_1, \ldots, h_n$, martingales $M_1, \ldots, M_n$ are defined by

$$M_i(t) = N_i(t) - \int_0^t (1 - N_i(s)) h_i(s)\, ds.$$

Let a process M be defined by

$$M(t) = N(t) - \int_0^t (1 - N(s))[h_1(s) + \cdots + h_n(s)]\, ds.$$

We have $M(t) = M_1(t) + \cdots + M_n(t)$ for $t \leq \tau$, and $M(t) = M(\tau)$ for $t \geq \tau$, so M is also a martingale. Thus $h_1 + \cdots + h_n$ is, by definition, the intensity process for τ, as asserted.

We would next like to characterize the "survival" probability $P(\tau \geq t)$, for a given t. The jump ΔY of a semi-martingale Y is defined by $\Delta Y(t) = Y(t) - \lim_{s \uparrow t} Y(s)$. The following proposition is likely to be well known among specialists; a proof is in any case provided for completeness.

Proposition 1. *Suppose τ is a stopping time with a bounded intensity process λ. Fixing some time $T > 0$, let*

$$Y(t) = E\left[\exp\left(-\int_t^T \lambda_u\, du\right) \,\Big|\, \mathcal{F}_t\right], \quad t \leq T.$$

If the jump $\Delta Y(\tau)$ is zero almost surely, then

$$P(\tau \geq T \,|\, \mathcal{F}_t) = Y(t), \qquad t < \tau,$$

almost surely.

Proof: Let Z be the martingale defined by

$$Z_t = E\left[\exp\left(-\int_0^T \lambda_u\, du\right) \,\Big|\, \mathcal{F}_t\right], \quad t \leq T.$$

Because

$$Y(t) = \exp\left(\int_0^t \lambda_u\, du\right) Z_t,$$

Ito's Formula implies that

$$dY_t = \lambda_t Y_t \, dt + \exp\left(\int_0^t \lambda_u \, du\right) dZ_t.$$

By definition of λ, there is a martingale M such that

$$dN_t = (1 - N_t)\lambda_t \, dt + dM_t.$$

Let U be defined by $U_t = Y(t)(1 - N_t)$. By Ito's Formula,

$$dU_t = -Y(t-) \, dN_t + (1 - N(t-)) \, dY_t - \Delta Y(t) \Delta N(t).$$

By the assumption that $\Delta Y(\tau) = 0$, we know that $\Delta Y(t)\Delta N(t) = 0$. Using our expressions above for dY_t and dN_t,

$$dU_t = -Y(t-) \, dM_t + (1 - N(t-)) \exp\left(\int_0^t \lambda_u \, du\right) dZ_t.$$

We thus find that U is a martingale with $U(T) = 1 - N(T)$. Thus, for any $t < \tau$,

$$Y(t) = U(t) = E(1 - N_T \,|\, \mathcal{F}_t) = P(\tau \geq T \,|\, \mathcal{F}_t),$$

as claimed.

The assumption that λ is bounded can be replaced with integrability conditions, as usual.[7]

Remark 1: *For any given predictable non-negative process λ satisfying, for each t, $\int_0^t \lambda_s \, ds < \infty$, we can always define a stopping time τ with the property that λ is its intensity and with the property assumed in Proposition 1, that $\Delta Y(\tau) = 0$ almost surely. This can be done, for example, by letting*

[7]For U to be a martingale, we want $\int Y(t-) \, dM_t$ and $(1 - N(t-)) \exp\left(\int_0^t \lambda_u \, du\right) dZ_t$ to be martingales. The first is a martingale as Y is bounded. For the second, letting, $\theta(t) = \exp\left(\int_0^t 2\lambda_s \, ds\right)$, it is enough that $E\left[\int_0^T \theta(t) \, d[Z]_t\right] < \infty$, where $[Z]$ is the "square-brackets" process for Z. For details, see, for example, Protter (1990).

Λ be an exponentially distributed random variable with mean 1, independent[8] *of Y, and by letting $\tau = \inf\{t : \int_0^t \lambda_s \, ds = \Lambda\}$.*

4 Formulation of Credit Events

If the hypotheses of Lemma 1 or Proposition 1 are not satisfied, it may be possible to re-formulate the definitions of the credit events so as to retain these properties. For instance, their failures can occur because, at some stopping time U, one or more designated credit events may perhaps occur only with probabilities (conditional on the information $\mathcal{F}_{U-}$ available just before U) that are neither zero nor one. For example, the onset of default by one counterparty can generate simultaneous default by another, perhaps contractually connected, with a conditional probability that is between 0 and 1.

A generally recommended principle is to reduce to primitive credit events, and to "build up" credit events derived from the primitives, by defining 0-or-1 random variables that indicate whether or not a derived credit event is instigated or not by the primitive event, as follows. The primitive event times are denoted $\tau_1, \ldots, \tau_n$. These should, if possible, have the "good" properties defined above. A derived event time $\hat{\tau}$ is then modeled by taking its point process $\hat{N}$ (that is, $\hat{N}(t) = 1_{t \geq \hat{\tau}}$) to be of the form

$$d\hat{N}(t) = (1 - \hat{N}(t-)) \sum_{i=1}^{n} A(i) \, dN_{it}, \tag{1}$$

where $A(1), \ldots, A(n)$ are random variables valued in $\{0, 1\}$, with $A(i)$ measurable with respect to $\mathcal{F}_{\tau(i)}$. We let a_i denote a predictable process with the property that, for $t < \tau$, $a_i(t) = E(A(i) | \mathcal{F}_t)$. The idea is that, at each stopping time τ_i, if the credit-event time $\hat{\tau}$ has not already occured, then

[8] In order to do this, it may be necessary to define a new probability space $\{A, \mathcal{A}, \alpha\}$, for example by letting $A = \Omega \times [0, \infty)$ with the usual product σ-algebra and product measure $\alpha = P \times \nu$, where ν is the distribution of an exponential density on $[0, \infty)$. Then a new filtration $\{\mathcal{A}_t\}$ is defined by letting $\mathcal{A}_t$ be the σ-algebra generated by the union of $\mathcal{F}_t$ and $\bigcup_{s \leq t} \{\omega : \tau(\omega) \geq s\}$. In terms of arriving at a reasonable economic model with a given intensity process, expanding the probability space and information in this manner, if necessary, seems innocuous. Our construction of τ allows for the possibility that $P(\tau = \infty) > 0$.

