EE102 exercises

1. *Optimizing gains in a two-stage amplifier.* Consider the two-stage amplifier described on page 2-12 of the lecture notes. In this problem you will determine optimal values for the two amplifier gains a_1 and a_2. The constraints and specifications are:

   - The amplifier gains can be varied from 5 (14dB) to 20 (26dB).
   - Each of the noises is a voltage with a magnitude no more than 100μV, *i.e.*, $|n_1| \leq 10^{-4}$, $|n_2| \leq 10^{-4}$.
   - The input signal voltage ranges between $\pm$100mV.
   - The gain from the input signal to the output signal must be 100, *i.e.*, if the noises were zero we would have $y = 100u$.
   - The maximum allowed voltage magnitude is 1V at the output of the first amplifier, and 10V at the output of the second amplifier. (Effects of the noise voltages can be ignored in this calculation.)

   Find choices for a_1 and a_2 that satisfy the specifications while minimizing the largest possible effect of the noise voltages at the output. Explain what you are doing, and what your reasonning is.

2. *Small signals.* In lecture 1 we mentioned several methods for determining the size of a signal. Intuition suggests that even though they are not the same, the measures shouldn't be too different. After all, a small signal is a small signal, right? In this problem we explore this issue.

   Consider a family of signals described by

$$u(t) = \begin{cases} 1/\sqrt{d}, & 0 \leq t \leq d \\ 0, & d < t < 1 \end{cases}$$

   for $0 \leq t < 1$, and periodic with period 1 (*i.e.*, the signal repeats every second). The parameter d, which satisfies $0 < d < 1$, is called the *duty cycle* of the periodic pulse signal.

   Sketch the signal for a few values of d. What is its peak, RMS, and average-absolute value? As the duty-cycle d approaches 0, is the signal getting smaller or larger?

3. A *sawtooth signal* u has the form $u(t) = at/T$ for $0 \leq t < T$, and is T-periodic (*i.e.*, repeats every T seconds). The constant a is called the *amplitude* of the signal, and the constant T (which is positive) is called the *period* of the signal. You can assume that $a > 0$.

   (a) Find the peak value of a sawtooth signal.

   (b) Find the RMS (root-mean-square) value of a sawtooth signal.

   (c) Find the AA (average-absolute) value of a sawtooth signal.

   (d) In the space below, sketch the derivative of a sawtooth signal. Be sure to label all axes, slopes, magnitudes of any impulses, etc.

4. *Sample and hold system.* A sample and hold (S/H) system, with sample time h, is described by $y(t) = u(h\lfloor t/h \rfloor)$, where $\lfloor a \rfloor$ denotes the largest integer that is less than or equal to a.

   Sketch an input and corresponding output signal for a S/H, to illustrate that you understand what it does.

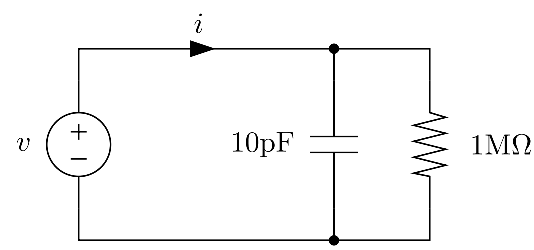
   Is a S/H system linear?

5. *CRT deflection circuit.* In a CRT (cathode ray tube) the horizontal and vertical deflection of the light spot are proportional to the currents in the horizontal and vertical deflection coils, respectively. In a simple raster scan, the spot scans across one row, left to right at a uniform speed, then rapidly moves down the next row (which is called *horizontal retrace*). After the bottom row is scanned, the spot rapidly moves from the bottom right corner of the screen to the top left (which is called *vertical retrace*) and starts scanning the next image or frame.

   A good electrical model of a deflection coil is an inductance in series with a resistance.

   Describe and sketch the current and voltage waveforms for a deflection coil. You can use the (not so good in practice) assumption that the horizontal and vertical retraces take neglible time.

6. A voltage source drives a load consisting of a resistance and a capacitance in parallel, as shown below.



   The voltage signal is a rectangular pulse:

$$v(t) = \begin{cases} 5, & 0 \le t \le 10 \\ 0, & t < 0 \text{ or } t > 10, \end{cases}$$

   where v is given in V and t is given in μsec.

