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IMPLEMENTING ARROW-DEBREU EQUILIBRIA BY CONTINUOUS TRADING OF FEW LONG-LIVED SECURITIES

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A two-period (0 and T) Arrow-Debreu economy is set up with a general model of uncertainty. We suppose that an equilibrium exists for this economy. The Arrow-Debreu economy is placed in a Radner (dynamic) setting; agents may trade claims at any time during $[0, T]$. Under appropriate conditions it is possible to implement the original Arrow-Debreu equilibrium, which may have an infinite-dimensional commodity space, in a Radner equilibrium with only a finite number of securities. This is done by opening the "right" set of security markets, a set which effectively completes markets for the Radner economy.

1. INTRODUCTION

FIGURE 1 DEPICTS A SIMPLE event tree information structure. Let's momentarily consider an exchange economy with endowments of and preferences for random time T consumption, depending on the state $\omega \in \Omega$ chosen by nature from the final five nodes of this event tree. A competitive equilibrium will exist under standard assumptions (Debreu [**5**, Chapter **7**]), including markets for securities whose time T consumption payoff vectors span R^5. This entails at least five security markets, while intuition suggests that, with the ability to learn information and trade during $[0, T]$, only three securities which are always available for trading, or *long-lived securities* [**13**], might be enough to effectively complete markets. This is the maximum number of branches leaving any node in the tree. The reasoning is given by Kreps [**13**] and in alternative more general form later in this paper. An early precurser to this work is Arrow [**1**], which showed the spanning effectiveness of financial securities when trade can occur twice in a two-period model.

One major purpose of this paper is to verify this intuition for a very general class of information structures, including those which cannot be represented by event trees, such as the filtration generated by continuous-time "state-variable" stochastic processes. In some cases, where an Arrow-Debreu style equilibrium would call for an infinite number of securities, we show how a continuous trading Radner [**20**] *equilibrium of plans, prices and price expectations* can implement the same Arrow-Debreu consumption allocations with only a finite number of long-lived securities. It is misleading, of course, to use the number of security markets alone as a measure of the efficiency of the market structure; the number of transactions which must be performed to achieve a given allocation must also be considered. Largely for want of a reasonable model to study this tradeoff, we have not addressed the issue of the efficiency of market structure.

A comparison of Event Trees A of Figure 1 and B of Figure 2, which are intended to correspond to the same two-period Arrow-Debreu economy, obviates the role of the information structure in determining the number of long-lived

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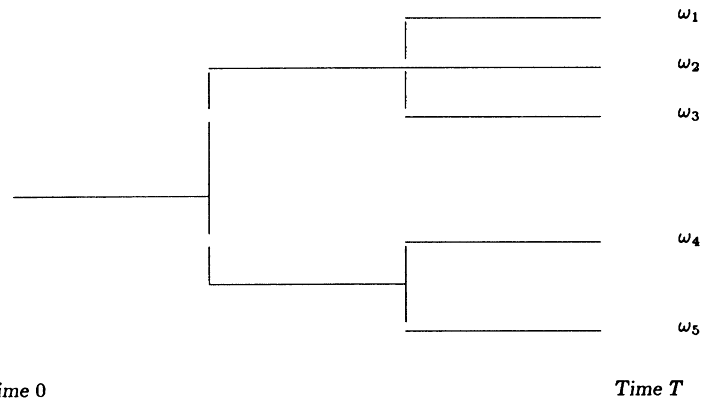
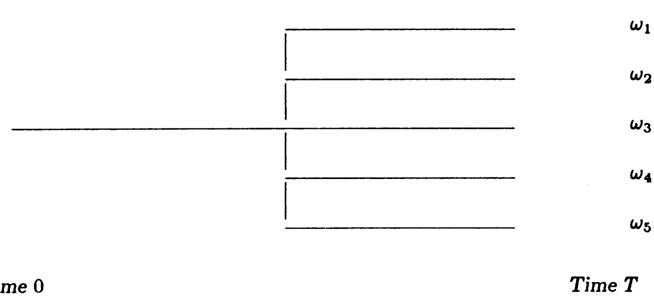


$$\Omega = \{\omega_1, \ldots, \omega_5\}$$

FIGURE 1.

securities required to dynamically "span" the consumption space, or the *spanning number*. We later give this term a more precise meaning. Since all uncertainty is resolved at once in Event Tree B, the spanning number is five, instead of three for Event Tree A. Intuitively speaking, the maximum number of "dimensions of uncertainty" which could be resolved at any one time is the key determining property. This vague notion actually takes a precise form as the *martingale multiplicity* of the information structure, defined in the Appendix. A key result of this paper is that the spanning number is the martingale multiplicity plus one. The "plus one" is no mystery; in addition to spanning uncertainty, agents must have the ability to transfer purchasing power across time.

The notion that certain securities are redundant because their payoffs can be replicated by trading other securities over time, yielding arbitrage pricing relationships among securities, was dramatized in the Black-Scholes [2] option pricing



$$\Omega = \{\omega_1, \ldots, \omega_5\}$$

FIGURE 2.

formula.[2] Provided the equilibrium price process for one security happens to be a geometric Brownian Motion, and for another is a (deterministic) exponential of time, then any contingent claim whose payoff depends (measurably) on the path taken by the underlying Brownian Motion, such as a call option on the risky security, is redundant and priced by arbitrage. This discovery curiously preceded its simpler logical antecedents, such as the corresponding results for event tree information structures. Only in the past few years have the implications of the spanning properties of price processes (e.g., [**13**]), the connection between martingale theory and equilibrium price processes (e.g., [**8**]), and the mathematical machinery for continuous security trading [**9**] been formalized.

In all of the above mentioned literature the takeoff point is a given set of security price processes, implicitly imbedded in a Radner equilibrium. Our second major goal is to begin more primitively with a given Arrow-Debreu equilibrium, one in which trading over time is not of concern since markets are complete at time zero. From that point we construct the consumption payoffs and price processes for a set of long-lived securities in such a way that agents may be allocated trading strategies allowing them to consume their original Arrow-Debreu allocations within a Radner style equilibrium. In short, we implement a given Arrow-Debreu equilibrium by continuous trading of a set of long-lived securities which is typically much smaller in number than the dimension of the consumption space. Merton [**19**; p. 666] recently predicted that results such as ours would appear.

The paper unfolds in the following order. First we describe the economy (Section 2) and an Arrow-Debreu equilibrium for it (Section 3). Section 4 provides a constructive proof of a Radner equilibrium which implements a given Arrow-Debreu equilibrium under stated conditions, based on a martingale representation technique. Section 5 characterizes the spanning number in terms of martingale multiplicity. Section 6 discusses the continuous trading machinery, some generalizations, and two examples of the model. Section 7 adds concluding remarks.

2. THE ECONOMY

Uncertainty in our economy is modeled as a complete probability space $(\Omega, \mathcal{F}, P)$. The set Ω constitutes all possible states of the world which could exist at a terminal date $T>0$. The *tribe* $\mathcal{F}$ is the σ-algebra of measurable subsets of Ω, or events, of which agents can make probability assessments based on the probability measure P. Events are revealed over time according to a *filtration*, $\boldsymbol{F}=\{\mathcal{F}_t, t\in[0, T]\}$, a right-continuous increasing family of sub-tribes of $\mathcal{F}$, where $\mathcal{F}_T=\mathcal{F}$ and $\mathcal{F}_0$ is almost trivial (the tribe generated by Ω and all of the P-null sets). We can interpret this by thinking of $\mathcal{F}_t$ as the set of all events which could occur at or before time t. The assumption that $\boldsymbol{F}$ is increasing, or $\mathcal{F}_t\subset\mathcal{F}_s$ for $s>t$, means simply that agents do not forget that an event has occurred once it

[2] Merton [**18**] is also seminal in this regard. Similar results were obtained by Cox and Ross [**3**] for other models of uncertainty.

is revealed. The above descriptions of $\mathcal{F}_0$ and $\mathcal{F}_T$ means that no information is known at time 0, and all uncertainty is resolved at time T.