it occurs at τ_i with probability $a_i(\tau_i-)$, conditional on the information $\mathcal{F}_{\tau_i-}$ "just before" τ_i. If we let $\hat{h}$ be defined by

$$\hat{h}(t) = \sum_{i=1}^{n} a_i(t) h_i(t),$$

then, under the property that $P(\tau_i = \tau_j) = 0$ for $i \neq j$, it can be shown that $\hat{h}$ is an intensity process for $\hat{\tau}$. If, however, we let

$$\hat{Y}(t) = E\left[\exp\left(\int_t^s -\hat{h}(u)\, du\right) \;\Big|\; \mathcal{F}_t\right],$$

it is *not* generally true that $\hat{Y}(t) = P(\tau > s \,|\, \mathcal{F}_t)$ for $t < \hat{\tau}$, even if the "good" hypotheses of Lemma 1 and Proposition 1 are satisfied for each (τ_i, h_i). Indeed, $\hat{Y}$ can jump at $\hat{\tau}$ with positive probability, because $a_i(t)$ jumps to 0 or 1 at τ_i, with may turn out to be $\hat{\tau}$. Depending on the context, this failure of good properties for $\hat{\tau}$ may not bo so inconvenient, as one may have the ability to model prices or probabilities in terms of the primitive credit events.

5 Valuation Modeling

This section presents our basic valuation models. Subsequent sections present applications and calculation methods.

We take as given some bounded short-rate process[9] r, and some associated equivalent martingale measure Q, defined, following Harrison and Kreps (1979), as follows.

First, there is a given collection of securities available for trade, with each security defined by its cumulative dividend process D. This means that, for each time t, the total cumulative payment of the security up to and including time t is $D(t)$. For our purposes, a dividend process will always be taken to be of the form $D = A - B$, where A and B are bounded increasing adapted right-continuous left-limits (RCLL) processes, and we suppose that there are no dividends after a fixed time $T > 0$, in that $D(t) = D(T)$ for all t larger

[9]The probability measure Q is equivalent to P, in that the two measures have the same events of probability zero. The process r is assumed to be predictable. Most results go through without a bound on r, for example under conditions such as $E^Q\left[\exp\left(-\int_0^T r_t\, dt\right)\right] < \infty$ for all T.

than T. The fact that Q is an equivalent martingale measure means that, for any such security D, the ex-dividend price process S for the security is given by

$$S_t = E^Q\left[\int_t^T \exp\left(-\int_t^s r_u\,du\right)\,dD_s \;\middle|\; \mathcal{F}_t\right], \quad 0 \le t < T, \tag{2}$$

where E^Q denotes expectation under Q. The ex-dividend terminal price $S(T)$ is of course zero. An example is a security whose price is always 1, and paying the short-rate as a dividend, in that $D(t) = \int_0^t r_s\,ds$. As pointed out by Harrison and Kreps (1979) and Harrison and Pliska (1981), the existence of an equivalent martingale measure implies the absence of arbitrage and, under technical conditions, is equivalent to the absence of arbitrage. For weak technical conditions supporting this equivalence, see Delbaen and Schachermeyer (1994, 1997).

In some cases, markets are incomplete, for example one may not be able to perfectly hedge losses in market value that may occur at default, and this would mean that there need not be a unique equivalent martingale measure.

Given an intensity process h for a default time τ, Artzner and Delbaen (1995, Appendix A1) show that there is also an intensity process λ for the same time τ under the equivalent martingale measure Q, and show how to obtain λ in terms of h and Q. One should beware of the fact that even if Proposition 1 applies under the original measure P, it may not apply under an equivalent martingale measure Q, as the conjectured solution Y^Q, defined by $Y_t^Q = E^Q\left[\exp\left(\int_t^s -\lambda_u\,du\right)\right]$, for the process describing the survival probability under Q may in fact jump at the event time τ, even though the corresponding process Y associated with the original measure P does not. Kusuoka (1998) provides such an example.

Fixing an equivalent martingale measure Q, let us consider the valuation of a security that pays an amount Z at time T in the event that a given credit event time τ is after T, and otherwise pays an amount[10] W at the event time τ.

The cumulative dividend process D of this security can therefore be defined by

$$dD(t) = W\,dN_t, \quad t < T,$$

[10] We take W to be a bounded $\mathcal{F}_\tau$-measurable random variable, and Z to be a bounded $\mathcal{F}_T$-measurable random variable.

and

$$D(T) = WN_T + (1 - N_T)Z,$$

where $N = 1_{\tau \geq t}$ is the point process associated with τ. We suppose that, under Q, the event time τ has a bounded intensity process λ, so that

$$dN_t = (1 - N_t)\lambda_t \, dt + dM_N(t), \tag{3}$$

where M_N is a Q-martingale.

One can refer to Brémaud (1980), for example, to see that

$$dD_t = \lambda_t(1 - N_t)f(t) \, dt + dM_D(t), \quad t < T, \tag{4}$$

for a Q-martingale M_D, where f is the compensator for W, in the following sense. If we let $g = E^Q(W \mid \mathcal{F}_{\tau-})$, we may think of g as the expected payment conditional on all information up to, but not including, the time τ of the credit event. According[11] to Dellacherie and Meyer (1978), Result IV.67(b), there exists a predictable process f such that $f(\tau) = g$. For each fixed time t, we may view $f(t)$ as the risk-neutral expected payment, conditional on all information up to but not including time t, and given no default before time t, that would apply if default were to occur at time t.

As an example, one could take the amount paid at default to be $W = F(\tau)$, for an adapted process F. If F has continuous sample paths,[12] then $f(t) = F(t)$. For another example, suppose that the amount paid is the loss, relative to par, on a bond at default. Suppose that, until default, this loss has a conditional distribution η (under Q) that is fixed, and given by a statistically estimated distribution based on the history of losses of related bonds, and perhaps some risk premia parameters. (For example, η could be the empirical distribution for the seniority class of the bond, with an assumption of no risk premia.) Then f is a constant on $[0, T]$, and simply the mean (under Q) of this distribution η. Likewise, if the contractually stipulated payment W is some bounded measurable function $w : I\!\!R \to I\!\!R$ of the loss $100 - Y(\tau)$ relative to par, given recovery Y. Under the same

[11] I am grateful to Freddy Delbaen for showing me this construction. Please note that there is a typographical error in Dellacherie and Meyer (1978), Theorem IV.67(b), in that the second sentence should read: "Conversely, if Y is an $\mathcal{F}^0_{T-}$-measurable..." rather than "Conversely, if Y is an $\mathcal{F}^0_T$-measurable...," as can be verified from the proof, or, for example, from their Remark 68(b).