   Sketch the current signal i. Label the axes and all critical values on your plot, giving physical units (*e.g.*, mA, μsec, ...). If there are impulses in the current, you must give the magnitude of each impulse. We want to know the *exact signal* i, not its *general form.*

7. *Some convolution systems.* Consider a convolution system,

$$y(t) = \int_{-\infty}^{+\infty} u(t-\tau)h(\tau)\, d\tau,$$

   where h is a function called the *kernel* or *impulse response* of the system.

   (a) Suppose the input is a unit impulse function, *i.e.*, $u = \delta$. What is the output y? (This explains the terminology above.)

   (b) Suppose $h = \delta$. What does the system do?

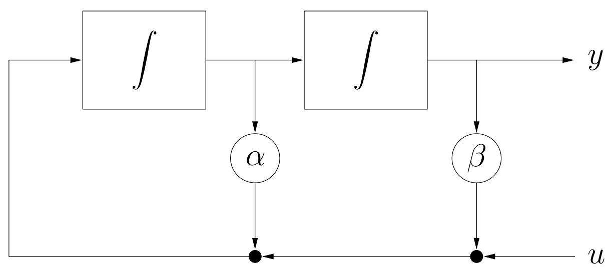
(c) Suppose h is a unit step function. What does the system do?

(d) Suppose $h = \delta'$. What does the system do?

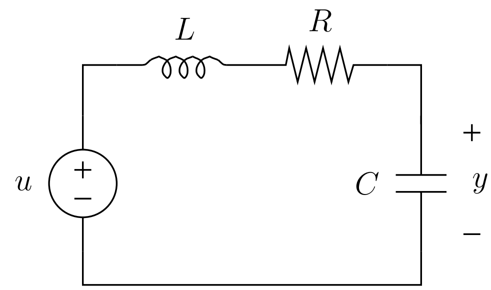
(e) Suppose $h(t) = \delta(t-1)$. What does the system do?

(f) Suppose h is a rectangular pulse signal that is one between 0 and 1. What does the system do?

8. Describe the system shown below as an LCCODE. *Hint:* first label all signals, then write down how they are related.



9. The circuit shown below is a simple model of a real wire in which we take into account its (presumably small) inductance, capacitance, and resistance. Describe this system as an LCCODE. The input is the driving voltage u, and the output is the voltage y at the other end of the wire.



10. *Block diagram from equations.* An interconnected set of systems is described by the following equations:

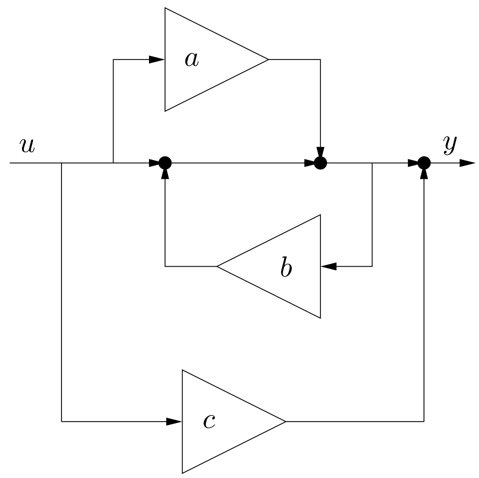
$$v = A(u - v), \quad w = B(v - z), \quad z = C(w).$$

Here u, v, w, z are signals and A, B, C are systems. You can consider u as the external input to the interconnected systems, and z as the external output of the interconnected system.

(a) Draw a pretty block diagram representing these equations. *Hint:* it usually takes two or three passes to get a pretty block diagram.

(b) Now suppose that the systems A, B, C are scaling systems with gains a, b, c, respectively. Express z in terms of u. (In other words, eliminate the signals v and w from the equations.)

11. Consider the block diagram shown below:



The triangle shaped blocks represent scaling systems with gains a, b, and c.

Find a *simple* mathematical expression for the output y in terms of the input u and the constants a, b, and c. Your answer should be *only* in terms of the input u and the scale factors a, b, and c.

12. *Commuting systems.* We say that two systems A and B *commute* if the composition or cascade connection of A and B is the same in either order, *i.e.*, $A(B(u)) = B(A(u))$ for any signal u. For example, a linear system commutes with any scaling system (that's what it means for a linear system to be homogenous).