Each agent in the economy is characterized by the following properties: (i) a known endowment of a perishable consumption good at time zero, (ii) a random, that is, state-dependent, endowment of the consumption good at time T, and (iii) preferences over consumption pairs (r, x), where r is time zero consumption and x is a random variable describing time T consumption, $x(\omega)$ in state $\omega \in \Omega$.

We will only consider consumption claims with finite variance. The consumption space is thus formalized as $V = R \times L^2(P)$, where $L^2(P)$ is the space of (equivalence classes) of square-integrable random-variables on $(\Omega, \mathcal{F}, P)$, with the usual product topology on V given by the Euclidean and L^2 norms.

The agents, finite in number, are indexed by $i = 1, \ldots, I$. The preferences of agent i are modeled as a complete transitive binary relation, or *preference order*, $\succcurlyeq_i$ on $V_i \subset V$, the ith agent's consumption set.

The whole economy can then be summarized in the usual way by the collection

$$\mathscr{E} = (V_i, \hat{v}_i, \succcurlyeq_i; i = 1, \ldots, I),$$

where $\hat{v}_i = (\hat{r}_i, \hat{x}_i) \in V_i$ is the ith agent's endowment. It is not important for this paper whether or not one assumes positive consumption or endowments.

3. ARROW-DEBREU EQUILIBRIUM

An Arrow-Debreu equilibrium for $\mathscr{E}$ is a nonzero linear (price) functional $\Psi: V \to R$ and a set of allocations $(v_i^* \in V_i; i = 1, \ldots, I)$ satisfying, for all i,

$$\begin{aligned} &\Psi(v_i^*) \leqslant \Psi(\hat{v}_i), \\ (3.1) \qquad &v \succ_i v_i^* \quad \Rightarrow \quad \Psi(v) > \Psi(v_i^*) \quad \forall v \in V_i, \\ &\sum_{i=1}^{I} v_i^* = \sum_{i=i}^{I} \hat{v}_i. \end{aligned}$$

We will assume that at least one agent i has strictly monotonic preferences. Specifically, if $v \in V_i$ and $v' \geqslant v$ (in the obvious product order on V), then $v' \in V_i$ and $v' \succ_i v$ provided $v \neq v'$. This ensures that in equilibrium Ψ is a strictly positive linear functional. Since V is a Hilbert lattice [**21**], this then implies that Ψ is a continuous linear functional on V, which can therefore be represented by some element (a, ξ) of V itself in the form:

$$\Psi(r, x) = ar + \int_\Omega x(\omega)\xi(\omega)\, dP(\omega) \quad \forall (r, x) \in V.$$

Without loss of generality we can normalize Ψ by a constant so that the positive random variable ξ has unit expectation, in order to construct the probability measure Q on $(\Omega, \mathcal{F})$ by the relation

$$Q(B) = \int_B 1_B(\omega)\xi(\omega)\, dP(\omega) \quad \forall B \in \mathcal{F}.$$

Equivalently, ξ is the Radon–Nikodym derivative dQ/dP. This leaves the simple representation

$$\Psi(r,x)=ar+E^*(x) \quad \forall (r,x)\in V, \tag{3.2}$$

where E^* denotes expectation under Q. Thus the equilibrium price of any random consumption claim $x\in L^2(P)$ is simply its expected consumption payoff under Q. For this reason we call Q an *equilibrium price measure.*

For tractability we will want any random variable which has finite variance under P to have finite variance under Q, and *vice versa.* A necessary and sufficient condition is that P and Q are *uniformly absolutely continuous,* denoted $Q\approx P$ (Halmos [7, p. 100]), or equivalently, that the Radon–Nikodym derivative dQ/dP is bounded above and below away from zero. Sufficient conditions for this can be given when preferences can be represented by von-Neumann–Morgenstern utility functions, in terms of bounds on marginal utility for time T consumption. We do not pursue this here since we are taking ξ as a primitive, rather than deriving it from preferences.[3]

A second regularity condition which comes into play is the separability[4] of $\mathcal{F}$ under P. This assumption should not be viewed as too restrictive. One can, for example, construct Brownian Motion on a separable probability space. Given $Q\approx P$ it is then easy to show the separability of $\mathcal{F}$ under Q by making use of the upper essential bound on dQ/dP.

Since uniform absolute continuity of two measures implies their equivalence (that is, they give probability zero to the same events), we can use the symbol a.s. for "almost surely" indiscriminately in this paper.

4. RADNER EQUILIBRIUM

A *long-lived security* is a consumption claim (to some element of $L^2(P)$) available for trade throughout $[0,T]$. A *price process* for a long-lived security is a semimartingale[5] on our given probability space adapted to the given information structure $\mathbf{F}$. In general the number of units of a long-lived security which are held by an agent over time defines some stochastic process θ. We will say θ is an *admissible* trading process for a long-lived security with price process S if it meets the following regularity conditions:

(i) *predictability,* defined in the Appendix and denoted $\theta\in\mathcal{P}$;

(ii) *square-integrability,* or $\theta\in L^2_P[S]\equiv\{\phi\in\mathcal{P}: E(\int_0^T \phi_t^2\, d[S]_t)<\infty\}$, where $[S]$ denotes the quadratic variation process for S (Jacod [**11**]); and

[3] Work subsequent to this paper shows extremely general continuity assumptions which yield these bounds [**6**].

[4] A tribe $\mathcal{F}$ is said to be separable under P if there exists a countable number of elements $B_1, B_2, \ldots$ in $\mathcal{F}$ such that, for any $B\in\mathcal{F}$ and $\varepsilon>0$ there exists B_n in the sequence with $P\{B\Delta B_n\}<\varepsilon$, where Δ denotes symmetric difference.

[5] See Jacod [**11**], for example, for the definition of a semimartingale. This is not at all a severe restriction on price processes if one is to obtain a meaningful model of gains and losses from security trades.

(iii) the *gains process* $\int \theta\, dS$ is well defined as a stochastic integral. We will be dealing with price processes in this paper for which square-integrability (ii) is sufficient for this condition. Memin [**16**] gives a full set of sufficient conditions in the general case.

The stochastic integral $\int_0^t \theta(s)\, dS(s)$ is a model of the gains or losses realized up to and including time t by trading a security with price process S using the trading process θ. Interpreted as a Stieltjes integral this model is obvious, but the integral is generally well defined only as a stochastic integral. This model, formalized by Harrison and Pliska [**9**], is discussed further in Section 6, as are the other regularity conditions on θ.

Taking $S=(S_1,\ldots,S_N)$, $N\leqslant\infty$, as the set of all long-lived security price processes, any corresponding set of admissible trading processes $\theta=(\theta_1,\ldots,\theta_N)$ must also meet the accounting identity:

$$\theta(t)^\top S(t)=\theta(0)^\top S(0)+\int_0^t \theta(s)^\top\, dS(s)\quad \forall t\in[0,T]\qquad \text{a.s.,} \tag{4.1}$$

meaning that the current value of a portfolio must be its initial value plus any gains or losses incurred from trading. The symbol $^\top$ denotes the obvious shorthand notation for summation from 1 to N. We'll adopt the notation $\Theta(S)$ for the space of trading strategies $\theta=(\theta_1,\ldots,\theta_N)$ meeting the regularity conditions (i)-(ii)-(iii) for each long-lived security and satisfying the "self-financing" restriction (4.1).