[12] If F is a semi-martingale with jumps only at non-predictable stopping times, then $f(t) = F(t-)$.

independence assumption, we would have f a constant equal to the expected payment $\int w(x)\, d\eta(x)$. For example, $w(x) = \max(x, 50)$ would stipulate a payment of the loss or 50, whichever is greater.

If an issuer has multiple obligations with priorities defined by maturity or subordination, then, for each obligation, $f(t)$ changes over time based on the prioritized allocation of assets to liabilities, in expectation (under Q), conditional on survival to date. For example, if an issuer has one short-maturity and one long-maturity bond, as t passes through the earlier maturity date, the Q-expected default-contingent payment $f(t)$ on the long-maturity bond would jump, possibly downward if the two bonds are of equal priority and assets are close in distribution to the level necessary to pay down the short-maturity bond. Jumps at predictable times such as a maturity date cause no difficulty.

In general, the price process S of this security is given from (2) and (4) by

$$S_t = E^Q \left[\int_t^T \delta_{t,s}^r (1 - N_s) \lambda_s f(s)\, ds + \delta_{t,T}^r (1 - N_T) Z \;\middle|\; \mathcal{F}_t \right], \quad t < T, \tag{5}$$

where, for $s \geq t$ and each predictable process α with $\int_0^T |\alpha_t|\, dt < \infty$ a.s.,

$$\delta_{t,s}^{\alpha} = \exp\left(- \int_t^s \alpha_u\, du \right).$$

This expression (5) for the price process S can in practice be difficult to evaluate, as N_t appears directly. For example, brute-force simulation, by discretization, of $(N_t, r_t, \lambda_t, f_t)$ can be extrmely tedious, and relatively precise estimates of the initial market value S_0 may call for a large number of independent scenarios if λ is small, which is typical in practice. The following result, along the lines of results found in Duffie, Schroder, Skiadas (1996), Lando (1997), and Schönbucher (1997), is more convenient.

Proposition 2. *Let*

$$V(t) = E^Q \left[\int_t^T \delta_{t,s}^{r+\lambda} \lambda_s f(s)\, ds + \delta_{t,T}^{r+\lambda} Z \;\middle|\; \mathcal{F}_t \right], \quad t < T, \tag{6}$$

and $V(T) = 0$. If the jump $\Delta V(\tau)$ of V at τ is zero almost surely, then $S(t) = V(t)$ for $t < \tau$.

The condition that $\Delta V(\tau) = 0$ (analogous to $\Delta Y(\tau) = 0$ in Proposition 1) is not restrictive in settings for which there are no sudden (jump) surprises, other than default, in the joint conditional distribution under Q of interest rates, arrival intensities, and the payoff variables f and Z. For example, if r, λ, f, and Z are functions of some diffusion process[13] then V is continuous up to T, and thus $\Delta V(t) = 0$ for any $t < \tau$. More generally, one can allow jumps in V provided they cannot happen at default times. As provided in Remark 1, for *any* such candidate value process V, one can always construct a model with $\Delta V(\tau) = 0$ and price process $S = (1 - N)V$. (In this case, the artificially introduced random variable Λ of the Remark is defined to be independent and exponential under Q.) If $\Delta V(\tau)$ need not be zero, a somewhat more complicated formula, provided in Duffie, Schroder, and Skiadas (1996), can be applied. Duffie, Schroder, and Skiadas (1996) also extend the model to treat cases in which the expected loss $f(t)$ or the intensity λ may depend on V itself, under additional technical regularity.

Proof: The proof is an extension of that of Proposition 1. Let M be the Q-martingale defined by

$$M_t = E^Q \left[\int_0^T \delta_{0,s}^{r+\lambda} \lambda_s f(s)\, ds + \delta_{0,T}^{r+\lambda} Z \;\middle|\; \mathcal{F}_t \right].$$

We have

$$M_t = \delta_{0,t}^{r+\lambda} V_t + \int_0^t \delta_{0,s}^{r+\lambda} \lambda_s f_s\, ds, \quad t < T, \tag{7}$$

and we may therefore write

$$V_t = \int_0^t \mu_V(s)\, ds + M_V(t), \quad t < T, \tag{8}$$

where M_V is a Q-martingale and, applying the fact that M is a martingale to (7),

$$\mu_V(t) = (r_t + \lambda_t) V_t - \lambda_t f_t, \quad t < T.$$

[13]That is, if λ, r, f, and Z are measurable with respect to some process that is a d-dimensional Brownian motion with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, Q)$, then, on the time interval $[0, T)$, V is an Ito process, and therefore has no jumps.

We will complete the proof by showing that $S_t = (1 - N_t)V_t$, which is true if and only if $H = J$, where

$$H_t = (1 - N_t)V_t\delta^r_{0,t} + \int_0^t \delta^r_{0,u}\, dD_u$$

and

$$J_t = S_t\delta^r_{0,t} + \int_0^t \delta^r_{0,u}\, dD_u.$$

Because $S_T = V_T = 0$, we have $J_T = H_T$. Because $\Delta S(T) = (1-N_T)\Delta V(T) = \Delta D(T) = (1 - N_T)Z$, we know that $\Delta H(T) = \Delta J(T) = 0$. We know from (2) that J is a Q-martingale. It therefore suffices to show that H is also a Q-martingale. This follows from an application of Ito's Formula,[14] (3), (4), and (8):

$$\begin{aligned}
dH_t &= -r_t\delta^r_{0,t}(1 - N_t)V_t\, dt - V_{t-}\delta^r_{0,t}\, dN_t + (1 - N_{t-})\delta^r_{0,t}\, dV_t + \delta^r_{0,t}\, dD_t \\
&= -(r_t + \lambda_t)\delta^r_{0,t}(1 - N_t)V_t\, dt + (1 - N_{t-})\delta^r_{0,t}\mu_V(t)\, dt \\
&\qquad +\delta^r_{0,t}(1 - N_t)\lambda_t f_t\, dt + dM_H(t) \\
&= dM_H(t), \quad t < T,
\end{aligned}$$

where M_H is the Q-martingale, defined by

$$dM_H(t) = (1 - N_{t-})\delta^r_{0,t}\, dM_V(t) + V_{t-}\delta^r_{0,t}\, dM_N(t) + \delta^r_{0,t}\, dM_D(t), \quad t < T.$$

(We used the assumption that $\Delta V(\tau) = 0$ when disregarding the term $\Delta V(t)\Delta N(t)$.) The result therefore follows as stated.