    Which pairs of the following systems commute?

    - a scaling system with gain 3
    - an inverter, *i.e.*, a scaling system with gain -1
    - a delay system with delay 2sec
    - a S/H operating at 1 sample/sec: $y(t) = u(\lfloor t \rfloor)$
    - an integrator
    - a 1-bit limiter
    - a 1-second averaging system: $y(t) = \int_{t-1}^{t} u(\tau)\, d\tau$
    - a square-law system, for which $y(t) = u(t)^2$

    (Every system commutes with itself. That leaves 28 other pairs of systems for you to think about ...)

    You might want to organize your answer in a table. You do not have to justify your answers.

13. Find the Laplace transform of the following functions.

    (a) $f(t) = (1 + t - t^2)e^{-3t}$.

    (b) $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ -1 & 2 \le t \end{cases}$

    (c) $f(t) = 1 - e^{-t/T}$ where $T > 0$.

14. *Laplace transform monotonicity properties.* Let f and g be two real-valued functions (or signals) defined on $\{t|t \geq 0\}$. Let F and G denote the Laplace transforms of f and g, respectively. We will assume that f and g are bounded, so the Laplace transforms are defined at least for all s with $\Re s > 0$.

    (a) Suppose that $f(t) \geq g(t)$ for all $t \geq 0$. Is it true that $F(s) \geq G(s)$ for all real, positive s? If true, explain why. If false, provide a counterexample.

    (b) Is the converse true? That is, if $F(s) \geq G(s)$ for all real positive s, is it true that $f(t) \geq g(t)$ for $t \geq 0$? If true, explain why. If false, provide a counterexample.

15. *Laplace transform of reversed signal.* So, you thought we covered all the Laplace transform rules in lecture 3? Suppose f is a signal that only lasts T seconds, *i.e.*, $f(t) = 0$ for $t > T$. Define the signal g by 'reversing' f, *i.e.*, $g(t) = f(T-t)$ for $0 \leq t \leq T$, and $g(t) = 0$ for $t > T$. Express G, the Laplace transform of g, in terms of F, the Laplace tranform of f.

16. *Convolution and the Laplace transform.*

    (a) Evaluate $h(t) = e^{-t} * e^{-2t}$ using direct itegration. (These signals are not defined for $t < 0$.)

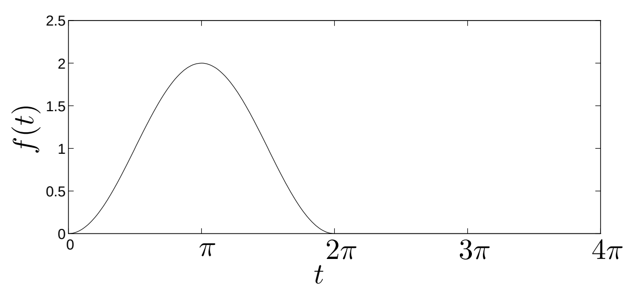
    (b) Find H, the Laplace transform of h, using the expression for h you found in part (a).

    (c) Verify that H is the product of the Laplace transforms of e^{-t} and e^{-2t}.

17. The "raised cosine pulse" is a signal used in applications such as radar and communications. It is defined by

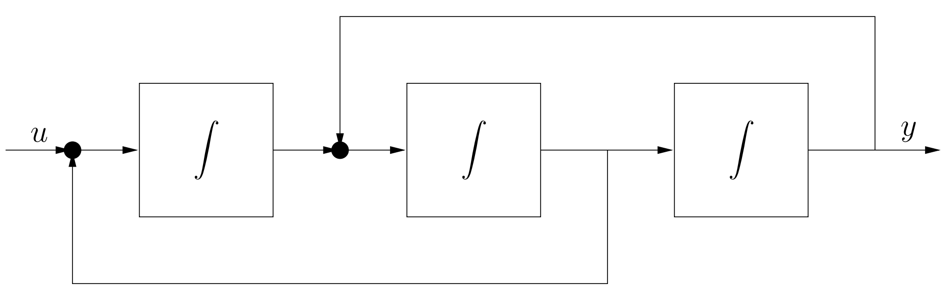
$$f(t) = \begin{cases} 1 - \cos t & 0 \leq t \leq 2\pi \\ 0 & t > 2\pi \end{cases}$$

    and plotted below.