A *Radner equilibrium* for $\mathscr{E}$ is comprised of:

(1) a set of long-lived securities claiming $d=(d_1,\ldots,d_N)$, $d_n\in L^2(P)$, $1\leqslant n\leqslant N\leqslant\infty$, with corresponding price processes $S=(S_1,\ldots,S_N)$;

(2) a set of trading strategies $\theta^i\in\Theta(S)$, one for each agent $i=1,\ldots,I$; and

(3) a price $a\in R_+$ for time zero consumption;

all of these satisfying:

(4) budget constrained optimality: for each agent i,

$$\left(\hat r_i-\frac{\theta^i(0)^\top S(0)}{a},\ \hat x_i+\theta^i(T)^\top d\right)$$

is $\succcurlyeq_i$-maximal in the budget set:

$$\left\{\left(\hat r_i-\frac{\theta(0)^\top S(0)}{a},\ \hat x_i+\theta(T)^\top d\right)\in V_i\colon\ \theta\in\Theta(S)\right\},$$

and

(5) market clearing:

$$\sum_{i=1}^{I}\theta^i(t)=0\quad \forall t\in[0,T].$$

The space of square-integrable martingales under Q, denoted $\mathscr{M}_Q^2$; its multiplicity, denoted $M(\mathscr{M}_Q^2)$; and a corresponding orthogonal 2-basis of martingales, $m=(m_1,\ldots,m_N)$, where $N=M(\mathscr{M}_Q^2)\leqslant\infty$; are all defined in the Appendix. The central concept is that any of the martingales associated with the given information structure can be represented as the sum of $M(\mathcal{M}_Q^2)$ stochastic integrals against the fixed 2-basis of martingales m, in the manner given by the following theorem. This result is a direct consequence of the definition of martingale multiplicity. The content of the result lies with specific examples in which martingale multiplicity is characterized. Details, beyond those in the Appendix, may be found in the fourth chapter of Jacod [**11**]. Kunita and Watanabe [**14**] as well as Davis and Varaiya [**4**] also include the essentials.

THEOREM 4.1: *For any $X \in \mathcal{M}_Q^2$ there exists $\theta = (\theta_1, \ldots, \theta_N)$, where $\theta_n \in L_Q^2[m_n]$ for all n, such that*

$$X_t = \int_0^t \theta(s)^\top \, dm(s) \quad \forall t \in [0, T] \quad \text{a.s.}$$

We should remark that when $Q \approx P$, the spaces $L_Q^2[m_n]$ and $L_P^2[m_n]$ are identical because of the bounds implied on dQ/dP. We also use the fact that a martingale under Q is a semimartingale under P given the equivalence of P and Q. Thus any element of $\mathcal{M}_Q^2$ is a valid price process. This can be checked in Jacod [**11**, Chapter 7], along with the existence of $\int \theta_n \, dm_n$ as a stochastic integral under P whenever $\theta_n \in L_P^2[m_n]$.

Now we have the main result.

THEOREM 4.2: *Suppose $(\Psi, v_i^*, i = 1, \ldots, I)$ is an Arrow-Debreu equilibrium for $\mathscr{E}$, where without loss of generality Ψ has the representation (a, Q) given by relation (3.2). Provided $Q \approx P$ and $\mathscr{F}$ is separable under P, there is a Radner equilibrium for $\mathscr{E}$ achieving the Arrow-Debreu equilibrium allocations.*

PROOF: The proof takes four steps:

(1) Specify a set of long-lived securities.

(2) Announce a price for time zero consumption and price processes for the long-lived securities.

(3) Allocate a trading strategy to each agent which generates that agent's Arrow-Debreu allocation and which, collectively, clears markets.

(4) Prove that no agent has any incentive to deviate from the allocated trading strategy.

STEP 1: Select the following elements of $L^2(P)$ as the claims of the available long-lived securities:

$$d_0 = 1_\Omega,$$

$$d_n = m_n(T), \quad 1 \leq n \leq N = M(\mathcal{M}_Q^2),$$

where 1_Ω is the random variable whose value is identically 1 (the indicator function on Ω), and $m = (m_1, \ldots, m_N)$ is an orthogonal 2-basis for $\mathcal{M}_Q^2$. Since $Q \approx P$, the final values of the martingales, $m_n(T)$, are elements of $L^2(P)$.

STEP 2: For $0 \leq n \leq N$ let $S_n(t)$, the price of d_n at time t, be announced as $E^*[d_n | \mathscr{F}_t]$. In other words, each long-lived security's current price is the conditional expectation under Q of its consumption value. For convenience we actually take RCLL[6] versions of these price processes. There is obviously some forethought here, for the result is $S_0 \equiv 1$ and $S_n = m_n$, $1 \leq n \leq N$, implying that the last N price processes are themselves an orthogonal 2-basis for $\mathscr{M}_Q^2$, suggesting their ability to "span" (in the sense of Theorem 4.1) all consumption claims not actually available for trading. The first security serves as a "store-of-value", since its price is constant. We also announce the positive scalar a given in the statement of the theorem as the price of time zero consumption.

STEP 3: For any agent i, for $1 \leq i \leq I-1$, let $e_i = x_i^* - \hat{x}_i$. Then the process

$$X_i(t) = E^*(e_i | \mathscr{F}_t) - E^*(e_i), \quad t \in [0, T], \tag{4.2}$$

is an element of $\mathscr{M}_Q^2$, given $Q \approx P$, which can be reconstructed via Theorem 4.1 as

$$X_i(t) = \sum_{n=1}^{N} \int_0^t \theta_n^i(s)\, dS_n(s), \quad \forall t \in [0, T] \quad \text{a.s.}, \tag{4.3}$$

for some $\theta_n^i \in L_P^2[S_n]$, $1 \leq n \leq N$.

In order to meet the accounting restriction (4.1), we set the following trading process for the "store-of-value" security:

$$\theta_0^i(t) = E^*(e_i) + \sum_{n=1}^{N} \int_0^t \theta_n^i(s)\, dS_n(s) - \theta_n^i(t) S_n(t), \quad t \in [0, T]. \tag{4.4}$$

Of course $\int \theta_0^i \, dS_0 \equiv 0$ since $S_0 \equiv 1$. A technical argument showing $\theta_0^i \in \mathscr{P}$ is given as Appendix Lemma A.1, which then implies $\theta_0^i \in L_P^2[S_0]$.

Substituting (4.4) into (4.3), noting that $m_n(0) = 0 \; \forall n$, we then have

$$\theta^i(t)^\top S(t) = \theta^i(0)^\top S(0) + \int_0^t \theta^i(s)^\top dS(s) \quad \forall t \in [0, T], \quad \text{a.s.}, \tag{4.5}$$

confirming (4.1). This yields the final requirement for claiming the trading strategy is admissible, or $\theta^i = (\theta_0^i, \ldots, \theta_N^i) \in \Theta(S)$. Evaluating (4.5) at times T and 0, using the definitions of e_i and X_i yields:

$$\theta^i(T)^\top d + \hat{x}_i = \theta^i(T)^\top S(T) + \hat{x}_i = x_i^* \quad \text{a.s.}$$

and

$$\theta^i(0)^\top S(0) = E^*(x_i^* - \hat{x}_i) = \Psi(0, x_i^*) - \Psi(0, \hat{x}_i) = (\hat{r}_i - r_i^*)a,$$

the last line making use of the budget constraint on the Arrow-Debreu allocation for agent i. Thus by adopting the trading strategy θ^i, and faced with the time-zero consumption price of a, agent i can consume precisely $(r_i^*, x_i^*) = v_i^*$.

