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Optimal Innovation of Futures Contracts

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This article presents a simple model of the innovation of new futures contracts by transaction volume-maximizing futures exchanges in incomplete markets under uncertainty, with mean-variance preferences and proportional transactions costs. We characterize the set of Nash equilibria for a number of exchanges simultaneously or sequentially choosing contracts. The optimal monopolistic contract design is shown to be Pareto-optimal. An example shows the failure of Pareto optimality for a particular Nash equilibrium. Likewise, in a monopolistic multiperiod setting, an example shows the failure of Pareto optimality given an incentive for the exchange to induce turnover.

Broadly speaking, this article addresses the role of security exchanges in determining the equilibrium structure of security markets. More narrowly, the article characterizes the futures contracts that will be offered by exchanges in order to maximize transaction volume, in an extremely restrictive general equilibrium setting. The article begins in a monopolistic setting, later turning to Nash equilibrium among exchanges in an oligopolistic setting.

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A futures exchange is typically a nonprofit organization; its governing members, acting as brokers, profit individually from providing transaction services, either directly in the form of brokerage or indirectly by making a market in order to profit from trades on their own accounts. A premise of the article is that exchange members prefer a futures contract choice that maximizes the volume of trade. Saloner (1984), for example, has provided a model supporting this argument in which Bertrand competition (in brokerage fees) among the exchange members results in a commonly set transaction fee T per contract and equally shared contract volume. In reality, some exchange memberships are held by large hedgers, who obviously have distinct objectives in the contract choice process. We will nevertheless proceed with the assumption that a futures exchange chooses a contract that maximizes the equilibrium volume of trade.

In abstract terms, a futures contract is a contingent claim $f: \Omega \rightarrow \mathbb{R}$ promising to pay $f(\omega)$ units of a *numeraire* good at a prearranged price, where ω is the state chosen at random from the set Ω of all states. Given a transaction fee $T \in [0, \infty)$ set by floor brokers, a contract f is optimal for the exchange if it maximizes the resulting transaction volume¹ $v(f, T)$. After examining the monopolistic case, we move to an oligopolistic setting. We define the *innovation* of a new futures contract f_n to be the unspanned portion of the contract's payoff, given the previously available contracts f_1, \dots, f_{n-1} . In our mean-variance setting, this definition is made precise by defining the unspanned portion of a random variable z given existing contracts f_1, \dots, f_{n-1} to be $\pi^\perp(z|f_1, \dots, f_{n-1}) = z - \pi(z|f_1, \dots, f_{n-1})$, where $\pi(z|f_1, \dots, f_{n-1})$ denotes the usual minimum-variance projection of z onto the span of f_1, \dots, f_{n-1} . Thus, the innovation of f_n can be thought of as the portion of f_n that cannot be "hedged" on the previously available markets.

Our restrictive model affords the luxury of unique equilibrium solutions that support a prevailing intuition in the field of finance: A popular new security is one whose innovation is highly correlated with a linear combination of the unspanned portions of individuals' endowments, in the following sense. Let x_k denote the endowment of agent k , and let r_k be the constant absolute risk aversion of agent k . We normalize the contract choice decision so that the standard deviation of the contract innovation is fixed. (Without normalization, volume can be made arbitrarily large by scaling down the contract and scaling up individual trades.) As we show, if a new contract f maximizes transaction volume, then the innovation of f is perfectly correlated with the unspanned portion of the *endowment differential* $e_S r_S - e_L r_L$, with $e_S = \sum_{k \in S} x_k$ denoting the total endowment of the set S of agents on the short side of the market, $r_S = [\sum_{k \in S} (1/r_k)]^{-1}$, the harmonic mean of the risk-aversion coefficients of the short side, and e_L

¹ If the exchange sets the transaction fee and contract in order to maximize transaction fee revenue, our results would not be affected in any major way. In the single-contract case, the problems $\max_{f,T} v(f, T)T$ and $\max_f v(f, T)$ (for fixed T) yield the same characterization of f reported here. In a multicounterpart setting, the solutions for the two cases are close for small transaction fees in the usual sense of limits.

and r_L similarly defined for the long side. This conclusion also gives us a characterization of pure strategy Nash equilibria (without redundant contracts) in a game among exchanges, each maximizing transaction volume in its choice of futures contracts given the choices of the other exchanges.

We should emphasize some of the limitations of our model. First, the objective of maximizing transaction volume does not incorporate individual exchange members' competing objectives. For example, members may have hedging motives. Furthermore, transactions fees are often, in effect, based on nonlinear pricing schedules. In short, we have not modeled the inner workings of the exchange in any detail. Second, we have placed no limits on the index defining the payoff of candidate futures contracts. The exchange in our model has complete knowledge of the agents' endowments and preferences and is not limited by the practical difficulties of choosing a legally unambiguous, nonmanipulable, and easily measured index. A more realistic model would impose practical limits on the menu of potential contracts. Third, we have largely ignored the role of liquidity in contract design. In multiperiod settings, a new contract can fail because potential orders may flow to other contracts with established liquidity ("ease" of taking and offsetting positions at a low bid-ask spread), even if the other contracts are inferior from our (Walrasian) perspective. Basically, the natural role of continual price volatility and sustained order flow in contract design cannot be properly captured in our simple setting. Despite these limitations, we feel our model takes "first things first" and will be a useful framework for further developments. Some empirical weight on the side of our approach has been given by Black (1986) as well as Johnston and McConnell (1989).

Our basic model is presented in the following section, where the monopolistic contract design problem is solved. In Section 2, we design the optimal additional contract for a given regime of futures contracts. As a by-product, we obtain a formula for open interest that extends to a similar multiperiod setting studied in Duffie and Jackson (1986). In Section 3, we examine the multi-exchange contract design problem. We characterize the set of Nash equilibria of a game of volume maximization among a number of exchanges simultaneously choosing futures contracts. We also show that the set of contracts resulting from sequential contract choice forms a Nash equilibrium.

The remainder of the article addresses efficiency issues. First, we show the static monopolistic design to be constrained Pareto-optimal, in the sense that no other contract with the same transaction fee yields a Pareto-dominating allocation in equilibrium. While there exist Pareto-optimal Nash equilibria, there also exist Nash equilibria that are not Pareto-optimal (as we show by example). We also overturn efficiency in a multiperiod setting by counterexample. With more than a single round of trade, a revenue-maximizing exchange has an incentive to generate a higher volume of trade by designing contracts that induce agents to reverse the directions of their trades. Given the social benefits of liquidity in non-

Walrasian markets, there may be offsetting social welfare advantages to the design of contracts that induce frequent trading by agents. Our objective is merely to bring to notice some of the most basic issues, and not to prescribe any regulatory policy.

For other discussion related to the innovation of security markets, see Anderson (1986), Carlton (1983a, 1983b), Allen and Gale (1988), Sandor (1973), Silber (1981, 1983), Miller (1986), Telser (1981), Townsend (1984), and Van Horne (1985). Anderson (1984) reviews the regulation of futures innovation. Anderson and Harris (1986) study a game of innovation between futures exchanges in which the choice variable is the time of introduction of a given contract rather than the choice of contract.