    Find F, the Laplace transform of f.

18. Consider the system shown below.



(a) Express the relation between u and y as an LCCODE. *Hint:* if the signal z is the output of an integrator, then its input is the signal z'.

(b) Assuming that $y(0) = y'(0) = y''(0) = 0$, derive an expression for Y (the Laplace transform of y) in terms of U (the Laplace transform of u).

19. Solve the following LCCODEs using Laplace transforms. Verify that the solution you find satisfies the initial conditions and the differential equation.

(a) $dv/dt = -2v + 3$, $v(0) = -1$.

(b) $d^2i/dt^2 + 9i = 0$, $i(0) = 1$, $di/dt(0) = 0$.

20. Four signals a, b, c, and d are related by the differential equations

$$a' + a = b, \qquad b' + b = c, \qquad c' + c = d,$$

where $a(0) = b(0) = c(0) = 0$. Express $A(s)$, the Laplace transform of a, in terms of $D(s)$, the Laplace transform of d.

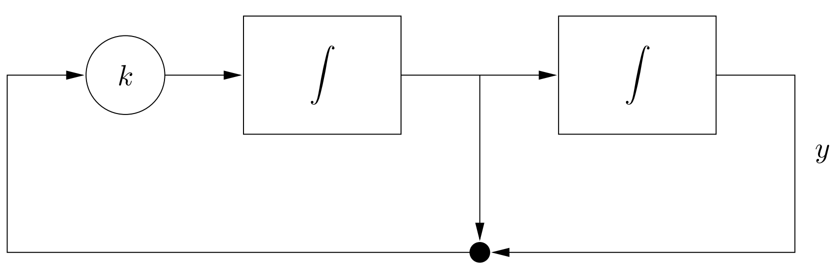
21. The Laplace transform of a signal q is given by

$$Q(s) = \frac{3 - 5e^{-s}}{s + 1}.$$

Find q (*i.e.*, describe it explicitly).

22. A signal z satisfies $z''' + z'' + z' + z = 0$, and $z(0) = z'(0) = 0$, $z''(0) = 1$. Find z.

23. In the system shown below, k is a gain.

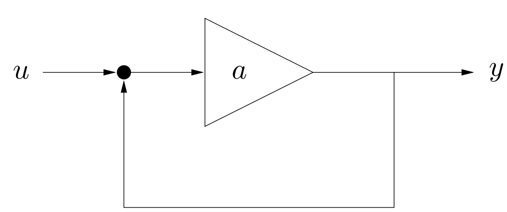


(a) For what values of k is this system stable?

(b) For what values of k is this system stable and critically damped?

(c) For what values of k do (nonzero) solutions y change sign infinitely often? (We *do not* require stability here.)

24. *Unity feedback around amplifier.* The system shown below consists of an amplifier of gain a, connected in what is sometimes called a *unity feedback* configuration. (We'll study feedback in more detail later in the course.)



(a) Solve for $y(t)$ in terms of $u(t)$. Your answer should come out in the form $y(t) = bu(t)$ for some scalar b, which is called the *closed-loop gain* of the feedback configuration. You can assume that $a \neq 1$.

(b) What is the closed-loop gain b, when $a = 0.9$? When a is positive and near but smaller than one, the system is sometimes described as *regenerative.* (Can you explain why this name is used?) This configuration is sometimes used to get more gain from a device or amplifier than you would otherwise have.

(c) The feedback configuration is often used when a is large and negative, say, $a \approx -10^4$. In this case the system is referred to as a *following amplifier.* (Can you guess why?) What is b is this case? How much does b vary as a varies over the 100-fold range from $a = -10^3$ to $a = -10^5$? The answer shows one of the reasons this configuration is used.

25. *Convolution and Laplace transforms.* Let u denote a unit rectangular pulse signal that starts at $t = 0$ and lasts until $t = 1$.

(a) Find the signal $v = u * u$ by evaluating the convolution integral directly. Sketch u and $v = u * u$. *Hint:* v is sometimes called a *triangular pulse signal.*

(b) Find U and V, the Laplace transforms of u and v, respectively, by directly evaluating the Laplace transform defining integrals using the expressions for u and v found in part a. *Hint:* the indefinite integral of te^{-st} is $-\left(t/s + 1/s^2\right) e^{-st}$.