The above construction applies for agents 1 through $I-1$. For the last agent, agent I, let $\theta^I = -\sum_{i=1}^{I-1} \theta^i$. By the Kunita-Watanabe inequality [**14**], $\Theta(S)$ is a

[6] An *RCLL* process is one whose sample paths are right continuous with left limits almost surely.

linear space, so $\theta^I \in \Theta(S)$. Market clearing is obviously met by construction. To complete this step it remains to show that θ^I generates the consumption allocation $(r_i^*, x_i^*) = v_i^*$, but this is immediate from the linearity of stochastic integrals and market clearing in the Arrow-Debreu equilibrium.

STEP 4: We proceed by contradiction. Suppose some agent j can obtain a strictly preferred allocation $(r, x) \succ_j (r_j^*, x_j^*)$ by adopting a different trading strategy $\theta \in \Theta(S)$. Then the Arrow-Debreu price of (r, x) must be strictly higher than that of (r_j^*, x_j^*), or

$$ar + E^*(x) > ar_j^* + E^*(x_j^*).$$

Substituting the Radner budget constraint for r and x,

$$a\hat{r}_j - \theta(0)^\top S(0) + E^*\left[\hat{x}_j + \theta(0)^\top S(0) + \int_0^T \theta(t)^\top \, dS(t)\right] > ar_j^* + E^*(x_j^*),$$

or

$$a\hat{r}_j + E^*(\hat{x}_j) > ar_j^* + E^*(x_j^*). \tag{4.6}$$

The last line uses the fact that $E^*[\int_0^T \theta(t)^\top \, dS(t)] = 0$ since $\int \theta^\top \, dS$ is a Q-martingale for any $\theta \in \Theta(S)$, from the fact that $\int \phi \, dS_n \in \mathcal{M}_Q^2 \; \forall \phi \in L_Q^2[S_n]$ [**11**, Chapter 4]. But (4.6) contradicts the Arrow-Debreu budget-constrained optimality of (r_j^*, x_j^*). This establishes the theorem. *Q.E.D.*

Of course, under the standard weak conditions ensuring that an Arrow-Debreu equilibrium allocation is Pareto optimal, the resulting Radner equilibrium allocation of this theorem is also Pareto optimal as it implements the Arrow-Debreu allocation.

5. THE SPANNING NUMBER OF RADNER EQUILIBRIA

The key idea of the last proof is that an appropriately selected and priced set of long-lived securities "spans" the entire final period consumption space in the sense that any $x \in L^2(P)$ can be represented in the form

$$E^*[x \mid \mathcal{F}_t] = \theta(t)^\top S(t) = \theta(0)^\top S(0) + \int_0^t \theta(s)^\top \, dS(s) \quad \forall t \in [0, T] \quad \text{a.s.,} \tag{5.1}$$

where $S = (S_0, \ldots, S_N)$ is the set of $(N+1)$ security price processes constructed in the proof and $\theta \in \Theta(S)$ is an appropriate trading strategy. In particular, $E^*[x \mid \mathcal{F}_T] = x$ a.s. As examples in the following section will show, this number of securities, $N+1$, or the multiplicity of $\mathcal{M}_Q^2$ plus one, can be considerably smaller than the dimension of $L^2(P)$. But is this the "smallest number" which will serve this purpose, or the "spanning number" in some sense? To be more precise, we will prove the following result, still assuming $Q \approx P$ and the separability of $\mathcal{F}$.

PROPOSITION 5.1: *Suppose long-lived security prices for $\mathscr{E}$ are square-integrable martingales under Q, the equilibrium price measure for $\mathscr{E}$. Then the minimum number of long-lived securities which completes markets in the sense of* (5.1) *is* $M(\mathscr{M}_Q^2)+1$.

PROOF: That $M(\mathscr{M}_Q^2)+1$ is a sufficient number is given by construction in the proof of Theorem 4.2. The remainder of the proof is devoted to showing that at least this number is required.

If $M(\mathscr{M}_Q^2)=\infty$ we are done. Otherwise, suppose $S=(S_1,\ldots,S_K)$, $K<\infty$, is a set of square-integrable Q-martingale security price processes with the representation property (5.1). By the definition of multiplicity it follows that $K\geqslant M(\mathscr{M}_Q^2)$. It remains to show that $K=M(\mathscr{M}_Q^2)$ implies a contradiction, which we now pursue.

Let $x=k+\mathbf{1}^\top S(T)\in L^2(P)$, where k is any real constant and $\mathbf{1}$ is a K-dimensional vector of ones. If S has the representation property (5.1) there exists some $\theta\in\Theta(S)$ satisfying (5.1) for this particular x. Furthermore, since S is a vector of Q-martingales,

$$E^*[x\,|\,\mathscr{F}_t]=k+\mathbf{1}^\top S(t)=k+\mathbf{1}^\top S(0)+\int_0^t \mathbf{1}^\top\, dS(s)\quad \forall t\in[0,T]\quad \text{a.s.} \tag{5.2}$$

Since $\theta(0)^\top S(0)=E^*(x)=k+\mathbf{1}^\top S(0)$, equating the right hand sides of (5.1) and (5.2) yields

$$\int_0^t \theta(s)^\top\, dS(s)=\int_0^t \mathbf{1}^\top\, dS(s)\quad \forall t\in[0,T]\quad \text{a.s.}$$

Appendix Lemma A.2 then implies

$$Q\{\exists t\in[0,T]:\ \theta(t)=1\}>0.$$

Since $Q\approx P$, the above event also has strictly positive P-probability, and equating the second members of (5.1) and (5.2) yields

$$P\{\exists t\in[0,T]:\ \mathbf{1}^\top S(t)=\mathbf{1}^\top S(t)+k\}>0,$$

an obvious absurdity if $k\neq 0$. *Q.E.D.*

The reader will likely have raised two points by now. First, having shown that the "spanning number" is $M(\mathscr{M}_Q^2)+1$ when long-lived securities are square-integrable martingales under Q, what do we know about the spanning number in general? From the work of Harrison and Kreps [**8**], we see that a "viable" Radner equilibrium *must* be of the form of security price processes which are martingales under *some* probability measure. Their framework, somewhat less general than ours, was extended in Huang [**10**] to a setting much like our own. Readers may wish to confirm that the same result can be proved in the same manner for the present setting. We have chosen to announce prices as martingales under Q, rather than some other probability measure, as this follows from the natural selection of a numeraire claiming one unit of consumption in every state, the security claiming d_0 in Theorem 4.2. Other numeraires could be chosen; if a random numeraire is selected then in equilibrium security prices will be martingales under some other probability measure, say $\hat{P}$, and the spanning number will be $M(\mathscr{M}^2_{\hat{P}})+1$, if the appropriate regularity conditions are adhered to. Does this number differ from $M(\mathscr{M}^2_Q)+1$; that is, can the martingale multiplicity for the same information structure change under substitution of probability measures? Within the class of equivalent probability measures, those giving zero probability to the same events, this seems unlikely. It is certainly not true for event trees. We put off a direct assault on this question to a subsequent paper. We will show later, however, that if the information is generated by a Standard Brownian Motion, then $M(\mathscr{M}^2_P)=M(\mathscr{M}^2_Q)$.