1. The Basic Model

We begin with the simplest possible model, in which a single contract is chosen, and later extend. A single consumption commodity acts as *numeraire*. A probability space (Ω, F, P) characterizes uncertainty, with Ω denoting the set of states of the world, F the events, and P the commonly held probability measure on events. [For a finite-dimensional example, $\Omega = \{1, \dots, S\}$ for S states, F is the set of subsets of Ω , and $P(\{s\})$ is the probability of any state $s \in \Omega$.] Let $L^2(P)$ denote the random variables on (Ω, F, P) with finite variance, treating random variables that are equal almost surely as the same. Each agent $k \in \{1, \dots, K\}$ has a random endowment $x_k \in L^2(P)$, representing $x_k(\omega)$ units of consumption in state $\omega \in \Omega$. The preferences of agent k are represented by a functional $U_k : L^2(P) \rightarrow \mathbb{R}$. For simplicity, we assume the mean-variance representation

$$U_k(x) = E(x) - r_k \text{var}(x)$$

for some scalar $r_k > 0$, where $\text{var}(x)$ denotes the variance of x . Agents' preferences are thus completely characterized by the risk-aversion coefficients r_1, \dots, r_K . We ignore the usual problem of mean-variance utility, supposing the risk-aversion coefficients to be sufficiently small relative to endowments.

In this basic model, a single futures contract is available for trade; later we cover multiple futures contracts. A futures contract is a contingent claim $f \in L^2(P)$, which may be thought of as a claim to $f(\omega)$ units of consumption contingent in state $\omega \in \Omega$. At a futures price $p \in \mathbb{R}$, holding one contract is actually a contingent claim to $f(\omega) - p$ in state ω . That is, rather than paying p initially and receiving $f(\omega)$ in state ω , the change in value $f(\omega) - p$ is credited to an agent's margin account in state ω . This abstract definition allows a futures contract to settle on any index of random variables.

Agent k takes a contract $f \in L^2(P)$, a per-contract transactions fee $T \in [0, \infty)$, and a futures price $p \in \mathbb{R}$ all as given and solves for an optimal futures position

$$y_k \in \text{argmax}_{y \in \mathbb{R}} U_k(x_k + y(f - p) - T|y|) \quad (1)$$

(Adding an initial consumption period and a margin account to the setting only obscures the results. In our setting, a futures contract and a forward contract are indistinguishable.) Provided $\text{var}(f) \neq 0$, the solution y_k to (1) is unique and satisfies

$$y_k = |\alpha_k| \left(\frac{-\text{cov}(x_k, f)}{\text{var}(f)} + \frac{E(f) - p + \alpha_k T}{2r_k \text{var}(f)} \right) \tag{2}$$

where $\alpha_k = -\text{sign}(y_k)$. The solution is obtained by checking for the correct case of (1) the right-hand side of Equation (2) is negative and $\alpha_k = 1$, (2) the right-hand side of Equation (2) is positive and $\alpha_k = -1$, or (3) $y_k = \alpha_k = 0$. One and only one of these three cases applies. Case (3) applies if and only if

$$\left| -2 \text{cov}(x_k, f) + \frac{E(f) - p}{r_k} \right| \leq \frac{T}{r_k}$$

The coefficient α_k is zero only when the transactions costs dominate the benefit of a nonzero futures position.

A collection $(y_1, \dots, y_K, p) \in \mathbb{R}^{K+1}$ is an equilibrium given $(f, T) \in L^2(P) \times [0, \infty)$ if, for all k , y_k solves Equation (1) given (p, f, T) , and if $\sum_{k=1}^K y_k = 0$.

Lemma 1. *For any given $(f, T) \in L^2(P) \times [0, \infty)$ there is a unique equilibrium allocation (y_1, \dots, y_K) . If $y_k \neq 0$ for some k , the corresponding equilibrium futures price is*

$$p = E(f) - \frac{2}{\gamma} \text{cov} \left(\sum_k |\alpha_k| x_k, f \right) + \frac{T \Gamma}{\gamma} \tag{3}$$

where $\alpha_k = -\text{sign}(y_k)$, $\gamma = \sum_k (|\alpha_k|/r_k)$, and $\Gamma = \sum_k (\alpha_k/r_k)$.

Proof. The given expression for p follows directly from Equation (2) and market clearing. We then substitute (3) into (2) to obtain

$$y_k = |\alpha_k| \left[\frac{-\text{cov}(x_k, f)}{\text{var}(f)} + \frac{2 \text{cov} \left(\sum_j |\alpha_j| x_j, f \right) - T \sum_j (\alpha_j/r_j) + \alpha_k T}{2r_k \text{var}(f) \sum_j (|\alpha_j|/r_j)} \right]$$

It is then easily verified that one and only one of the cases 1, 2, or 3 [described following Equation (2)] applies. ■

Lemma 1 shows that the expected price change in the futures contract is related to the covariance of the futures price with the endowments of the market participants. In particular, if there are no transaction costs (T

= 0), then Equation (3) reduces to the CAPM, since $\sum_j \alpha_j x_j$ represents the aggregate endowment of investors with nonzero futures position.

As is to be expected with transaction costs, finding the equilibrium is a combinatorial problem. One must guess whether $\text{sign}(y_k)$ is zero or the same as in the equilibrium without transactions costs, then solve Equation (3) for p , next solve Equation (2) for y_k , and finally verify the sign assumptions.

Let $S \subset \{1, \dots, K\}$ denote the subset of short ($y_k < 0$) agents and L the subset of long agents. Define $\beta_S(f) = \text{cov}(f, e_S) / \text{var}(f)$ as the beta of the total short endowment position $e_S = \sum_{k \in S} x_k$ with respect to the contract, and similarly define $\beta_L(f) = \text{cov}(f, e_L) / \text{var}(f)$. Let $\gamma_S = \sum_{k \in S} (1/r_k)$ and $\gamma_L = \sum_{k \in L} (1/r_k)$ represent the risk tolerances of the short side and long side of the market, respectively. Substituting Equation (3) into Equation (2) then gives the volume² calculation

$$v(f, T) = -\beta_L(f) + \beta_S(f) + [\beta_L(f) + \beta_S(f)] \frac{\gamma_L - \gamma_S}{\gamma} - \frac{2T\gamma_S\gamma_L}{\text{var}(f)\gamma}$$

Ignoring transaction costs ($T = 0$) and assuming a common level of risk aversion, there are two polar cases of practical interest:

1. If hedging is evenly split between shorts and long, specifically if the number $|L|$ of long agents is equal to the number $|S|$ of short, or if $\sum_{k=1}^K x_k = 0$, we have $v(f, 0) = \beta_S(f) - \beta_L(f)$.
2. If all hedging is on one side of the market, say the short side (meaning $\sum_{k \in L} x_k = 0$), then $\beta_L(f) = 0$ and we have $v(f, 0) = 2\beta_S(f) |L|/K$, since the aggregate risk tolerance of the long and short side is proportional to these numbers.

For the multicontract case, see section 1.1. For a multiperiod setting, Duffie and Jackson (1986) show (without transaction costs) that similar calculations yield open interest at any date under statistical assumptions on price movements.

1.1 Contract design

By the equilibrium lemma, there is a function $y_k : L^2(P) \times [0, \infty) \rightarrow IR$ mapping any given contract f and transactions fee T to the equilibrium futures position $y_k(f, T)$ of agent k . We have the following from Equations (2) and (3):

Lemma 2 (homogeneity). For any agent k , any $(f, T) \in L^2(P) \times [0, \infty)$, and any scalar $\delta \in (0, \infty)$,

$$y_k(\delta f, \delta T) = \frac{1}{\delta} y_k(f, T)$$

² In our static setting, open interest is simply half the volume.