(c) Find V from U using the convolution formula for Laplace transforms and verify you get the same result.

26. *Thermal runaway.* A conductor with resistance R carries a fixed positive current i, and hence dissipates a power $P = i^2 R$. This causes the conductor to heat up (hopefully not too much) above the ambient temperature.

Let $T(t)$ denote the temperature of the conductor above the ambient temperature at time t. T satisfies the equation

$$aT' = -bT + P$$

where $a > 0$ is the thermal capacity of the conductor (in J/°C), $b > 0$ is the thermal conductivity (in W/°C), and P is the power (in W) dissipated in the conductor.

The resistance R of the conductor changes with temperature according to

$$R = R_0(1 + cT)$$

where the constant c (which has units 1/°C) is called the *resistance temperature coefficient* (or just 'tempco') of the conductor, and $R_0 > 0$ is the resistance of the conductor at ambient temperature. Depending on the material of the conductor, the tempco c can be positive or negative; for example, for metal wires the tempco is positive. (The formula above is valid only over a range where $1 + cT > 0$.)

(a) Consider a metal wire, for which $c > 0$. If the current i is smaller than a critical value i_{crit}, the temperature T converges to a steady-state value as $t \to \infty$. If the current i is larger than this critical value of current, then the temperature T converges to ∞ as $t \to \infty$. (In practice, the temperature increases until the conductor is destroyed, *e.g.*, melted.) This phenomenon is called *thermal runaway.*

Find the critical value i_{crit}, above which thermal runaway occurs. Express the answer in terms of the other constants in the problem (a, b, R_0, c).

(b) Suppose the wire is initially at ambient temperature, *i.e.*, $T(0) = 0$, and the constants have the values

$$a = 1\text{J}/^\circ\text{C}, \quad b = 0.5\text{W}/^\circ\text{C}, \quad i = 10\text{A}, \quad R_0 = 1\Omega, \quad c = 0.01/^\circ\text{C}$$

Find $T(t)$ for $t \geq 0$

27. A voltage $v(t)$ is applied to a DC motor. A simple electrical model of the motor is an inductance L in series with a resistance R, so the motor current $i(t)$ satisfies

$$L di/dt + Ri = v.$$

The motor shaft angle is denoted $\theta(t)$, and the shaft angular velocity $\omega(t)$ (so we have $\omega = d\theta/dt$). The motor current puts a torque on the shaft equal to $ki(t)$, where k is the *motor constant*. The shaft rotational inertia is J and the damping coefficient is b. Newton's equation is then:

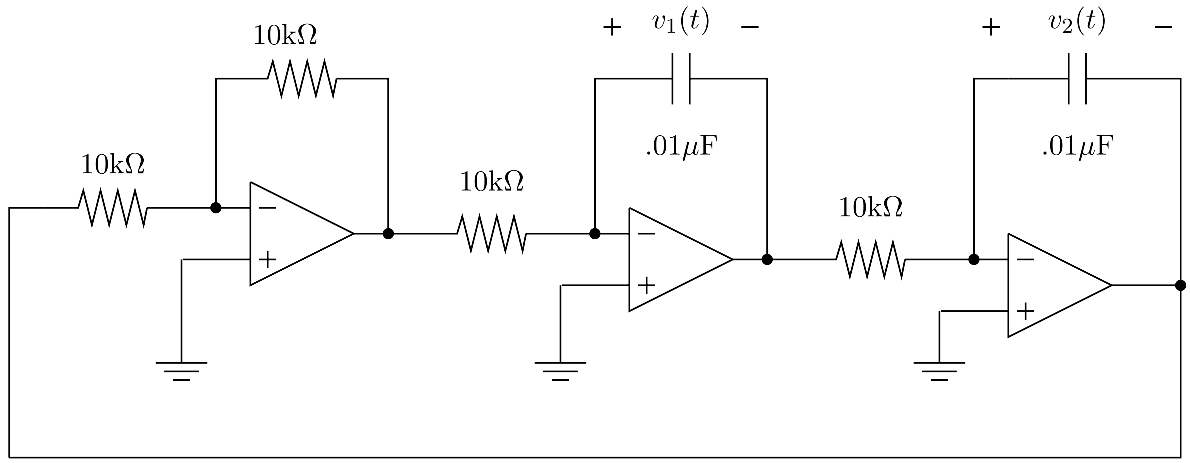
$$J d\omega/dt = ki - b\omega.$$

Assuming that $i(0) = 0$, $\theta(0) = 0$, and $\omega(0) = 0$, express Θ (the Laplace transform of θ) in terms of V (the Laplace transform of v).