The second point which ought to have been raised is the number of securities required to implement an Arrow-Debreu equilibrium in a Radner style model, *dropping the requirement for complete markets.* For example, with only two agents, a single security which pays the difference between the endowment and the Arrow-Debreu allocation of one of the agents will obviously allow the two to trade to equilibrium at time zero. This is not a very robust regime of markets, of course. By fixing such agent-specific securities, any perturbation of agents' endowments or preferences which preserves Arrow-Debreu prices will generally preclude an efficient Radner equilibrium. Agents will generally be unable to reach their perturbed Arrow-Debreu allocations without a new set of long-lived securities. A set of long-lived securities which completes markets in the dynamic sense of relation (5.1) is constrastingly robust, although the selection still depends endogenously on Arrow-Debreu prices. It remains a formidable challenge to show how markets can be completed by selecting the claims of long-lived securities entirely on the basis of the exogenous information structure F. (As an aside, however, this is easily done for event trees. From this proof of the Proposition it is apparent that a selection of consumption payoffs for long-lived securities can be designed which (generically) completes markets for any Arrow-Debreu equilibrium prices.) There are no economic grounds, of course, precluding the selection of security markets from being an endogenous part of the equilibrium. One would in fact expect this to be the case, an interesting problem for future theoretical and empirical research.

Nothing precludes the fact that some of the martingale price processes in our model may take negative values, even if a positive constant (which is innocuous) is added. For a "spanning" set of positive price processes, one could split each of the original spanning martingales into its positive and negative parts, for a set of $2M(\mathscr{M}^2_Q)+1$ price processes in all. The existence of the required stochastic integrals follows easily.

6. DISCUSSION

In this section we discuss some definitional issues, generalizations of the model, and some specific examples.

6.1. *The Gains Process and Admissible Trading Strategies*

Why is $L_P^2[S]$ an appropriate restriction on trading strategies for a security with price process S? Why is the stochastic integral $\int \theta\, dS$, for $\theta \in L_P^2[S]$, then the appropriate definition of gains from such a strategy? These are questions raised earlier by Harrison and Pliska [**9**].

Following Harrison and Pliska [**9**], we will say that a predictable trading strategy is *simple*, denoted $\theta \in \Lambda$, if there is a partition $\{0 = t_0, t_1, \ldots, t_{n-1}, t_n = T\}$ of $[0, T]$ and bounded random variables $\{h_0, \ldots, h_{n-1}\}$, where h_i is measurable with respect to $\mathscr{F}_{t_i}$, satisfying

$$\theta(t) = h_i, \quad t \in (t_i, t_{i+1}].$$

A simple trading strategy θ, roughly speaking, is one which is piecewise constant and for which $\theta(t)$ can be determined by information available up to, *but not including*, time t. The latter restriction is the basic content of *predictability*. This is not an unreasonable abstraction of "real" trading strategies. The gains process $\int \theta\, dS$, for $\theta \in \Lambda$, is furthermore defined path by path as a Stieltjes integral. That is, the gains at time t_i are

$$\int_0^{t_i} \theta(s)\, dS(s) = \sum_{j=0}^{i-1} \theta(t_j)[S(t_{j+1}) - S(t_j)],$$

simply the sum of profits and losses realized at discrete points in time.

We will give the space $\mathscr{M}_Q^2$ the norm

$$\|m\|_{\mathscr{M}_Q^2} = [E^*([m]_T)]^{1/2} \quad \forall m \in \mathscr{M}_Q^2,$$

and give $L_P^2[S]$ the semi-norm

$$\|\theta\|_{L_P^2[S]} = \left[E\left(\int_0^T \theta^2(t)\, d[S]_t\right)\right]^{1/2}.$$

PROPOSITION 6.1: *For every trading strategy $\theta \in L_P^2[S]$ there exists a sequence (θ_n) of simple trading strategies converging to θ in $L_P^2[S]$ (in the given semi-norm). For any such sequence, the corresponding gains processes $\int \theta_n\, dS$ converge to $\int \theta\, dS$ in $\mathscr{M}_Q^2$ in the given norm.*

PROOF: This is the way Ito originally extended the definition of stochastic integrals. His theorem uses the fact that Λ is dense in $L_Q^2[S]$ and shows that $\theta \mapsto \int \theta\, dS$, $\theta \in \Lambda$, extends uniquely to an isometry of $L_Q^2[S]$ in $\mathscr{M}_Q^2$. These facts can be checked, for instance, in Jacod [**11**, Chapter 4]. Since dQ/dP is assumed to be bounded above and below away from zero, the semi-norms $\|\cdot\|_{L_P[S]}$ and $\|\cdot\|_{L_Q^2[S]}$ are equivalent, and the result is proved. *Q.E.D.*

Interpreting this result: for any admissible strategy $\theta \in \Theta(S)$ there is a sequence of simple trading strategies converging (as agents are able to trade more and more frequently) to θ, with the corresponding gains processes converging to that generated by θ. The sequence of simple trading strategies can also be chosen to be self-financing (4.1) by using the same construction shown in the proof of Theorem 4.2 for the "store-of-value" strategy. A store-of-value security, one whose price is identically one for instance, is again called for. The minimal requirements for a "store-of-value" security price process have not been fully explored.

In what way have we limited agents by restricting them to $L_P^2[S]$ trading strategies? It is known, for instance, that by removing this constraint the so-called "suicide" and "doubling" strategies may become feasible, as discussed by Harrison and Pliska [**9**] and Kreps [**12**]. A suicide strategy makes nothing out of something almost surely, which no one would want to do anyway. A doubling strategy, however, generates a "free lunch," which shouldn't happen in equilibrium. More precisely, an equilibrium can't happen if doubling strategies are allowed. There are no doubling strategies in $L_P^2[S]$ since these strategies only generate martingales (under Q). There is also some comfort in knowing that, since $L_P^2[S]$ is a complete space, there is no sequence of simple or even general $L_P^2[S]$ strategies which converges to a doubling strategy in the sense of Proposition 6.1.

6.2. *Some Generalizations*

There is of course no difficulty in having heterogeneous probability assessments, provided all agents' subjective probability measures on $(\Omega, \mathcal{F})$ are uniformly absolutely continuous. This preserves the topologies on the consumption and strategy spaces across agents.

As a second generalization we could allow the consumption space to be $R \times L^q(P)$ for any $q \in [1, \infty)$, relaxing from $q = 2$. The allowable trading strategies should be generalized to $L_P^q[\underline{S}]$, as defined by Jacod [**11**, (4.59)], since there is then no guarantee of an orthogonal q-basis for $\mathcal{M}_Q^2$. It is a straightforward task to carry out all of the proofs in this paper under both of these generalizations. All interesting models of uncertainty we are aware of, however, are for $q = 2$.

It is also easy, but cumbersome, to extend our results to an economy with production and with a finite number of different consumption goods.

6.3. *Example: Economies on Event Trees*

If the information structure $\boldsymbol{F}$ is such that $\mathcal{F}_t$ contains only a finite number of events at each time t, then it can be represented in the form of an event tree, as in Figure 1.

For finite horizon problems, the terminal nodes of the tree can be treated as the elements of Ω. They are equal in number with the contingent claims forming a complete regime of Arrow-Debreu "simple securities." Yet, as the following proposition demonstrates, a complete markets Radner equilibrium can be established with far fewer securities, except in degenerate cases. Since integrability is not a consideration when Ω is finite, we characterize martingale multiplicity directly in terms of the "finite" filtration $\boldsymbol{F}$, limiting consideration to probability measures under which each $\omega \in \Omega$ has strictly positive probability.