Because of the homogeneity lemma, if transaction costs are proportional [$T = \tau SD(f)$, where $\tau \in \mathbb{R}$ and $SD(f)$ denotes the standard deviation of f], it is natural to assume that the contract is chosen from those with fixed scale, say the subset F of $L^2(P)$ with standard deviation equal to 1. We then take it that the exchange chooses f to solve the problem

$$\max_{f \in F} v(f, T) = \sum_{k=1}^K |y_k(f, T)| \tag{4}$$

A contract $f \in L^2(P)$ with nonzero standard deviation is defined to be volume maximizing if the normalized contract $f/SD(f)$ solves Equation (4).

The homogeneity lemma implies that, if there is any per-contract cost to the exchange for performing transactions services and there are no integer constraints on trades, then there is no optimal contract design size, since scaling up a contract reduces costs to the exchange and leaves revenue unaffected. This indicates that integer transaction constraints are a necessary part of any model determining optimal contract size.

Proposition 1. *Let $(y_1, \dots, y_K) \neq 0$ be the equilibrium contract allocation for a given futures contract f . Let $d(f) = \gamma_L e_S - \gamma_S e_L$, where $e_S = \sum_{k \in S} x_k$ is the sum of the endowments of the agents who have short positions (in f), and likewise, where $e_L = \sum_{k \in L} x_k$, $\gamma_L = \sum_{k \in L} (1/r_k)$, and $\gamma_S = \sum_{k \in S} (1/r_k)$. Then f is volume maximizing only if f is perfectly correlated with $d(f)$.*

The proposition states that a futures contract f is volume maximizing only if it is perfectly correlated with $d(f)$, which we will call the endowment differential, the endowment of the short side of the market less the endowment of the long side of the market, each weighted by the risk tolerance of the other side of the market. Provided both S and L are not empty, we can replace $d(f)$ with $d(f)/(\gamma_L \gamma_S)$ and redefine the endowment differential as the more intuitive index $e_S r_S - e_L r_L$, with $r_S = [\sum_{k \in S} (1/r_k)]^{-1}$, the harmonic mean of the risk-aversion coefficients of the short side, and r_L similarly defined for the long side. In other words, we can think of the determinant of volume-maximizing futures contracts to be the difference between the short and long endowments of the market, each weighted by the harmonic average of the risk aversion of that side. This provides support for the adage that the popularity of a contract is determined by hedging motives.

Proof of Proposition 1. From Equations (2), (3), and simple calculations, Equation (4) is equivalent to

$$\max_{f \in F} \text{cov} \left[f, \sum_k x_k \left(\alpha_k - |\alpha_k| \frac{\Gamma}{\gamma} \right) \right] - \frac{T}{2\gamma} (\gamma^2 - \Gamma^2) \tag{5}$$

It is easily verified that $\sum_k x_k (\alpha_k - |\alpha_k| \Gamma / \gamma) = 2d(f) / \gamma$. For constant γ and Γ , it follows that a solution to Equation (5) must be of the form $f = d(f) / \text{SD}[d(f)] + c$ for some constant c . In order to account for changes in γ and Γ , the result is verified by checking that the Gâteaux derivatives of the volume with respect to contract choice are nonpositive in all directions only for contracts of the given form. The constant c is irrelevant. ■

Proposition 1 gives only a necessary condition in terms of an endogenous variable, the endowment differential $d(f)$, which depends, via the sets S and L of short and long agents, on the choice of futures contract. In the case of zero transaction costs, we can give necessary and sufficient conditions. From the proof of Proposition 1, the volume-maximizing futures contract f must give the largest possible variance to $d(f)$.

Corollary 1. *If $T = 0$, then the futures contract f is volume maximizing if and only if 1. f is perfectly correlated with the endowment differential $d(f)$. 2. For any futures contract f' , $\text{var}[d(f)] \geq \text{var}[d(f')]$.*

Roughly, condition 2 requires that the contract be perfectly correlated with the largest (in standard deviation) possible endowment differential. The following example helps to illustrate the role of condition 2.

Example 1. (sufficient conditions for volume maximization). Consider an economy with four agents who have a common level of risk aversion r and random endowments described by $x_1 = -x_2 = w$ and $x_3 = -x_4 = z$, where w and z are random variables with $\text{var}(w) = \text{var}(z) = 1$ and $\text{cov}(w, z) = 0$.

For this economy, there are four types of contracts that satisfy the necessary conditions for volume maximization. They are $f = \delta[(2/r)w] + c$, $f = \delta[(2/r)z] + c$, $f = \delta[(4/r)(w + z)] + c$, and $f = \delta[(4/r)(w - z)] + c$, with respective endowment differentials $\pm(2/r)w$, $\pm(2/r)z$, $\pm(4/r)(w + z)$, and $\pm(4/r)(w - z)$. The four types of contracts, however, do not lead to the same volume of transactions. Let us consider representative (normalized) contracts of each type: $f = w$, $f = z$, $f = (w + z)/\sqrt{2}$, and $f = (w - z)/\sqrt{2}$. The transaction volumes associated with these four contracts are 2, 2, $2\sqrt{2}$, and $2\sqrt{2}$, respectively. The third and fourth types of contracts result in larger transaction volumes since they involve all agents in trading, while the first two types of contracts only induce half of the agents to trade. The endowment differential $d(f)$ associated with contracts of the third and fourth types have the highest variance (since more agents are involved in trading); therefore, only contracts of the third and fourth types satisfy condition 2 in Corollary 1.

2. The Optimal Additional Contract

A richer model allows the exchange to design a contract adding to those currently available for trade. We will characterize the optimal design of

the additional contract. For tractability, we assume in the following that the transaction fee is zero.

2.1 Equilibrium

Let $f = (f_1, \dots, f_n)^\top$ denote n given contracts, where \top denotes transpose, and where $f_i \in L^2(\mathcal{P})$ for $i \in \{1, \dots, n\}$. For a vector $v = (v_1, \dots, v_n)$ of scalars or random variables, let v_{-i} denote the $(n - 1)$ -dimensional vector $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$. The contract f_i is redundant given f if there exists $\beta \in \mathbb{R}^{n-1}$ such that $f_i - E(f_i) = \beta^\top [f_{-i} - E(f_{-i})]$. Given a price vector $p \in \mathbb{R}^n$, agent k chooses a futures position $y_k \in \mathbb{R}^n$ so as to maximize

$$U_k[x_k + y_k^\top(f - p)] \tag{6}$$

A collection $(y_1, \dots, y_K, p) \in \mathbb{R}^{n(K+1)}$ is an equilibrium given f if y_k maximizes Equation (6) for all $k \in \{1, \dots, K\}$, and if $\sum_k y_k = 0$.

Let $\text{cov}(f)$ denote the $n \times n$ covariance matrix of the contract set f . Assuming there are no redundant contracts, which is true if and only if $\text{cov}(f)$ is positive definite, the optimal futures position for agent k is

$$y_k = [\text{cov}(f)]^{-1} \left[-\text{cov}(f, x_k) + \frac{E(f - p)}{2r_k} \right] \tag{7}$$

where $\text{cov}(f, x_k) \in \mathbb{R}^n$ has i th element $\text{cov}(f_i, x_k)$. Since $r_k > 0$ and $\text{cov}(f)$ is positive definite, Equation (7) is also a sufficient condition for y_k to maximize Equation (6).

Proposition 2. *Given the futures contracts f , there is an equilibrium (y_1, \dots, y_K, p) (which is unique if there are no redundant futures contracts), and*

$$p = E(f) - \frac{2}{\gamma} \text{cov}\left(f, \sum_k x_k\right)$$

where $\gamma = \sum_k (1/r_k)$.