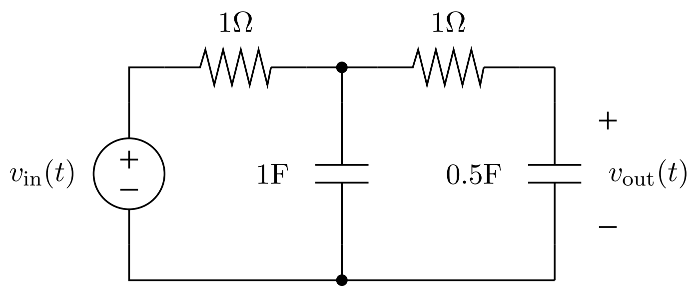
The numbers L, R, k, J, b are all positive constants.

28. What does the following circuit do? Assume $v_1(0) = 0$ and $v_2(0) = 1$.

Hint: Show that $d^2v_1(t)/dt^2 + \omega^2 v_1(t) = 0$, where $\omega = 1/(10\text{k}\Omega \cdot 0.01\mu\text{F})$.



29. *An RCRC circuit.* Consider the circuit shown below. For $t < 0$, $v_{\text{in}}(t) = 1\text{V}$ and the circuit was in static conditions. For $t \geq 0$, $v_{\text{in}}(t) = 0\text{V}$.

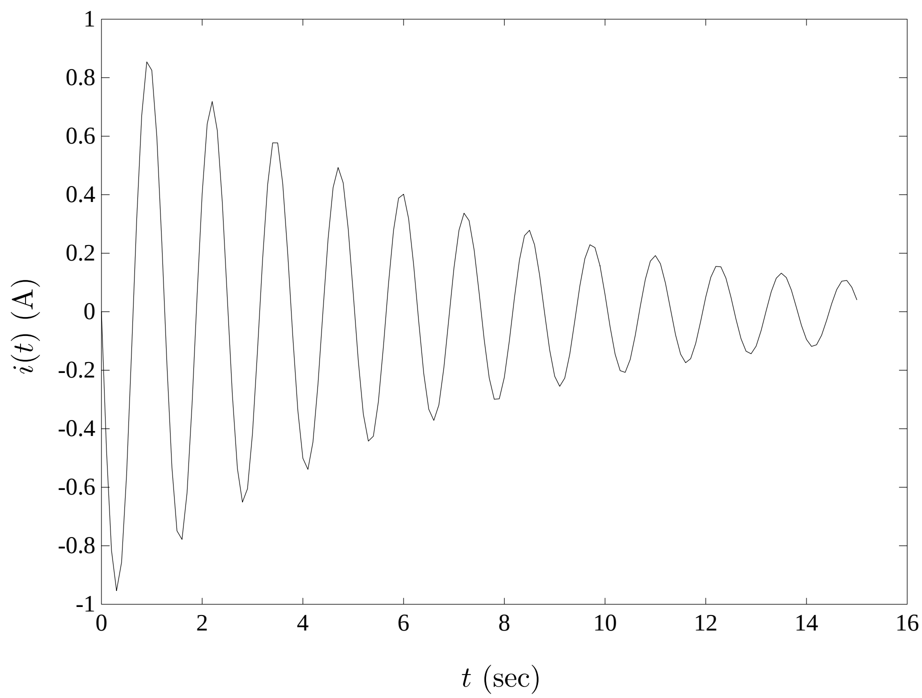


(a) Find v_{out} at $t = 0$, immediately after v_{in} has switched to 0V.

(b) Find $\dfrac{dv_{\text{out}}}{dt}$ at $t = 0$, immediately after v_{in} has switched to 0V.

(c) Find v_{out} at $t = 1$.