PROPOSITION 6.2: *The multiplicity of a finite filtration* $\boldsymbol{F}$, *under any of a set of equivalent probability measures, is the maximum number of branches leaving any node of the corresponding event tree, minus one.*

The proof, given in the Appendix, presents an algorithm for constructing an orthogonal martingale basis. Just as in Section 4, a complete markets Radner equilibrium can be constructed from any given Arrow-Debreu equilibrium provided there are markets for long-lived securities paying the terminal values of such a Q-martingale basis in time T consumption, and one store-of-value security paying one unit of consumption at time T in each state $\omega \in \Omega$.

By drawing simple examples of event trees, however, it soon becomes apparent that many other choices for the spanning securities will work. This is consistent with Kreps [**13**]. His Proposition 2 effectively states that a necessary and sufficient condition for a complete markets Radner economy is that at any node of the event tree the following condition is met: The dimension of the span of the vectors of "branch-contingent" prices of the available long-lived securities must be the number of branches leaving that node. Kreps goes on to state that the number of long-lived securities required for implementing an Arrow-Debreu equilibrium in this manner must be at least K, the maximum number of branches leaving any node, consistent with our "spanning number" (the martingale multiplicity plus one), as demonstrated by the previous proposition. Kreps also obtains the genericity result: except for a "sparse" set of long-lived securities, a set of measure zero in a sense given in the Kreps article, *any* selection of K or more long-lived security price processes admits a complete markets Radner economy. (The economy needn't be in equilibrium of course.) This result seems exceedingly difficult to extend to a general continuous-time model.

One should beware of taking the "limit by compression" of finite filtrations and expecting the spanning number to be preserved. For example, we have seen statements in the finance literature to the following effect: "In the Black-Scholes option pricing model it is to be expected that continuous trading on *two* securities can replicate any claim since Brownian Motion is the limit of a normalized sequence of coin-toss random walks, each of which has only two outcomes at any toss." If this logic is correct it hides some unexplained reasoning. For example, two simultaneous independent coin-toss random walks generate a martingale space of multiplicity *three* (four branches at each node, minus one), whereas the corresponding Brownian Motion limits (Williams [**22**, Chapter 1]) generate a martingale space of multiplicity *two*. Somehow one dimension of "local uncertainty" is lost in the limiting procedure.

6.4. *A Brownian Motion Example*

This subsection illustrates an infinite dimensional consumption space whose economy (under regularity conditions) has a complete markets Radner equilibrium including only two securities!

Suppose uncertainty is characterized, and information is revealed, by a Standard Brownian Motion, W. To be precise, each $\omega \in \Omega$ corresponds to a particular sample path chosen for W from the continuous functions on $[0, T]$, denoted $C[0, T]$, according to the Wiener probability measure P on $\mathcal{F}$, the completed Borel tribe on $C[0, T]$. The probability space, then, is the completed Wiener triple $(\Omega, \mathcal{F}, P)$ and the filtration is the family $\mathbf{F} = \{\mathcal{F}_t, t \in [0, T]\}$, where $\mathcal{F}_t$ is the completion of the Borel tribe on $C[0, t]$. For conciseness, we'll call $(\Omega, \mathbf{F}, P)$ the completed filtered Wiener triple. More details on this construction are given in the first chapter of Williams [**22**].

To construct a complete markets Radner equilibrium from a given Arrow-Debreu equilibrium, as in Section 4, we need an orthogonal 2-basis for $\mathcal{M}_Q^2$, where Q is an equilibrium price measure for the Arrow-Debreu economy. In this case we can actually show that a particular Standard Brownian Motion on $(\Omega, \mathbf{F}, Q)$ is just such a 2-basis!

It is a well known result (e.g. [**14**]) that the underlying Brownian Motion W is a 2-basis for $\mathcal{M}_P^2$. Assuming $Q \approx P$, the process

$$Z(t) = E\left[\frac{dQ}{dP}\middle| \mathcal{F}_t\right], \quad t \in [0, T],$$

is a square-integrable martingale on $(\Omega, \mathbf{F}, P)$, with $E[Z(T)] = 1$. Then, by Theorem 4.1, there exists some $\rho \in L_P^2[W]$ giving the representation:

$$Z(t) = 1 + \int_0^t \rho(s)\, dW(s) \quad \forall t \in [0, T] \quad \text{a.s.}$$

It follows from Ito's Lemma that, defining the process $\eta(t) = \rho(t)/Z(t)$, we have the alternative representation:

$$Z(t) = \exp\left\{\int_0^t \eta(s)\, dW(s) - \tfrac{1}{2}\int_0^t \eta^2(s)\, ds\right\} \quad \forall t \in [0, T] \quad \text{a.s.}$$

From this, the new process

$$W^*(t) = W(t) - \int_0^t \eta(s)\, ds, \quad t \in [0, T], \tag{6.1}$$

defines a Standard Brownian Motion on $(\Omega, \mathbf{F}, Q)$ by Girsanov's Fundamental Theorem (Lipster and Shiryayev [**15**, p. 232]). It remains to show that W^* is itself a 2-basis for $\mathcal{M}_Q^2$, but this is immediate from Theorem 5.18 of Liptser and Shiryayev [**15**], using the uniform absolute continuity of P and Q. This construction is summarized as follows.

PROPOSITION 6.3: *Suppose W is the Standard Brownian Motion underlying the filtered Wiener triple (Ω, F, P) and $Q \approx P$. Then W^* defined by* (6.1) *is a Standard Brownian Motion under Q which is a 2-basis for $\mathcal{M}^2_Q$. In particular, $M(\mathcal{M}^2_P) = M(\mathcal{M}^2_Q) = 1$.*

By a slightly more subtle argument, we could have reached the same conclusion under the weaker assumption that P and Q are merely equivalent, but $Q \approx P$ is needed for other reasons in Theorem 4.1.

In short, by marketing just two long-lived securities, one paying $W^*(T)$ in time T consumption, the other paying one unit of time T consumption with certainty, and announcing their price processes as $W^*(t)$ and 1 (for all t), a complete markets Radner equilibrium is achieved.

This example can be extended to filtrations generated by vector diffusion processes. Under well known conditions (e.g. [**8**]) a vector diffusion generates the same filtration as the underlying vector of independent Brownian Motions. An orthogonal 2-basis for $\mathcal{M}^2_P$ is then simply these Brownian Motions themselves [**14**]. By generalizing the result quoted from Lipster and Shiryayev [**15**, Theorem 5.18], one can then demonstrate a vector of equally many Brownian Motions under Q which form a martingale basis for $\mathcal{M}^2_Q$ in the sense of Theorem 4.1. Since the manipulations are rather involved, and because the results raise some provocative issues concerning the "inter-temporal capital asset pricing models" (e.g., [**17**]) which are also based on diffusion uncertainty, we put off this development to a subsequent paper. It is also known that the filtration generated by a Poisson process corresponds to a martingale multiplicity of one [**11**].

7. CONCLUDING REMARKS

We are working on several extensions and improvements suggested by the results of this paper.

The first major step will be to demonstrate the existence of continuous trading Radner equilibria "from scratch," that is, taking endowments and preferences as agent primitives and proving the existence of an equilibrium such as that demonstrated in Theorem 4.2. In particular, the existence of an Arrow-Debreu equilibrium and the condition $Q \approx P$ must be proven from exogenous assumptions, rather than assumed. A full-blown Radner economy is also being examined, one with consumption occurring over time rather than at the two points 0 and T.

The Brownian Motion example of Section 6.4, as suggested there, is being extended to the case in which uncertainty and information are characterized by a vector of diffusion "state-variable" processes. This will allow us to tie in with, and provide a critical evaluation of, the inter-temporal capital asset pricing models popular in the financial economics literature.