We can reinterpret Proposition 2 so as to apply to the valuation of a loss at a given default (or other credit event) that may be brought on, or not, by a primitive credit events. That is, suppose a given credit event is derived from τ in the sense that its stopping time $\hat{\tau}$ is modeled, as with, (1), by $\hat{N}(t) = 1_{\hat{\tau}\geq t}$ with

$$d\hat{N}(t) = A\, dN(t),$$

where $N_t = 1_{\tau\geq t}$ is the indicator for a stopping time τ with Q-intensity λ, and A is an $\mathcal{F}_\tau$-measurable random variable valued in $\{0, 1\}$ indicating whether

[14] We use the fact that, for any bounded semi-martingale U, we have $\int_0^T U(t)\, dt = \int_0^T U(t-)\, dt$ almost surely, as U can have at most a countable number of jumps.

or not the arrival of τ causes the event in question to occur. We let α be a process valued in $[0,1]$ such that, for $t < \tau$, we have $\alpha(t) = E^Q(A \mid \mathcal{F}_t)$. Then a Q-intensity $\hat{\lambda}$ of $\hat{\tau}$ is defined by $\hat{\lambda}(t) = \alpha(t)\lambda(t)$. As noted earlier, $(\hat{\tau}, \hat{\lambda})$ may not have the "good" property exploited in Proposition 1. We can nevertheless proceed by modeling in terms of (τ, λ). Suppose that $\hat{W}$ is the amount paid at $\hat{\tau}$. (We assume that $\hat{W}$ is bounded and $\mathcal{F}_\tau$-measurable.) Let $W = A\hat{W}$ and f be a predictable process such that, for $t < \tau$, we have $f(t) = E(W \mid \mathcal{F}_t)$. It follows that, under the hypotheses of Proposition 2, the market value of receiving $\hat{W}$ at $\hat{\tau}$ (and Z at T if $\hat{\tau} > T$) is given at any time $t < \min(\tau, T)$ by $V(t)$ of (9), and at any time $t \geq \min(\tau, T)$ by zero.

6 First-to-Default Valuation

Now we consider the valuation of a contingent claim that pays off at $\tau = \min(\tau_1, \ldots, \tau_n)$, the first of n credit events, a contingent amount W_i if $\tau = \tau_i$. That is, the amount paid is explicitly dependent on the identity of the first credit event to occur.[15] We suppose that the n credit events have stopping times $\tau_1, \ldots, \tau_n$ with respective bounded intensity processes $\lambda_1, \ldots, \lambda_n$ under Q, and that $\tau_i \neq \tau_j$ almost surely for $i \neq j$. By Lemma 1, the intensity process of τ under Q is $\lambda = \lambda_1 + \cdots + \lambda_n$. Our candidate security pays a random variable[16] Z at T if $\tau > T$. The dividend process D of this security is therefore defined by

$$\begin{aligned} dD(t) &= (1 - N_{t-}) \sum_i W_i \, dN_i(t), \qquad t < T, \\ dD(T) &= (1 - N_T) Z, \end{aligned}$$

where $N_t = 1_{\tau \geq t}$. We let f_i denote the predictable projection of a measure-valued adapted process η_i with the property that, for $t < \tau$, $\eta_i(t)$ is the conditional distribution of W_i given $\mathcal{F}_t$. Using the fact that

$$dD_t = (1 - N_{t-}) \sum_i \lambda_i(t) f_i(t) \, dt + dM_D(t), \quad t < T,$$

where M_D is a Q-martingale, we have the following.

[15] The random variables $W_1, \ldots, W_n$ are bounded, and W_i is measurable with respect to $\mathcal{F}_{\tau(i)}$.

[16] Again, we take Z to be bounded and $\mathcal{F}_T$-measurable.

Proposition 3. *Let*

$$V(t) = E^Q\left[\int_t^T \delta_{t,s}^{r+\lambda} \sum_{i=1}^n \lambda_i(s) f_i(s)\, ds + \delta_{t,T}^{r+\lambda} Z \;\middle|\; \mathcal{F}_t\right], \quad 0 \le t < T, \tag{9}$$

and $V(T) = 0$. If the jump $\Delta V(\tau)$ of V at τ is zero almost surely, then $S_t = V_t$ for $t < \tau$.

The proof is identical to that of Proposition 2. One merely exploits the fact that $dV_t = \mu_V(t)\, dt + dM_V(t)$, where M_V is a Q-martingale and

$$\mu_V(t) = V_t(r_t + \lambda_t) - \sum_i f_i(t)\lambda_i(t).$$

This implies that

$$\delta_t^r V_t(1 - N_t) + \int_0^t \delta_s^r (1 - N_s) \sum_i \lambda_i(s) f_i(s)\, ds, \quad t < T,$$

is, under the given hypotheses, a Q-martingale, which leaves $V(1 - N) = S$, as with the proof of Proposition 2.

Kusuoka (1998) gives examples in which the timing risk-premium process (loosely, the difference between the intensities of the default arrivals under the original measure P and the equivalent martingale measure Q) may jump unexpectedly at a default arrival. This would be the case, for example, if default by one obligor causes a sudden re-assessment of the equilibrium risk premium for another default by another obligor. In such cases, a somewhat more complicated pricing formula arises, as in Duffie, Schroder, and Skiadas (1996).

7 Analytical Solutions in Affine Settings

This section proposes parametric examples in which the first-to-default pricing formula (9) can be computed either explicitly, or by numerically solving relatively simple ordinary differential equations, in an "affine" setting. That is, these examples are based on a a "state" process X valued in $\mathbb{R}^k$ that (under Q) is a k-dimensional affine jump-diffusion, in the sense of Duffie and Kan (1996). That is, X is valued in some appropriate domain $D \subset \mathbb{R}^k$, with

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t + dJ_t,$$

where B is a standard brownian motion in $I\!R^d$, J is a pure jump process with jump-arrival intensity $\{\kappa(X_t) : t \geq 0\}$ and jump distribution ν on $I\!R^k$, and where $\kappa : D \to [0, \infty)$, $\mu : D \to I\!R^k$, and $C \equiv (\sigma\sigma^\top) : D \to I\!R^{k\times d}$ are affine functions.[17] We delete time dependencies to the coefficients for notational simplicity only; the approach outlined below extends to that case in a stragihtforward manner. A classical special case is the "multi-factor CIR state process" X, for which $X^{(1)}$, $X^{(2)}, \ldots, X^{(k)}$ are independent (or Q-independent, in a valuation context) processes of the "square-root" type[18] introduced into term-structure modeling by Cox, Ingersoll, and Ross (1985). For many other examples of affine models, see Duffie, Pan, and Singleton (1997).