The pricing formula given above is again equivalent to the capital asset pricing model (CAPM).

Proof of Proposition 2. The proof follows directly from substituting p into Equation (7) and using $\sum_k y_k = 0$. ■

2.2 Contract design

We now examine the problem of an exchange that takes the contracts f_{-i} chosen by other exchanges as given and chooses f_i so as to maximize the volume of trade in contract f_i subject to a scale normalization. We recall that the innovation of f_i given the existing contracts f_{-i} is the unspanned portion $\pi^\perp(f_i | f_{-i}) = f_i - \pi(f_i | f_{-i})$, where $\pi(z | f_{-i})$ denotes the usual min-

imum-variance projection of a random variable $z \in L^2(P)$ onto the span of f_{-i} , and can be thought of as the portion of z that can be “hedged” on the previously available markets. For a set of nonredundant contracts f , the minimum-variance projection of a random variable z onto f_{-i} is $\pi(z|f_{-i}) \equiv \text{cov}(z, f_{-i})\text{cov}(f_{-i})^{-1}f_{-i}$. Extending from the normalization chosen in the single-contract case, we restrict the exchange to a contract whose innovation has a fixed standard deviation of 1. The reason for choosing this normalization is straightforward. The size of any given agent’s position in contract i depends only on the innovation of contract i . (See the proof of Proposition 3 and the Appendix.) Another normalization may distort the incentives of the exchange. This is illustrated in Example 2, which follows shortly.

In summary, we have arrived at the innovation problem:

$$\max_{f_i \in L^2(P)} \sum_k |y_{ki}(f)| \quad \text{subject to} \quad N(f_i) = 1$$

where $N(z) = \text{SD}[\pi^\perp(z|f_{-i})]$.

As stated in the following lemma, the normalization defined here by $N(\cdot)$ satisfies the natural requirement of homogeneity (of degree -1) in scale, showing no essential economic scale distortions or incentives to rescale caused by the normalization constraint $N(f_i) = 1$.

Lemma 2 (homogeneity). For nonredundant futures contracts $f = (f_1, \dots, f_n)$,

$$y_{ki}(f_1, \dots, f_{i-1}, \delta f_i, f_{i+1}, \dots, f_n)N(\delta f_i)$$

is constant over $\delta \in (0, \infty)$.

It follows from Lemma 2 that the innovation problem described above is equivalent to

$$\max_{f_i \in L^2(P)} \sum_k |y_{ki}(f)| N(f_i) \tag{8}$$

Let (y_1, \dots, y_K, p) be an equilibrium given a set of nonredundant futures contracts (f_1, \dots, f_n) . Let $S(i)$ denote the set of agents short in contract i , that is, $S(i) = \{k: y_{ki}(f) < 0\}$, and likewise define $L(i) = \{k: y_{ki}(f) \geq 0\}$ for the agents long in contract i . The endowment differential for contract i is $d(f)_i = \gamma_{L(i)}e_{S(i)} - \gamma_{S(i)}e_{L(i)}$, where $e_{S(i)} = \sum_{k \in S(i)} x_k$ and likewise for $e_{L(i)}$ and the risk tolerances $\gamma_{S(i)}$ and $\gamma_{L(i)}$. We define α_{ki} by $\alpha_{ki} = -1$ when $y_{ki}(f) \geq 0$ and $\alpha_{ki} = 1$ when $y_{ki}(f) < 0$. (This convention is slightly different from the one used in the single-contract design case. The two conventions are equivalent, however, when there are no transaction costs as we assume in this section.) We define $\Gamma_i = \sum_k (\alpha_{ki}/r_k)$.

The following result shows that the volume-maximizing contract f_i must have an innovation perfectly correlated with the unspanned portion of the endowment differential $d(f)_i$.

Proposition 3. *Suppose that $f = (f_1, \dots, f_n)$ includes no redundant contracts. If f_i is volume maximizing given f_{-i} , then the innovation of f_i is perfectly correlated with $\pi^\perp[d(f)_i|f_{-i}]$, the unspanned portion of the endowment differential.*

Proof. We use the fact that $d(f)_i$ and $\sum_k |y_k(f)| N(f_i)$ are both invariant with respect to adjustments in the contract choice defined by addition and multiplication by nonzero scalars. This implies that solutions come in the form of an equivalence class. That is, if f_i is an optimal contract design, then $\delta f_i + \beta^\top f_{-i} + c$ is also an optimal contract design for any $\beta \in \mathbb{R}^{n-1}$, $\delta \neq 0$, and c . (On the other hand, volume in the other contracts is affected by β , which may be important if the other contracts are controlled by the same exchange choosing f_i .)

We solve the problem for $i = 1$, without loss of generality. Defining $V(f) = \sum_k |y_k(f)|$ as the vector of total volumes in the n contracts, the innovation problem of Equation (8) is equivalent to $\max_{f_1} V(f)_1 N(f_1)$. It is shown in the Appendix that

$$V(f)_1 = \frac{2}{\gamma} (\text{var}[\pi^\perp(f_1|f_{-1})])^{-1} \text{cov}(\pi^\perp(f_1|f_{-1}), \pi^\perp[d(f)_1|f_{-1}]) \quad (9)$$

Hence,

$$V(f)_1 N(f_1) = \frac{2}{\gamma} \text{cov}\left(\frac{\pi^\perp(f_1|f_{-1})}{\text{SD}[\pi^\perp(f_1|f_{-1})]}, \pi^\perp[d(f)_1|f_{-1}]\right)$$

It follows that a solution to Equation (8) must be of the form

$$f_1 = \delta \pi^\perp[d(f)_1|f_{-1}] + \beta^\top f_{-1} + c$$

for $\delta \in \mathbb{R}$, $\delta \neq 0$, $\beta \in \mathbb{R}^{n-1}$, and $c \in \mathbb{R}$, proving the claim. ■

The volume expression given by Equation (9) can be written in a “beta” form analogous to that shown in the monopolistic case.

Example 2 (choice of normalization). This example is designed to show that normalization by standard deviation is not appropriate in our setting. As we see in Equation (9), the volume of trade in a given contract depends only on the innovation of that contract. Hence, normalizing by $\text{SD}(f_i)$ distorts the incentives of the exchange, since this normalization introduces terms that are not in the volume expression.

Recall the economy described in Example 1. There are four agents with a common level of risk aversion, r , and endowments described by $x_1 = -x_2 = w$ and $x_3 = -x_4 = z$, where w and z are random variables such that $\text{var}(w) = \text{var}(z) = 1$ and $\text{cov}(w, z) = 0$.

Suppose that $f_2 = z$. We examine the choice of f_1 by exchange 1. Agents 3 and 4 have a perfect hedge available by trading contract f_2 , and so will only trade contract f_1 if $E(f_1) \neq p_1$. Given the symmetry of the agents’ endowments, however, it follows that $E(f_1) = p_1$ for any choice of f_1 . Agents

3 and 4 will therefore not trade contract f_i , and so only agents 1 and 2 will trade contract f_i . In this case, it is intuitively clear that a contract of the form $\delta w + c$ (with $\delta \neq 0$) should be a volume-maximizing choice for f_i .