We left off in Section 5 by characterizing the spanning number in terms of (endogenous) Arrow-Debreu prices through the equilibrium price measure Q. Our next efforts will be directed at showing that, subject to regularity conditions, martingale multiplicity is invariant under substitution of equivalent probability measures. In that case the spanning number can be stated to be the exogenously given number, $M(\mathcal{M}_P^2)+1$.

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APPENDIX

MARTINGALE MULTIPLICITY

What follows is a heavily condensed treatment, taken mainly from the fourth chapter of Jacod [**11**].

A square-integrable martingale on the filtered probability space $(\Omega, \boldsymbol{F}, P)$ is an $\boldsymbol{F}$-adapted[7] process $X=\{X_t;\, t\in[0,T]\}$ with the properties: (i) $E[X(t)^2]<\infty$ for all $t\in[0,T]$, and (ii) $E[X(t)\mid\mathcal{F}_s]=X(s)$ a.s. for all $t\geq s$.

We will also assume without loss of generality for this paper that each martingale is an RCLL process. The first property (i) is *square-integrability*, the second (ii) is *martingale*, meaning roughly that the expected future value of X given current information is always the current value of X.

The space of square-integrable martingales on $(\Omega, \boldsymbol{F}, P)$ which are null at zero (or $X(0)=0$) is denoted $\mathcal{M}_P^2$. The spaces $\mathcal{M}_P^2$ and $L^2(P)$ are in one to one correspondence via the relationship, between some $X\in\mathcal{M}_P^2$ and $x\in L^2(P)$:

$$X(t)=E[x\mid\mathcal{F}_t],\quad t\in[0,T],$$

where all RCLL versions of the conditional expectation process are indistinguishable and therefore identified.

An $\boldsymbol{F}$-adapted process is termed *predictable* if it is measurable with respect to the tribe $\mathcal{P}$ on $\Omega\times[0,T]$ generated by the left-continuous $\boldsymbol{F}$-adapted processes. At an intuitive level, θ is a predictable process if the value of $\theta(t)$ can be determined from information available up to, *but not including*, time t, for each $t\in[0,T]$.

Two martingales X and Y are said to be *orthogonal* if the product XY is a martingale. From this point we'll assume that $\mathcal{F}$ is a separable tribe under P. In that case the path breaking work of Kunita and Watanabe [**14**] shows the existence of an orthogonal 2-*basis* for $\mathcal{M}_P^2$, defined as a minimal set of mutually orthogonal elements of $\mathcal{M}_P^2$ with the representation property stated in Theorem 4.1. By "minimal," we mean that no fewer elements of $\mathcal{M}_P^2$ have this property. The number of elements of a 2-basis, whether countably infinite or some positive integer, is called the *multiplicity* of $\mathcal{M}_P^2$, denoted $M(\mathcal{M}_P^2)$.

The following lemma makes a technical argument used in the proof of Theorem 4.2.

LEMMA A.1: *Suppose the process X is defined by*

$$X(t)=\sum_{n=1}^{N}\left[\int_0^t \theta_n(s)\,dS_n(s)-\theta_n(t)S_n(t)\right],\quad t\in[0,T],$$

where $\int\theta_n\,dS_n$ is the stochastic integral of a predictable process θ_n with respect to a semi-martingale S_n, for $1\leq n\leq N<\infty$. Then X is predictable.

PROOF: For any left-limits process Z, let $Z(t-)$ denote the left limit of Z at $t\in[0,T]$, and denote the "jump" of Z at t by $\Delta Z(t)=Z(t)-Z(t-)$, where we have used the convention that $Z(0-)=Z(0)$. Then we can write

$$X(t)=\sum_{n=1}^{N}\left[\int_0^{t-}\theta_n(s)\,dS_n(s)-\theta_n(t)S_n(t)+\theta_n(t)\,\Delta S_n(t)\right]$$

[7] A stochastic process $X=\{X_t;\, t\in[0,T]\}$ is adapted to a filtration $\boldsymbol{F}=\{\mathcal{F}_t;\, t\in[0,T]\}$ if X_t is measurable with respect to $\mathcal{F}_t$ for all $t\in[0,T]$.

since

$$\Delta\left(\int_0^t \theta_n(s)\, dS_n(s)\right) = \theta_n(t)\, \Delta S_n(t)$$

by the definition of a stochastic integral. Then, using

$$\theta_n(t)S_n(t) = \theta_n(t)[S_n(t-) + \Delta S_n(t)],$$

we have

$$X(t) = \sum_{n=1}^{N}\left[\int_0^{t-} \theta_n(s)\, dS_n(s) - \theta_n(t)S_n(t-)\right], \quad t \in [0, T].$$

Since $\int_0^{t-} \theta_n\, dS_n$ and $S_n(t-)$ are left-continuous processes, and therefore predictable, and $\theta_n(t)$ is predictable, we know X is predictable because sums and products of measurable functions are measurable. *Q.E.D.*

For any two elements X and Y of $\mathcal{M}_P^2$, let $\langle X, Y\rangle$ denote the unique predictable process with the property that $XY - \langle X, Y\rangle$ is a martingale and $\langle X, Y\rangle_0 = 0$.

LEMMA A.2: *Suppose* $(m_1, \ldots, m_N)$ *constitute a finite set of elements of* $\mathcal{M}_Q^2$ *with the representation property given in the statement of Theorem* 4.1, *where* $N = M(\mathcal{M}_Q^2)$. *If* θ_n *and* ϕ_n *are elements of* $L_Q^2[m_n]$, *for* $1 \leq n \leq N$, *satisfying* (*with the obvious shorthand*)

$$\text{(a.1)} \qquad \int_0^t \theta^\top dm = \int_0^t \phi^\top dm \quad \forall t \in [0, T] \quad \text{a.s.,}$$

then

$$Q\{\exists t \in [0, T]\colon \theta(t) = \phi(t)\} > 0.$$

PROOF: Jacod [**11**] shows the existence of a predictable positive semi-definite $N \times N$ matrix valued process c and an increasing predictable process C with the property, for any α_n and β_n in $L_Q^2[m_n]$, $1 \leq n \leq N$,

$$\text{(a.2)} \qquad \left\langle \int \alpha^\top dm, \int \beta^\top dm \right\rangle_t = \int_0^t \alpha(s)^\top c(s)\beta(s)\, dC(s) \quad \forall t \in [0, T] \quad \text{a.s.}$$

The process C also defines a Doléans measure (also denoted C) on $(\Omega \times [0, T], \mathcal{P})$ according to

$$C(B) = \int_\Omega \int_{[0,T]} 1_B(\omega, s)\, dC(\omega, s)\, dQ(\omega) \quad \forall B \in \mathcal{P}.$$

By (4.43) of Jacod [**11**], the matrix process c reaches full rank, and is thus positive definite, on some set $B^* \in \mathcal{P}$ of strictly positive C-measure. But, by (a.1) and (a.2),

$$\text{(a.3)} \qquad \int_0^t [\theta(\omega, s) - \phi(\omega, s)]^\top c(\omega, s)[\theta(\omega, s) - \phi(\omega, s)]\, dC(\omega, s) = 0 \quad \forall t \in [0, T] \quad \text{a.s.}$$

Ignoring without loss of generality the Q-null set on which (a.3) does not hold, this implies that $\theta(\omega, t) = \phi(\omega, t)$ for all time points of increase of C on B^*, which have strictly positive Q-probability since the projection of B^* on Ω must have strictly positive Q-measure for B^* to have strictly positive C-measure. *Q.E.D.*

PROOF OF PROPOSITION 6.2—MULTIPLICITY OF AN EVENT TREE: Let N denote the maximum number of branches leaving any node of the event tree, minus one. The proposition will be proved by constructing an orthogonal martingale basis on this filtration consisting of the N processes $m_1, \ldots, m_N$.