We can take advantage of this setting if we suppose short rates, intensities, and payoffs are of the affine[19] form

$$\begin{aligned}
r(t) &= a_r(t) + b_r(t) \cdot X_t \\
\lambda_i(t) &= a_\lambda(i,t) + b_\lambda(i,t) \cdot X_t \\
f_i(t) &= \exp\left(a_f(i,t) + b_f(i,t) \cdot X(t-)\right) \\
Z &= \exp\left(a_Z + b_Z \cdot X_T\right),
\end{aligned}$$

where $a_Z \in I\!R$ and $b_Z \in I\!R^k$ are given constants and where, for each i in $\{1, \ldots, n\}$:

- $a_r, a_\lambda(i, \cdot)$, and $a_f(i, \cdot)$ are bounded measurable real-valued deterministic functions on $[0, T]$.

[17] The generator $\mathcal{D}$ for X is defined by

$$\mathcal{D}f(x,t) = f_t(x,t) + f_x(x,t)\mu(x) + \frac{1}{2}\sum_{ij} C_{ij}(x) f_{x_i x_j}(x,t) + \kappa(x)\int [f(x+z,t) - f(x,t)]\, d\nu(z).$$

One can add time dependencies to these coefficients. Conditions must be imposed for existence and uniqueness of solutions, as indicated by Duffie and Kan (1996).

[18] That is,

$$dX_t^{(i)} = \kappa_i(\overline{x}_i - X_t^{(i)})\, dt + \sigma_i \sqrt{X_t^{(i)}}\, dB_t^{(i)},$$

for some given constants $\kappa_i > 0$, $\overline{x}_i > 0$, and σ_i.

[19] One can proceed in more or less the same fashion if one generalizes by allowing quadratic terms for $r(t)$ and the jump intensity κ, subject of course to technical conditions. In such cases, higher-order terms will appear in the solution polynomial. One can also extend in a similar fashion to yet higher-order polynomials.

- $b_r, b_\lambda(i,\,\cdot\,)$, and $b_f(i,\,\cdot\,)$ are bounded measurable deterministic $I\!R^k$-valued functions on $[0,T]$.

Except in degenerate cases, the boundedness assumptions used in Propositions 1 through 3 do not apply, and integrability conditions must be assumed or established. For the special case in which $f_i(t)$ represents the risk-neutral expected loss (relative to par, say) on a floating-rate note (a typical application in practice, say first-to-default swaps), one might reasonably take $f_i(t)$ to be deterministic. (For this case, $b_f = 0$.)

For analytical approaches based on the affine structure just described, one can repeatedly use the following calculation, regularity conditions for which are provided in Duffie, Pan, and Singleton (1997).

Let X be an affine jump-diffusion. For a given time s, and for each $t \le s$ let $R(t) = a_R(t) + b_R(t) \cdot X(t)$, for bounded measurable $a_R : [0,s] \to I\!R$ and $b_R : [0,s] \to I\!R^k$. For given coefficients a in $I\!R$, and b in $I\!R^k$, let

$$g(X_t, t) = E\left[\exp\left(\int_t^s -R(u)\,du\right) e^{a+b\cdot X(s)} \;\Big|\; X_t\right]. \tag{10}$$

Under technical conditions, there are specified ODEs for $\alpha : [0,s] \to I\!R$ and $\beta : [0,s] \to I\!R^k$ such that

$$g(x,t) = \exp\left(\alpha(t) + \beta(t)\cdot x\right),$$

with boundary conditions $\alpha(s) = a$ and $\beta(s) = b$. In addition, for given A in $I\!R$, and B in $I\!R^k$, let

$$G(X_t, t) = E\left[\exp\left(\int_t^s -R(u)\,du\right) e^{a+b\cdot X(s)}(\hat{a} + \hat{b}\cdot X_s) \;\Big|\; X_t\right]. \tag{11}$$

Then, under technical conditions, there are specified ODEs for $\hat{\alpha} : [0,s] \to I\!R$ and $\hat{\beta} : [0,s] \to I\!R^k$ such that

$$G(x,t) = e^{\alpha(t)+\beta(t)\cdot x}(\hat{\alpha}(t) + \hat{\beta}(t)\cdot x),$$

with boundary conditions $\hat{\alpha}(s) = \hat{a}$ and $\hat{\beta}(s) = \hat{b}$.

Details, with illustrative numerical examples and empirical applications, can be obtained in Duffie, Pan, and Singleton (1997). For our application to

(9), we would be assuming that X is an affine jump-diffusion under Q, and the expectations in (10) and (11) would be under Q. With solutions for α, β, $\hat{\alpha}$, and $\hat{\beta}$ in hand, the pricing formula (9) reduces to a one-dimensional numerical integral, which is a relatively fast exercise. Monte-Carlo based approaches are illustrated in the following section.

For the special multi-factor CIR case, explicit closed-form solutions for α and β can be deduced from Cox, Ingersoll, and Ross (1985) (for the case $a = \hat{a} = 0$ and $b = \hat{b} = 0$), and Duffie, Pan, and Singleton (1997), who also provide analytical solutions for the Fourier transforms of X in the general affine setting, and related calculations that lead to analytical solutions for option pricing via Lévy inversion of the transforms. The Fourier-based option-pricing results can be applied in this setting for cases in which

$$f_i(t) = [\exp(a_f(i,t) + b_f(i,t) \cdot X(t-)) - K]^+ ,$$

for some exercise price K. Some explicit results for option pricing are available in certain cases, as shown by by Bakshi, Cao, and Chen (1996), Bakshi and Madan (1997), Bates (1996), and Chen and Scott (1995). These results can be applied in the present setting for valuation of defaultable options with affine structure, or with recovery determined by collateralization with an instrument whose price can be described in an exponential-affine form. Collateralization with equities, foreign currency, or notes (in domestic or foreign currency) would be natural examples for this.

8 Simulating The First to Default

In some cases, explicit solutions may be complicated, but Monte Carlo simulation may be straightforward. This section presents an algorithm for simulation based on explicit, or easily computed, distributions for the time to the first credit event, and for the identity of that event. The setup is that of Section 5.