Consider, however, what happens if we use $SD(f_i)$ as our normalization. Under this assumption, exchange 1 chooses f_i to maximize $V(f)_1SD(f_i)$. We show that this problem has no solution. Substituting for $V(f)_1$ from Equation (9), we find that

$$V(f)_1SD(f_i) = \frac{2}{\gamma} SD(f_i) \text{cov} \left(\frac{\pi^\perp(f_i|z)}{\text{var}[\pi^\perp(f_i|z)]}, \pi^\perp[d(f)_1|z] \right)$$

Consider a contract of the form $f_i = w + \lambda z$, with $d(f)_1 = (2/r)w$. For this contract choice

$$V(f)_1SD(f_i) = \frac{2}{\gamma} (1 + \lambda^2)^{1/2} \text{cov}[w, d(f)_1] = \frac{4}{\gamma r} (1 + \lambda^2)^{1/2}$$

which can be made arbitrarily large by setting λ to be arbitrarily large. There is no solution to the problem when we use the normalization $SD(f_i)$, since λz , although affecting the normalization, is irrelevant in determining the optimal position of any agent (and thus the volume of trade of contract f_i).

We now show that the normalization defined by $N(f_i) = SD[\pi^\perp(f_i|f_{-i})]$ generates the intuitive contract choice, $f_i = \delta w + c$, which we suggested before. In this case, exchange 1 chooses f_i to maximize $V(f)_1SD[\pi^\perp(f_i|z)]$. Substituting from Equation (9), we write this problem as

$$\max_{f_i \in L^2(P)} \frac{2}{\gamma} \text{cov} \left(\frac{\pi^\perp(f_i|z)}{SD[\pi^\perp(f_i|z)]}, \pi^\perp[d(f)_1|z] \right)$$

It follows that a contract with innovation $\pi^\perp[d(f)_1|z]$ and maximal $\text{var}(d(f)_1)$ is a solution to this problem. Hence, any contract of the form $\delta w + \beta z + c$, for constants $\delta \neq 0$, β , and c , is a solution. In particular a contract of the form we suggested, $\delta w + c$, is a solution.

Just as in the single-contract case (treated in Corollary 1 and Example 1), as a sufficient condition for f_i to be volume maximizing, the variance of $\pi^\perp[d(f)_i|f_{-i}]$ (which depends on f_i via the identity of long and short agents) must be maximal.

Corollary 2. *Adding to the statement of Proposition 3, f_i is volume maximizing given f_{-i} if and only if*

1. *The innovation of f_i is perfectly correlated with $\pi^\perp[d(f)_i|f_{-i}]$, the unspanned portion of the endowment differential.*
2. *The variance of $\pi^\perp[d(f)_i|f_{-i}]$ is maximal.*

Proof. This follows from the fact that condition 1 is necessary and sufficient

for optimality in Equation (8) given the sets $S(i)$ and $L(i)$ of long and short agents in contract f_i , and from the fact that contract volume for any contract f_i satisfying 1 is directly proportional to the variance of $\pi^\perp[d(f)_i, |f_{-i}]$, based on the calculations in the proof of Proposition 3. ■

3. Nash Equilibria in Multi-exchange Contract Design

We now consider the broader problem of contract design when a number n of exchanges simultaneously choose contracts to maximize their respective trading volumes. Without redundant contracts, it must be the case that the number of contracts is less than the number of agents with different endowments in the economy. One could also imagine a more complicated game of entry with setup costs and the resulting completely endogenous market structure. Because of the long contract approval time of the Commodity Futures Trading Commission (CFTC) and the costs of contract setup, we feel it is natural to consider only pure strategies.

With each exchange assumed to choose a single contract (for simplicity), the game is defined as follows. We use the notation of Section 2 and define the payoff to exchange i for strategies $f = (f_1, \dots, f_n) \in [L^2(P)]^n$ in this game, denoted G , to be the normalized volume

$$\sum_{k=1}^K |y_{ki}(f)| N(f_i)$$

The following proposition characterizes the set of Nash equilibria:

Proposition 4. *Under the assumptions of Proposition 3, the set of contracts forms a (pure strategy) Nash equilibrium of the game G if and only if, for all i , conditions 1. and 2. of Corollary 2 are satisfied.*

Proof. This follows directly from Proposition 3. ■

The corollary below follows from Proposition 4 and the fact that the volume associated with a given contract depends only on the innovation of the contract, as indicated in Equation (9).

Corollary 3. *Under the assumptions of Proposition 3, if a set f of contracts forms a (pure strategy) Nash equilibrium, then there is a corresponding set f' of contracts that forms a Nash equilibrium with the same volume of trade and the same agent utilities, with identity covariance matrix, that is, $cov(f') = I$.*

Given the advantages of being the first exchange to file a contract design with the CFTC, it is worthwhile noting that the set of contracts determined by picking the contracts in order of “best first” also forms a Nash equilibrium of a game of sequential design under the same assumptions.

Proposition 5. Under the conditions of Proposition 3, contracts with the sequential property $f_1 = d(f)_1, f_2 = \pi^+[d(f_1, f_2)_2 | f_1], f_3 = \pi^+[d(f_1, f_2, f_3)_3 | (f_1, f_2)]$, and so on, where each contract is volume maximizing given the previous contracts, constitute a Nash equilibrium.

Proof. Each contract is volume maximizing given the contracts previously existing since the contracts are chosen to satisfy conditions 1. and 2. of Corollary 2. The contracts also satisfy the conditions of Proposition 4, indicating that exchanges do not wish to change their contract design given contracts that will be introduced later. ■

A salient feature of the Nash equilibrium described in Proposition 5 is that each exchange can ignore the actions of exchanges that follow it in the innovation sequence as well as the number of contracts to follow. We illustrate the proposition with an example.

Example 3 (a sequential design Nash equilibrium). We reconsider the economy described in Examples 1 and 2. Recall that there are four agents with identical risk-aversion coefficients $r_k = r$ who have nonzero spot commitments $w, -w, z,$ and $-z,$ respectively. In addition, recall that $SD(w) = SD(z) = 1$ and $cov(w, z) = 0$. Two contracts are to be offered, one by each of two exchanges. It is easy to verify that $f_1 = w$ and $f_2 = z$ satisfy the conditions of Proposition 4 and form a Nash equilibrium. Proceeding along the lines of the sequential offering, $f_1 = d(f)_1 = w + z$ and $f_2 = d(f_1, f_2)_2 = w - z$ also form a Nash equilibrium. It is important to verify that each contract is volume maximizing given the previously existing contracts. For instance, we could consider $f_1 = w + z$ and $f_2 = 0$, although $f_2 = \pi^+[d(f_1, f_2)_2 | f_1]$, f_2 is not volume maximizing.

Under our assumptions, redundant contracts cannot constitute a Nash equilibrium. For a game that permits contract duplication, see Anderson and Harris (1986). In our setting, in order to study equilibrium with contract duplication, we would need a mechanism for allocating volume of trade to redundant contracts based, for example, on liquidity.

4. Pareto Optimality

As one might suspect, in the static model, utility is monotonic in the size of optimal trades, which provides the intuition for the following result. In the static (one-period) model we define the contract design f' to be Pareto-optimal if, for any transactions fee $T \geq 0$, there exists no contract f with $SD(f) = SD(f')$ that increases the equilibrium expected utility of any agent (relative to the expected utility associated with f') without reducing the expected utility of some other agent, even when the transactions revenue is returned to the agents by any scheme. (We have not specified how the profits of the exchange are redistributed.)