Any martingale on a finite filtration is characterized entirely by its right-continuous jumps at each node in the corresponding event tree. Denote the jump of m_j at a generic node with L departing branches by the vector $\delta_j = (\delta_{j1}, \ldots, \delta_{jL})$. That is, $\delta_j \in R^L$ represents the random variable which takes the real number δ_{jl} if branch l is the realized event at this node. Let $p=(p_1,\dots,p_L)\in R^L$ denote the vector of conditional branching probabilities at this node.

The processes $m_1,\dots,m_n$ are then mutually orthogonal martingales if they satisfy the following two conditions at each node:

(i) $p^{\mathsf{T}}\delta_j=0$, $j=1,\dots,N$ (zero mean jumps, the martingale property), and

(ii) $\delta_j^{\mathsf{T}}[p]\delta_k=0$, $\forall_j\neq k$, where $[p]$ denotes the diagonal matrix whose lth diagonal element is p_l (mutually uncorrelated jumps, implying mutually orthogonal martingales).

We construct the processes $m_1,\dots,m_N$ by designing their jumps at each node of the event tree, in any order, taking $m_j(0)=0\ \forall j$. At a given node (with L branches), it is simple to choose nonzero vectors $\delta_1,\dots,\delta_{L-1}$ in R^L satisfying

(a.4) $$\Delta_j[p]\delta_j=0,\quad 1\leq j\leq L-1,$$

where Δ_j is the $j\times L$ matrix whose first row is a vector of ones and whose kth row is, for $k\geq 2$, $\delta_{k-1}^{\mathsf{T}}$. This cannot be done for $j\geq L$ since $\Delta_L[p]^{1/2}$ is a full rank $L\times L$ matrix (its rows are nonzero and mutually orthogonal). Instead, let $\delta_L,\dots,\delta_N$ each be zero vectors. One can quickly verify that this construction meets the conditions (i)-(ii) for $m_1,\dots,m_N$ to be mutually orthogonal martingales. They are nontrivial since there is at least one node with $N+1$ branches, by definition of N. They form a basis (in the sense of Theorem 4.1) for all martingales since at each node the subspace $\{\delta\in R^L\colon \delta^{\mathsf{T}}p=0\}$ has linearly independent spanning vectors $\delta_1,\dots,\delta_{L-1}$. That is, the jump of any given martingale at this node is a linear combination of the jumps of the first $L-1$ martingales of the set $\{m_1,\dots,m_N\}$. At least N martingales are needed for a martingale basis since at some node this subspace has dimension N, by definition of N. *Q.E.D.*

EXAMPLE: A MARKOV CHAIN: As a simple example, consider a finite state-space Markov chain information structure. Transition probabilities are given by the matrix

$$\Pi=(\pi_{\alpha\beta})\quad 1\leq\alpha\leq n;\ 1\leq\beta\leq n;$$

where $\pi_{\alpha\beta}$ denotes the one-step transition probability from state α to state β. Let π^α denote the αth row of Π and $\delta_j^\alpha\in R^n$ denote the vector of jumps of the process m_j at any node corresponding to state α, for $1\leq j\leq n-1$. We will assume at least one row of Π has no zero elements. Then the multiplicity of the space of martingales on this Markov chain is $n-1$, and the processes $m_1,\dots,m_{n-1}$ form an orthogonal martingale basis provided, for $\alpha=1,\dots,n$,

$$\pi^{\alpha\mathsf{T}}\delta_j^\alpha=0,\quad 1\leq j\leq n-1,$$

and

$$\delta_j^{\alpha\mathsf{T}}[\pi^\alpha]\delta_k^\alpha=0,\quad j\neq k,$$

corresponding to conditions (i)-(ii) above, and $\delta_j^\alpha\neq 0$ for all α and all j.

If, for instance,

$$\Pi=\begin{pmatrix}0.3 & 0.3 & 0.4\\ 0.3 & 0.3 & 0.4\\ 0.3 & 0.3 & 0.4\end{pmatrix},$$

then the two martingales m_1 and m_2 are an orthogonal martingale basis, where, at any node, m_1 jumps $+2$ if state 1 follows, jumps $+2$ if state 2 follows, and jumps -3 if state 3 follows, or $\delta_1=(2,2,-3)$; and similarly m_2 is characterized by jumps $\delta_2=(1,-1,0)$. To be even more explicit, if state 2 occurs at time 1, state 3 at time 2, and the chain terminates at time 2.5, the sample path for m_1 is

$$m_1(t)=0,\quad 0\leq t<1,$$

$$m_1(t)=2,\quad 1\leq t<2,$$

$$m_1(t)=-1,\quad 2\leq t\leq 2.5.$$

REFERENCES

[1] ARROW, K.: "Le rôle des valeurs boursières pour la repartition la meillure des risques," *Econometrie*, 40(1952), 41-48; translated in *Review of Economic Studies*, 31(1964), 91-96.

[2] BLACK, F., AND M. SCHOLES: "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81(1973), 637-654.

[3] COX, J., AND S. ROSS: "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, 3(1976), 145-166.

[4] DAVIS, M. H., AND P. VARAIYA: "The Multiplicity of an Increasing Family of σ-Fields," *The Annals of Probability*, 2(1974), 958-963.

[5] DEBREU, G.: *Theory of Value.* Cowles Foundation Monograph 17. New Haven: Yale University Press, 1959.

[6] DUFFIE, J. D.: "Advances in General Equilibrium Theory," Ph.D. Dissertation, Stanford University, 1984.

[7] HALMOS, P.: *Measure Theory.* Princeton: Van Nostrand, 1950.

[8] HARRISON, J., AND D. KREPS: "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20(1979), 381-408.

[9] HARRISON, J., AND S. PLISKA: "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications*, 11(1981), 215-260.

[10] HUANG, C.: "Information Structure and Equilibrium Asset Prices," *Journal of Economic Theory*, 35(1985), 33-71.

[11] JACOD, J.; *Calcul Stochastique et Problémes de Martingales, Lecture Notes in Mathematics, No. 714.* Berlin: Springer-Verlag, 1979.

[12] KREPS, D.: "Three Essays on Capital Markets," Technical Report 298, Institute for Mathematical Studies in The Social Sciences, Stanford University, 1979.

[13] ———: "Multiperiod Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model," *The Economics of Uncertainty and Information*, ed. by J. McCall. Chicago: University of Chicago Press, 1982.

[14] KUNITA, H., AND S. WATANABE: "On Square-Integrable Martingales," *Nagoya Mathematics Journal*, 30(1967), 209-245.

[15] LIPTSER, R., AND A. SHIRYAYEV: *Statistics of Random Processes I: General Theory.* New York: Springer-Verlag, 1977.

[16] MEMIN, JEAN: "Espaces de semi Martingales et Changement de Probabilité," *Zeitschrift für Wahrscheinlichkeitstheorie*, 52(1980), 9-39.

[17] MERTON, R.: "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41(1973), 867-888.

[18] ———: "The Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4(1973), 141-183.

[19] ———: "On the Microeconomic Theory of Investment Under Uncertainty," in *Handbook of Mathematical Economics, Vol. II*, ed. by K. Arrow and M. Intriligator. Amsterdam: North-Holland Publishing Company, 1982.

[20] RADNER, R.: "Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets," *Econometrica*, 40(1972), 289-303.

[21] SCHAEFER, H. H.: *Banach Lattices and Positive Operators.* New York: Springer-Verlag, 1974.

[22] WILLIAMS, D.: *Diffusions, Markov Processes, and Martingales, Vol. 1.* New York: Wiley, 1979.