That is, $\tau = \min(\tau_1, \ldots, \tau_n)$, and $\lambda = \sum_i \lambda_i$. For any $t \geq 0$ and $s \geq t$, we let

$$Y(t,s) = E^Q \left[\exp\left(-\int_t^s \lambda_u \, du \right) \;\middle|\; \mathcal{F}_t \right].$$

We suppose that the jump $Y(\tau, s) - Y(\tau-, s)$ is zero almost surely. From Proposition 1,

$$Q(\tau \geq s \,|\, \mathcal{F}_t) = Y(t,s), \qquad t < \tau,$$

almost surely.

Let us suppose that we have a effective method for computing $Y(t,s)$, for each $s > t$. A special case is deterministic intensities, for which $Y(t,s)$ is of course given explicitly or by numerical integration. More generally, the affine structure proposed in Section 6 provides such a method. We may then simulate a random variable R that is equivalent in distribution to τ (to be precise, equivalent in $\mathcal{F}_t$-conditional distribution under Q) by independently simulating a random variable U that is uniformly distributed on $[0,1]$, and (provided $Y(t,s)$ ranges from 1 to 0 as s ranges from t to ∞) by then letting R be chosen[20] so that $Y(t,R) = U$. In this case

$$Q(R \geq s \,|\, \mathcal{F}_t) = Q(Y(t,R) \leq Y(t,s) \,|\, \mathcal{F}_t) = Q(U \leq Y(t,s) \,|\, \mathcal{F}_t) = Y(t,s).$$

If $\underline{Y}(t) = \lim_{s\to\infty} Y(t,s) > 0$, then we simply let $R = \infty$ (the credit event never happens) in the event that $U \leq \underline{Y}(t)$.

Having simulated the first arrival time τ, we wish to simulate which of the n events occured at that time. Given an outcome t for τ, we will simulate a random variable I with outcomes in $\{1,\ldots,n\}$, and with probability $q(i,\tau) = Q(I = i \,|\, \tau)$ for the outcome i equal to the conditional probability, under Q, that $\tau(i) = \tau$ given τ.[21] We apply the following calculation, relying on a formal applications of Bayes' Rule:

$$q(i,t) = Q(\tau = \tau_i \,|\, \tau \in (t, t+dt)) = \frac{\gamma(i,t)\,dt}{Q(\tau \in (t,t+dt))}, \tag{12}$$

where[22]

$$\gamma(i,t)\,dt = Q(\tau = \tau_i \text{ and } \tau \in (t,t+dt)). \tag{13}$$

Because $q(1,t) + \cdots + q(n,t) = 1$, it is enough to calculate $\gamma(i,t)$ for each i and t, leaving

$$q(i,t) = \frac{\gamma(i,t)}{\gamma(1,t) + \cdots + \gamma(n,t)}. \tag{14}$$

[20] As $Y(t,\,\cdot\,)$ is continuous and monotone, some such measurable R is well defined. If $Y(t,\,\cdot\,)$ is not strictly monotone, any measurable selection of its inverse will suffice for choosing $R(\omega)$ given $U(\omega)$.

[21] To be precise, we are looking for a regular version $q(i,\,\cdot\,) : [0,\infty) \to [0,1]$ of this conditional probability.

[22] That is, $\gamma(i,\,\cdot\,)$ is the density of the measure Γ_i on $[0,\infty)$ defined by $\Gamma_i(A) = Q(\tau = \tau_i \text{ and } \tau \in A)$.

Now, for bounded and right-continuous $(\lambda_1, \ldots, \lambda_n)$ and using dominated convergence and the assumption that $\tau_i \neq \tau_j$ a.s.,

$$\gamma(i,t) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \Gamma(t,i,\epsilon), \tag{15}$$

where

$$\Gamma(t,i,\epsilon) = Q(\tau \geq t - \epsilon \text{ and } \tau_i \leq t). \tag{16}$$

Next, we calculate that

$$\begin{aligned}
\Gamma(t,i,\epsilon) &= E^Q[1_{\tau \geq t-\epsilon} \ 1_{\tau(i) \leq t}] \\
&= E^Q[1_{\tau \geq t-\epsilon} \ E^Q[1_{\tau(i) \leq t} \mid \mathcal{F}_{t-\epsilon}]] \\
&= E^Q\left[1_{\tau \geq t-\epsilon} \left(1 - 1_{\tau(i) \geq t-\epsilon} E^Q\left[\exp\left(\int_{t-\epsilon}^t -\lambda_i(s)\, ds\right) \Bigm| \mathcal{F}_{t-\epsilon}\right]\right)\right] \\
&= E^Q\left[1_{\tau \geq t-\epsilon} \left(1 - E^Q\left[\exp\left(\int_{t-\epsilon}^t -\lambda_i(s)\, ds\right) \Bigm| \mathcal{F}_{t-\epsilon}\right]\right)\right].
\end{aligned}$$

By dominated convergence, the assumption that λ_i is bounded and right-continuous (which can be weakened), and the fact that, letting $H(x) = 1 - e^{-\delta x}$, for a given constant δ, we have $H'(0) = \delta$, it then follows that

$$\begin{aligned}
\gamma(t,i) &= E^Q\left[\lim_{\epsilon \downarrow 0} 1_{\tau \geq t-\epsilon} \ \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left(1 - E^Q\left[\exp\left(\int_{t-\epsilon}^t -\lambda_i(s)\, ds\right) \Bigm| \mathcal{F}_{t-\epsilon}\right]\right)\right] \\
&= E^Q\left[1_{\tau \geq t} \lambda_i(t)\right].
\end{aligned}$$

Finally, we can apply Proposition 2 to see that, under our usual "no-jump-at-τ" regularity,[23]

$$\gamma(t,i) = E^Q\left[\exp\left(-\int_0^t \lambda(s)\, ds\right) \lambda_i(t)\right].$$

For the affine structure described earlier, we have an efficient method for computing $\gamma(i,t)$, and thereby $q(i,t)$. Simulation of the first to default, both the time τ and the identity $I(\tau)$ of the first to default, is therefore straightforward in an affine setting, including of course deterministic intensities.

[23] Letting

$$\gamma(t,i;s) = E^Q\left[\exp\left(-\int_0^t \lambda(s)\, ds\right) \lambda_i(t) \Bigm| \mathcal{F}_s\right],$$

we would impose the hypothesis that, for each fixed t and i, $\gamma(t,i;\tau) - \gamma(t,i;\tau-) = 0$ almost surely.