4.1 Monopolistic Pareto optimality

Proposition 6. *A volume-maximizing contract (in the sense of Corollary 1) is Pareto-optimal.*

Proof. We can write agent k 's expected utility as

$$U_k = E(x_k) - r_k \text{var}(x_k) + r_k y_k \left[-2 \text{cov}(x_k, f) + \frac{E(f) - p + \alpha_k T}{r_k} - y_k \text{var}(f) \right] \quad (10)$$

Substituting from Equation (2), we rewrite (10) as

$$U_k = E(x_k) - r_k \text{var}(x_k) + r_k [y_k(f)]^2 \text{var}(f) \quad (11)$$

where $y_k(f)$ is the optimal position for agent k described by Equation (2). Suppose that a volume-maximizing contract $f = \delta d(f) + c$ is not Pareto-optimal. Let $H \equiv \{f' \in L^2(P) : \text{var}(f') = \text{var}(f)\}$. (Recall that volume maximization is subject to an arbitrary but fixed variance for the contract choice.) From Equation (11) (and the homogeneity lemma) it then follows that there exists an $f' \in H$ such that $[y_k(f')]^2 \geq [y_k(f)]^2$ for all k with strict inequality for some k . Hence, $\sum_k T |y_k(f')| > \sum_k T |y_k(f)|$ [or likewise $\sum_k |y_k(f')| > \sum_k |y_k(f)|$], which contradicts the fact that f is volume maximizing, proving the result. ■

4.2 The oligopolistic case

Without transaction costs, we extend our definition of Pareto optimality to more than one contract. A set of n futures contracts $f = (f_1, \dots, f_n)$ is said to be Pareto-optimal if there exists no other set of n contracts that increases the equilibrium expected utility of any agent (relative to the equilibrium expected utility associated with f) without reducing the equilibrium expected utility of some other agent.

Given a set of nonredundant futures contracts $f = (f_1, \dots, f_n)$ and its associated equilibrium, (y_1, \dots, y_n, p) , agent k 's consumption is $x_k + y_k^\top (f - p)$. The same equilibrium consumption allocations can be achieved with any set of contracts $f' = \beta f$, where β is a nonsingular $n \times n$ matrix. [The equilibrium is $(y'_1, \dots, y'_n, p') = (\beta^{-1\top} y_1, \dots, \beta^{-1\top} y_n, \beta p)$ with $x_k + y_k^\top (f' - p') = x_k + y_k^\top (f - p)$.] Furthermore, it is easy to see that any Pareto-optimal set of contracts is nonredundant, provided, of course, that the number of linearly independent endowments is at least n . Thus, without essential loss of generality for efficiency considerations, we can represent any set of contracts by a set of contracts in F' , where $F' = \{f : \text{cov}(f) = I\}$ and where I denotes the identity matrix.

For $f \in F'$, we can express the expected utility of agent k as

$$U_k = E[x_k + y_k^\top (f - p)] - r_k \text{var}[x_k + y_k^\top (f - p)]$$

or

$$U_k = E(x_k) - r_k \text{var}(x_k) + r_k y_k^T \left[\frac{E(f - p)}{r_k} - \text{cov}(f) y_k - 2 \text{cov}(x_k, f) \right]$$

Substituting from Equation (7) the optimal position $y_k(f)$ of agent k given f , it follows that

$$U_k = E(x_k) - r_k \text{var}(x_k) + r_k [y_k(f)]^T y_k(f) \tag{12}$$

Although there are economies for which there exist Nash equilibria that are Pareto-optimal, we do not know if, for a specified number of contracts, there always exists a Pareto-optimal Nash equilibrium. We know, however, that there exist Nash equilibria that are inefficient, as the example below illustrates.

Example 4 (Nash equilibrium inefficiency). Let v , w , and z be nonzero, mutually orthogonal, random variables of unit variance. Consider an economy with four agents having identical risk-aversion coefficients $r_k = r$ for $k \in \{1, \dots, 4\}$ and endowments $x_1 = -x_3 = v + w$ and $x_2 = -x_4 = v + z$. We examine the case of $n = 2$ contracts.

Consider $f = (f_1, f_2)^T$ defined by

$$f_1 = \frac{v}{\text{SD}(v)} \quad \text{and} \quad f_2 = \frac{w + z}{\text{SD}(w + z)}$$

It is easily verified that f satisfies the conditions of Proposition 4 and is a Nash equilibrium of the two-contract design game. [By the pairwise symmetry of the agents, $d(f)_i = (\gamma/2) \sum_k \alpha_{ki} x_k$. Since $f \in F'$, in equilibrium $y_{ki}(f) = -\text{cov}(f_i, x_k)$, and so we can determine the α_{ki} 's and find that $d(f)_1 = d(f)_2 = (4/r)(2v + w + z)$. It follows that $f_1 = (r/8)\pi^+[d(f)_1|f_2]$ and $f_2 = (r/4\sqrt{2})\pi^+[d(f)_2|f_1]$. It is easily checked that condition 2 of Corollary 2 holds for each of the contracts.] Define a second set of contracts f' by

$$f'_1 = \frac{v + w}{\text{SD}(v + w)} \quad \text{and} \quad f'_2 = \frac{2z + v - w}{\text{SD}(2z + v - w)}$$

It follows that $f' \in F'$. Note also that f' is a Nash equilibrium of the two-contract design game. We now show that f' Pareto dominates f . The equilibrium futures positions are

$$y_1(f) = -y_3(f) = \left(\frac{-\text{var}(v)}{\text{SD}(v)}, \frac{-\text{var}(w)}{\text{SD}(w + z)} \right)$$

$$y_2(f) = -y_4(f) = \left(\frac{-\text{var}(v)}{\text{SD}(v)}, \frac{-\text{var}(z)}{\text{SD}(w + z)} \right)$$

$$y_1(f') = -y_3(f') = \left(\frac{-\text{var}(v + w)}{\text{SD}(v + w)}, 0 \right)$$

$$y_2(f') = -y_4(f') = \left(\frac{-\text{var}(v)}{\text{SD}(v + w)}, \frac{-2 \text{ var}(z) - \text{var}(v)}{\text{SD}(2z + v - w)} \right)$$

For any set $z = (z_1, z_2)$ of contracts, let $Y_k(z) \equiv [y_{k1}(z)]^2 + [y_{k2}(z)]^2$. Since $\text{cov}(f)$ and $\text{cov}(f')$ are both the identity matrix, it follows from Equation (12) that if $Y_k(f') > Y_k(f)$, then $U_k(f') > U_k(f)$. We have

$$Y_1(f) = Y_3(f) = Y_2(f) = Y_4(f) = \frac{3}{2}$$

$$Y_1(f') = Y_3(f') = Y_2(f') = Y_4(f') = 2$$

It follows that f' Pareto dominates f .

In this example, one set f of Nash equilibrium contracts are Pareto-dominated by a second set of Nash equilibrium contracts f' . In fact, since the span of f' is the same as the span of the endowments, the contracts f' are Pareto-optimal. Notice, however, that the exchanges have differing views over which set of contracts is “better” in terms of maximizing transaction volume. Exchange 1 would like to see the equilibrium f' , since f'_1 generates more volume than f_1 , while exchange 2 would rather see the equilibrium f since f_2 generates more volume than f'_2 . In general, if we examine Nash equilibria when there is a sufficient number of contracts offered to span the endowment set, we can find a Pareto-optimal Nash equilibrium. However, this case is of little interest and, as mentioned earlier, outside of this case we do not know if there always exists a Pareto-optimal set of Nash equilibrium contracts.

4.3 Multiperiod inefficiency example

The following example shows that the static monopolistic efficiency result does not carry over to a multiperiod setting. An exchange has an incentive to offer a contract causing agents to change their positions in intermediate periods, increasing volume and transactions revenue but typically decreasing agents’ utilities. The model consists of a two-period world in which agents have a spot commitment at the end of the second period. Trading in futures is permitted in both periods. We begin by defining trade and a notion of Pareto domination. We specialize to a situation in which there are two agents with equal and opposite spot commitments, providing an example in which an exchange, when faced with two possible contracts to offer for trade, chooses one Pareto-dominated by the other.

At time $t = 0$, futures trading is initiated at a futures price f_0 . Agent k undertakes a position y_k^1 held through the first period. At time $t = 1$, some information is revealed and a futures price f^1 results. Agents trade again, with agent k choosing a futures position y_k^2 . The final state of the world is realized at time $t = 2$, determining the clearing futures price f^2 , which is the contract specified by the exchange. The wealth of agent k at time $t = 2$ is represented by

$$W_k = x_k + y_k^1(f^1 - f^0) + y_k^2(f^2 - f^1) - |y_k^1|T - |y_k^2 - y_k^1|T \quad (13)$$

where $T \geq 0$ is the transaction cost per contract traded.

We now define our notion of Pareto domination. Let $W_k[f]$ denote the equilibrium terminal wealth of agent k associated with contract f . Consider two contracts f and f' , constrained to have the same variance. A contract f' is said to Pareto dominate a contract f for transactions cost $T \geq 0$ if, for all k , $U_k^0(W_k[f']) \geq U_k^0(W_k[f])$ and if

$$\begin{aligned} &U_k^0(W_k[f']) + T|y_k^1(f')| + T|y_k^2(f') - y_k^1(f')| \\ &\geq U_k^0(W_k[f]) + T|y_k^1(f)| + T|y_k^2(f) - y_k^1(f)| \end{aligned}$$

with strict inequality for some k , where U_k^0 indicates that conditional expectations are taken at time 0 and $y_k^i(f)$ indicates the dependence of y_k^i on f . Our notion is that a contract is suboptimal if it distorts agents' actions so that they choose inferior hedging positions. A simple comparison of expected utility levels does not reveal whether this occurs, since it may be the case that one contract simply costs agents more in transaction payments without distorting their hedging. If, however, we find a lower expected utility, even after returning the transaction payments to the agents, then the contract causes agents to take suboptimal hedging positions.

Example 5 (multiperiod monopolistic inefficiency). We specialize to two agents with a common risk-aversion coefficient $r > 0$. Agent 1 has endowment $x_1 = x$ at time $t = 2$, while agent 2 has endowment $x_2 = -x$ at time $t = 2$. We consider a world with the states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ in the second period, each occurring with probability $\frac{1}{4}$. After the first period, it is known whether the state is in $\{\omega_1, \omega_2\}$ or in $\{\omega_3, \omega_4\}$, as shown in Figure 1.

Two different futures contracts are considered, both of which offer perfect hedges in a world with no transaction costs. The first claims x at time 2, the endowment of agent 1, and the second claims z , where x and z are the random variables shown in Figure 1; that is, $x(\omega_4) = 2, x(\omega_3) = 0, x(\omega_2) = 0, x(\omega_1) = -2, z(\omega_4) = 0, z(\omega_3) = 2, z(\omega_2) = -2, z(\omega_1) = 0$. Contract x allows agents to hold the same position throughout both periods, while contract z requires that they reverse their positions after the first period. The second contract, z , is preferred by the exchange since it generates greater trading volume and transaction revenue than the first, even though the first contract offers greater expected utility to all agents, for any transaction fee $T > 0$. To assure that the contract positions are not degenerate, we assume that $T < r$.

Claim. *If $0 < T < r$, then contract z generates more transaction revenue than contract x , while x Pareto dominates z .*

The claim is proved by calculating and comparing the agents' optimal positions and utility associated with each of the two contracts. The proof appears in the Appendix.

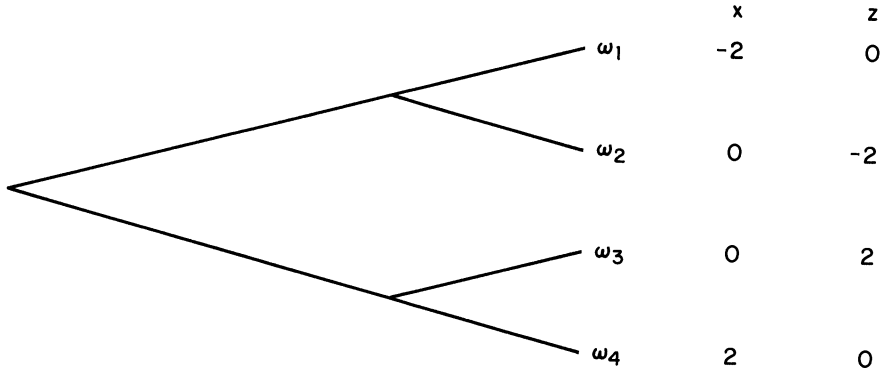


Figure 1
Two-period event tree

5. Summary

We have examined the choice of futures contracts by exchanges seeking to maximize their transaction volumes. We found that a contract that maximizes volume of trade for a monopolistic exchange is one that is perfectly correlated with the endowment differential, the difference between the endowments on the long and the short sides of the market, each weighted by the risk tolerance of the other side of the market. Correspondingly, if more than one contract is offered, a volume-maximal contract is one whose innovation (the portion of the contract unspanned by the other contracts) is perfectly correlated with the unspanned portion of the endowment differential. Using this result, we characterize the Nash equilibria of a multiple exchange contract design game. Finally, we examined the efficiency properties of the optimal contracts. The monopolistic volume-maximal contract is Pareto-optimal in our static setting. We showed, however, that this result does not carry over to a dynamic setting. We also presented a set of contracts that forms a Nash equilibrium of the multiple contract design game but is not Pareto-optimal. We do not know if, for general contract design games, there always exists a set of Nash equilibrium contracts that is Pareto-optimal.

Appendix

The Appendix contains the derivation of Equation (9) and the proof of the claim in Example 5.

Derivation of Equation (9)

Substituting the expression for p from Proposition 2 into Equation (7), we find that the optimal position of agent k is described by

$$y_k(f) = [\text{cov}(f)]^{-1} \left[-\text{cov}(f, x_k) + \frac{\gamma^{-1}}{r_k} \text{cov} \left(f, \sum_j x_j \right) \right]$$

It follows that

$$V(f)_1 = \left\{ [\text{cov}(f)]^{-1} \text{cov} \left[f, \frac{2}{\gamma} d(f)_1 \right] \right\}_1$$

noting that $(2/\gamma)d(f)_1 = \sum_k x_k [\alpha_{k1} - (\Gamma_1/\gamma)]$. We can write $\text{cov}(f)$ in the partitioned form

$$\begin{bmatrix} \text{var}(f_1) & \text{cov}(f_1, f_{-1}) \\ \text{cov}(f_{-1}, f_1) & \text{cov}(f_{-1}) \end{bmatrix} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

It follows that $[\text{cov}(f)]^{-1}$ is of the similarly partitioned form

$$\begin{bmatrix} J & K \\ L & M \end{bmatrix}$$

where J is the scalar $(A - BD^{-1}B^T)^{-1}$ and $K = -BD^{-1}J$. We then have

$$\begin{aligned} V(f)_1 &= \frac{2}{\gamma} J \text{cov}[f_1, d(f)_1] + \frac{2}{\gamma} K \text{cov}[f_{-1}, d(f)_1] \\ &= \frac{2}{\gamma} J \text{cov} \left(f_1, d(f)_1 - f_{-1}^T \text{cov}(f_{-1})^{-1} \text{cov}[f_{-1}, d(f)_1] \right) \\ &= \frac{2}{\gamma} [\text{var}(f_1 - \pi(f_1|f_{-1}))]^{-1} \text{cov} \left(f_1 - \pi(f_1|f_{-1}), d(f)_1 - \pi[d(f)_1|f_{-1}] \right) \\ &= \frac{2}{\gamma} (\text{var}[\pi^\perp(f_1|f_{-1})])^{-1} \text{cov}(\pi^\perp(f_1|f_{-1}), \pi^\perp[d(f)_1|f_{-1}]) \end{aligned}$$

which is the desired expression.