9 Simulating with Mortality for Discounts

This section shows how to compute the market value of the first to default, including the effect of discounting for interest rates. In order to do this, we will introduce a fictitious event with "risk-neutral" arrival intensity $\lambda_0 = r$, the short rate. (For this section, we assume that r is non-negative.) The contingent payment at the associated[24] stopping time τ_0 is $W_0 = 0$. The first-to-arrive event time, now including τ_0, is $\tau^* = \min(\tau_0, \tau_1, \ldots, \tau_n)$, with Q-intensity process $\lambda^* = \lambda_0 + \lambda_1 + \cdots + \lambda_n$. Then, under the hypotheses of Proposition 3, we have the price of the first to default given by

$$\begin{aligned} Y_0 &= E^Q\left[\int_0^T \exp\left(\int_0^t -(r_s + \lambda_s)\, ds\right) \sum_{i=1}^n \lambda_{it} f_{it}\, dt\right] \\ &= E^Q\left[\int_0^T \exp\left(\int_0^t -\lambda_s^*\, ds\right) \sum_{i=0}^n \lambda_{it} f_{it}\, dt\right] \\ &= E^Q[1_{\tau^* \leq T} W_{I^*}], \end{aligned}$$

where I^* is the random variable valued in $\{0, 1, \ldots, n\}$ that is the identity of the credit event that happens at τ^*. That is, $\tau^* = \tau_{I^*}$.

By our previous calculations, we can estimate Y_0 by simulation of τ^* and then I^*. We can draw τ^* by simulating a uniformly distributed random variable and then, as above, using the inverse cumultative distribution function for τ^*, defined, under the hypotheses of Proposition 1 (under Q) by

$$Q(\tau^* \geq t) = E^Q\left[\int_0^T \exp\left(\int_0^t -\lambda_s^*\, ds\right)\right].$$

As for I^*, once we have the outcome t of τ^*, we simulate a number from $\{0, 1, \ldots, n\}$, drawing i with probability

$$q^*(i,t) = \frac{\gamma^*(i,t)}{\gamma^*(0,t) + \gamma^*(1,t) + \cdots + \gamma^*(n,t)}, \tag{17}$$

where

$$\gamma^*(i,t) = E^Q\left[\exp\left(-\int_0^t \lambda^*(s)\, ds\right) \lambda_i(t)\right]. \tag{18}$$

[24] In order to complete the story, we could introduce a standard (unit intensity) Poisson process Λ, independent under Q, of all previously defined variables, and let $\tau_0 = \inf\{t : \Lambda\left(\int_0^t r_s\, ds\right) = 1\}$. There is of course no need to actually construct such a process.

Again, affine dynamics (including deterministic intensities) make computations straightforward.

Finally, given the outcomes of τ^* and I^*, we would let $F = 0$ if $\tau^* > T$ and otherwise let $F = f_{I^*}(\tau^*)$. Let $F_1, F_2, F_3, \ldots$ denote an *iid* sequence of random variables, all (simulated, in practice) with the distribution of F. We have, by the law of large numbers,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_i = Y_0 \quad a.s., \tag{19}$$

giving us the desired price, or an approximation for a large number N of draws. This procedure is certainly effective for deterministic $f_i(\,\cdot\,)$, or, for the case in which $f_i(t) = g_i(X_t)$, if we can compute the conditional distribution of $X(\tau^*)$ given τ^* and I^*, to which we next turn.

10 Conditional State Distribution at Events

In a state-space setting such as the affine setting described earlier, we will now focus on the conditional distribution of the state $X(\tau)$ given the first event time τ. This can be applied, for example, to the computation of the conditional expected projected payoff $E^Q[f_i(\tau) \,|\, \tau]$, or to the simulation of $X(\tau)$ conditional on τ, which is in turn useful for simulating event times after τ. In some cases, we are interested in the joint distribution of $X(\tau)$ and I (the identity of the first credit event), after conditioning on τ. We will begin by working under the measure P, but after re-formulation, the results can be applied under Q.

Suppose the state process X is valued in some domain $D \subset I\!R^k$. We are looking for some $\pi : D \times [0, T] \to [0, \infty)$ with the property that, for any bounded measurable $F : D \to I\!R$, we have

$$E[F(X(\tau)) \,|\, \tau] = \int_D F(x)\pi(x, \tau)\, dx \qquad a.s.$$

We begin by supposing that τ has intensity process $\{H(X_t) : t \geq 0\}$, where H is bounded, continuous, and strictly positive. We adopt the hypothesis of "no jumping at τ" for conditional expectations, wherever called for. Again, by a formal application of Bayes's Rule, one finds that, for each

t,

$$E[F(X(\tau)) \mid \tau = t] = \frac{F^*(X(t), t)}{E\left[\exp\left(\int_0^t -H(X_s)\, ds\right) H(X_t)\right]},$$

where

$$F^*(X_t, t) = E\left[\exp\left(\int_0^t -H(X_s)\, ds\right) H(X_t) F(X_t)\right].$$

We have

$$F^*(x, t) = \int_D F(x) H(x) \pi^*(x, t)\, dx,$$

where π^* is the fundamental solution of

$$\mathcal{D}g(x, t) - H(x) g(x, t) = 0, \tag{20}$$

for $\mathcal{D}$ the infinitesimal generator associated with X. Under regularity, it follows that

$$\pi(x, t) = \frac{\pi^*(x, t) H(x)}{p(t)}, \tag{21}$$

where p is the density of τ, given by

$$p(t) = E\left[\exp\left(\int_0^t -H(X_s)\, ds\right) H(X_t)\right].$$

There are several special cases of the affine models, for affine $H(\,\cdot\,)$, for which the fundamental solution π^* is known explicitly, including the multi-factor CIR model and the model of Chen (1994). (Such π^* are sometimes called "Green's functions.") As for $p(t)$, it can be readily computed in an affine setting, as discussed previously.

From (25), we are in a position to "re-start" the state process at a simulated first-credit-event time τ, by simulating the re-started state $X(\tau)$ with density $\pi(\,\cdot\,, t)$.

Simulation of $X(\tau)$, conditional on τ, might be simpler if the coordinate processes $X^{(1)}, \ldots, X^{(k)}$ were independent, as with the multi-factor CIR model. Even if this is true, however, it is not generally true after conditioning on τ, as can be seen from the form of the joint density $\pi(\,\cdot\,, t)$, which is not of a product form because of the appearance of $H(x)$ in (25).