Proof of the claim in Example 5

Given the symmetry of the agents and the structure of the two contracts in question, we restrict our attention to agent 1 without loss of generality. (To simplify the notation we drop the subscript referring to agent 1.) We begin by rewriting Equation (13) as

$$W = x + y^1(f^2 - f^0) + \Delta y(f^2 - f^1) - |y^1|T - |\Delta y|T \quad (A1)$$

where $\Delta y = (y^2 - y^1)$. At time $t = 1$, the agent's problem is

$$\max_{\Delta y \in \mathbb{R}} U^1(W)$$

where U^t indicates that conditional expectations are taken at time t . The solution to this problem is

$$\Delta y = \frac{|\alpha^2|}{\text{var}^1(f^2)} \left[-\text{cov}^1(f^2, x + y^1 f^2) + \frac{E^1(f^2) - f^1 + \alpha^2 T}{2r} \right] \quad (\text{A2})$$

where α^2 is the sign of $-\Delta y$. [The sign convention is analogous to that discussed following Equation (2).] The symmetry of the agents implies that $f^1 = E^1(f^2)$, and direct calculation reveals that $\text{var}^1(f^2) = 1$ for either $f^2 = x$ or $f^2 = z$. Expression (A2) thus simplifies to

$$\Delta y = |\alpha^2| \left[-\text{cov}^1(f^2, x) - y^1 + \frac{\alpha^2 T}{2r} \right] \quad (\text{A3})$$

Note that $\text{cov}^1(f^2, x)$ is known at time 0 for both contracts $f^2 = z$ and $f^2 = x$, and so Δy is known at time 0 (given a value of y^1). We can therefore substitute (A3) into (A1) and solve for the optimal choice of y^1 at time 0. By the symmetry of the agents, $E^0(f^2) = E^0(f^1) = f^0 = 0$ (for $f^2 = x$ or $f^2 = z$), so the solution is

$$y^1 = \frac{|\alpha^1|}{\text{var}^0[f^2 - |\alpha^2|(f^2 - f^1)]} \cdot \left(-\text{cov}^0 \left(f^2 - |\alpha^2|(f^2 - f^1), x + |\alpha^2| \left[-\text{cov}^1(f^2, x) + \frac{\alpha^2 T}{2r} \right] (f^2 - f^1) \right) + \frac{(\alpha^1 - \alpha^2) T}{2r} \right)$$

To account for the effect of y^1 on α^2 , one can verify that the directional derivatives of utility are nonpositive in each direction. Noting that, for both contracts, $\text{var}^0(f^1) = 1$, $\text{var}^0(f^2) = 2$, and $\text{cov}^0(f^1, f^2) = 1$, this simplifies to

$$y^1 = \frac{|\alpha^1|}{2 - |\alpha^2|} \left(-\text{cov}^0[f^2 - |\alpha^2|(f^2 - f^1), x] + \frac{T(\alpha^1 - \alpha^2)}{2r} \right) \quad (\text{A4})$$

The (unique) solutions to Equations (A3) and (A4) are $y^1 = -1 + (T/4r)$ and $\Delta y = 0$ if $f^2 = x$; and $y^1 = -1 + (T/r)$ and $\Delta y = 2 - (3T/2r)$ if $f^2 = z$. The exchange then prefers to offer $f^2 = z$, since the total trading volume, $2|y^1| + 2|\Delta y|$, is greater for $f^2 = z$ than for $f^2 = x$. However, the expected utilities of the agents are greater for $f^2 = x$, as we see below. For $f^2 = x$

$$U^0(W) = -T - \frac{T^2}{8r}$$

while for $f^2 = z$

$$U^0(W) = -3T - \frac{9T^2}{4r}$$

If we return the transaction revenue to the agents, the contract $f^2 = x$

provides expected utility $-3T^2/8r$, which is still greater than the expected utility, $-19T^2/4r$ provided by $f^2 = z$. ■

References

- Allen F., and D. Gale, 1988, "Optimal Security Design," *Review of Financial Studies*, 1(3), 229–263.
- Anderson, R., 1984, "The Regulation of Futures Contract Innovations in the United States," *Journal of Futures Markets*, 4, 297–332.
- Anderson, R., 1986, "Exchange Competition in the Futures Industry," mimeo, Department of Economics, City University of New York.
- Anderson, R., and C. J. Harris, 1986, "A Model of Financial Innovation," Oxford Economic Papers.
- Black, D., 1986, "Success and Failure of Futures Contracts: Theory and Evidence," NYU Business School Monograph Series in Finance and Economics.
- Carlton, D., 1983a, "Futures Volume, Market Interrelationships and Industry Structure," *American Journal of Agricultural Economics*, 65, 380–387.
- Carlton, D., 1983b, "The Cost of Eliminating a Futures Market and the Effect of Inflation on Market Interrelationships," Working Paper 69, University of Chicago.
- Duffie, D., and M. Jackson, 1986, "Optimal Hedging and Equilibrium in a Dynamic Futures Market," Research Paper 814, Stanford Graduate School of Business; forthcoming, *Journal of Economic Dynamics and Control*.
- Johnston, E., and J. McConnell, 1987, "Requiem for a Market: An Analysis of the Rise and Fall of a Financial Futures Contract," *Review of Financial Studies*, 2, 1–23.
- Miller, M., 1986, "Financial Innovation: The Last Twenty Years and the Next," *Journal of Financial and Quantitative Analysis*, 21, 459–471.
- Saloner, G., 1984, "Self Regulating Commodity Futures Exchanges," in R. Anderson, *Industrial Organization of Futures Markets*, Lexington Books, Cambridge, Mass.
- Sandor, R., 1973, "Innovation by an Exchange: A Case Study of the Development of the Plywood Futures Contract," *Journal of Law and Economics*, 16, 119–136.
- Silber, W. L., 1981, "Innovation, Competition, and New Contract Design in Futures Markets," *Journal of Futures Markets*, 1, 123–155.
- Silber, W. L., 1983, "The Process of Financial Innovation," *AEA Papers and Proceedings*, 73, 89–95.
- Telser, L., 1981, "Why There Are Organized Futures Markets," *Journal of Law and Economics*, 24, 1–22.
- Townsend, R. M., 1984, "Theories of Contract Design and Market Organization: Conceptual Bases for Understanding Futures Markets," in R. Anderson, *Industrial Organization of Futures Markets*, Lexington Books, Cambridge, Mass.
- Van Horne, J., 1985, "Of Financial Innovations and Excesses," *Journal of Finance*, 40, 621–631.