Simplification is possible, however, in the case that $X^{(1)}, \ldots, X^{(k)}$ are independent and $H(x) = \zeta(x_j)$ for some $\zeta(\,\cdot\,)$ and some particular $j \in \{1, \ldots, n\}$. In this case,

$$\pi(x,t) = \frac{\pi_j^*(x,t)\zeta(x_j)}{p(t)} \prod_{i \neq j} \pi_i^*(x_i, t),$$

where, for $i \neq j$, $\pi_i^*(\,\cdot\,, t)$ is the fundamental solution of

$$\mathcal{D}_i g(x_i, t) = 0, \tag{22}$$

for $\mathcal{D}_i$ the infinitesimal generator associated with $X^{(i)}$, and where $\pi_j^*(\,\cdot\,, t)$ is the fundamental solution of

$$\mathcal{D}_j g(x_j, t) - \zeta(x_j) g(x_j, t) = 0, \tag{23}$$

for $\mathcal{D}_j$ the infinitesimal generator associated with $X^{(j)}$. Here, one can simulate $X(\tau)$ given τ by simulating, given τ, $\{X^{(i)}(\tau) : 1 \leq i \leq n\}$ conditionally independently, with τ-conditional density $\pi_i^*(\,\cdot\,, \tau)$ for $i \neq j$ and, for $i = j$, with τ-conditional density $\pi_j^*(\,\cdot\,, \tau)\zeta(\,\cdot\,)/p(\tau)$, based on simulating n independent uniform-[0,1] variables.

As for the distribution of $X(\tau)$ given both $\tau = \min(\tau_1, \ldots, \tau_n)$, and the identity I of the first of n credit event times $\tau_1, \ldots, \tau_n$, we take again a setting in which τ_i has intensity process $\{H_i(X_t) : t \geq 0\}$, for H_i bounded and continuous, and with $H = \sum_{i=1}^n H_i$ strictly positive. We suppose that $P(\tau_i = \tau_j) = 0$ for $i \neq j$, and continue to assume the principal of "no jumping at τ" of the appropriate conditional expectations, wherever called for.

We have the calculation of $\pi : D \times [0, T] \times \{1, \ldots, n\} \to [0, \infty)$ with the property that, for any bounded measurable $F : D \to \mathbb{R}$, we have

$$E[F(X_\tau) \mid \tau, I\,] = \int_D F(x)\pi(x, \tau, I)\, dx \qquad a.s.$$

By a formal application of Bayes's Rule, one finds that, for each t and i,

$$E[F(X_\tau) \mid \tau = t, I = i\,] = \frac{F^*(X(t), t, i)}{E\left[\exp\left(\int_0^t -H(X_s)\, ds\right) H_i(X_t)\right]},$$

where

$$F^*(X_t, t) = E\left[\exp\left(\int_0^t -H(X_s)\, ds\right) H_i(X_t) F(X_t)\right].$$

We have

$$F^*(x,t) = \int_D F(x) H_i(x) \pi^*(x,t)\, dx,$$

where π^* is again the fundamental solution of (20). Under regularity, it follows that

$$\pi(x,t,i) = \frac{\pi^*(x,t) H_i(x)}{p_i(t)}, \tag{24}$$

where

$$p_i(t) = E\left[\exp\left(\int_0^t -H(X_s)\, ds\right) H_i(X_t)\right].$$

Explicit or straightforward calculation of $\pi(\,\cdot\,,t,i)$ is possible in certain affine settings, as explained above. As shown by Duffie, Pan, and Singleton (1997), we can always re-sort to analytical methods for computing the Fourier transform of the distribution of $X(\tau)$ given (τ, I), and many calculations can be done "on the Fourier side."

Now, for example, in order to compute V_0 of Proposition 3, for the case in which $f_i(t) = G(X_t, i)$ for some G that does not allow a solution entirely by analytical methods, it suffices to proceed as follows, in an affine setting (under Q) for the state vector X, the short rate $r = \rho(X)$, and, for each i, the intensity $\lambda_i = \Lambda_i(X)$. Under the no-jump-at-τ condition on the candidate value process V of Proposition 3, the estimated price V_0 at time 0 of a payment of W_i at τ in the event that $\min(\tau, T) = \tau_i$ is given by (19), where $F_1, F_2, \ldots,$ are independently generated with a distribution equivalent to F, simulated by the following algorithm:

1. Let $a_R(t) + b_R(t) \cdot X_t = r_t + \lambda_1(t) + \cdots + \lambda_n(t)$.

2. Simulate a uniform-[0,1] random variable U, and note that a random variable R with the Q-distribution of τ^* is obtained by[25] letting $R(\omega)$ satisfy

$$E^Q\left[\exp\left(\int_0^{R(\omega)} -[a_R + b_R \cdot X_t]\, dt\right)\right] = U(\omega),$$

   taking $R(\omega) = \infty$ if there is no solution.

3. If $R(\omega) > T$, let $F(\omega) = 0$, and stop.

[25] That is, R has the distribution under Q of a stopping time with intensity $a_R + b_R \cdot X_t$.

4. Otherwise, simulate ι from $\{0, 1, \ldots, n\}$ by drawing i with probability $q^*(i, t)$ given by (17) and (18). These probabilities are, in an affine setting, either explicitly computed or obtained by numerical solution of ODEs, as indicated previously.

5. If $\iota(\omega) = 0$, let $F(\omega) = 0$, and stop.

6. Otherwise, if $\iota(\omega) = i$, let

$$\pi_Q(x, t, i) = \frac{\pi_Q^*(x, t)\Lambda_i(x)}{q^*(i, t)}, \tag{25}$$

   where π_Q^* is the fundamental solution of

$$\mathcal{D}_Q g(x, t) - [a_R + b_R \cdot x] g(x, t) = 0,$$

   and $\mathcal{D}_Q$ denotes the infinitesimal generator associated with X under Q.

7. Let

$$F(\omega) = \int_D \pi_Q(x, R(\omega), \iota(\omega)) G(x, R(\omega), \iota(\omega))\, dx, \tag{26}$$

   estimated by numerical methods if necessary, and stop. One should bear in mind that the Fourier transform of $\pi_Q^* \Lambda_i$ can be computed by relatively straightforward methods, or in several cases explicitly, as explained in Duffie, Pan, and Singleton (1997), and approximate Fourier methods may be more efficient than direct numerical integration of (26).

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