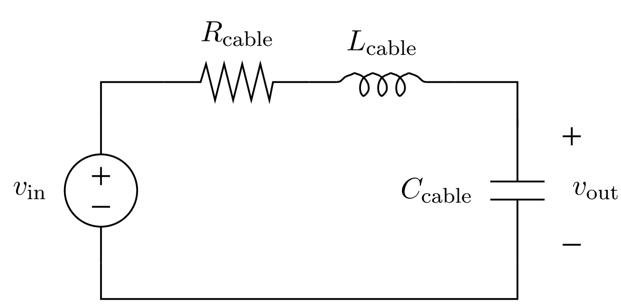
30. The waveform shown below is the current in a series RLC circuit. The value of the resistor is 100Ω.



(a) Estimate L and C.

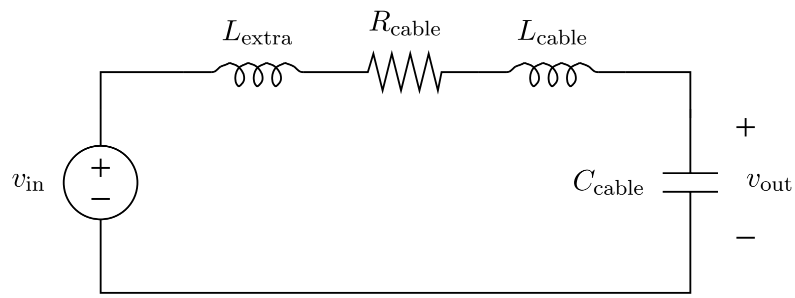
(b) About how long will it be before 99% of the initial stored energy in the circuit has dissipated?

31. *Adding inductance to improve decay rate of a cable.* A long cable, driven by a voltage source at one end and with a high impedance load at the other, is modeled as a resistance R_{cable}, inductance L_{cable}, and capacitance C_{cable}, as shown below.



As a measure of how fast the cable can react to the driving voltage, we use the *decay rate* of the system, when the voltage source v_{in} switches to 0V and stays at 0V. Recall that the decay rate is defined as $D = -\max\{\Re\lambda_1, \Re\lambda_2\}$, where λ_1 and λ_2 are the roots of the characteristic polynomial of the system (with $v_{\text{in}} = 0$). Thus, the system has positive decay rate, and a larger decay rate means that v_{out} converges to zero faster.

An engineer suggests that adding an extra inductance L_{extra} in series with the driving voltage, as shown below, might increase the decay rate of the cable.



A second engineer responds: "Adding inductance is crazy! It's precisely the inductance that slows the cable, by fighting changes in current. Adding inductance will make the cable system slower, *i.e.*, decrease the decay rate."

For the rest of this problem you can use the specific values

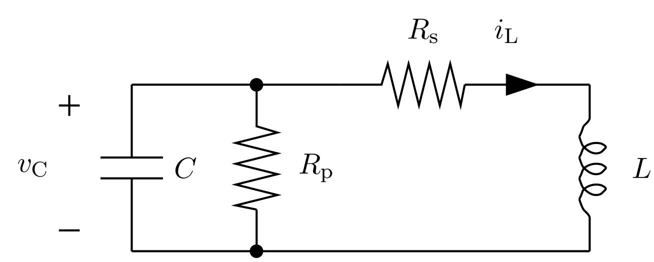
$$R_{\text{cable}} = 2000\Omega, \quad L_{\text{cable}} = 100\mu\text{H}, \quad C_{\text{cable}} = 0.01\mu\text{F}.$$

Finally, the problem: find the value of L_{extra} that yields the maximum possible decay rate for the cable. (Of course you must have $L_{\text{extra}} \geq 0$.)

If your choice is $L_{\text{extra}} = 0$, you are agreeing with the second engineer quoted above.

Explain the reasoning behind your choice of L_{extra}. We want a specific number for L_{extra}, not a formula involving other problem parameters.

32. *The RLRC circuit.* In the series RLC circuit, current flow causes power to be dissipated in the resistor as heat. In the parallel RLC circuit, voltage causes power to be dissipated. In the RLRC circuit shown below, power is dissipated by both mechanisms.



(a) Find an LCCODE of the form

$$av_c'' + bv_c' + cv_c = 0$$

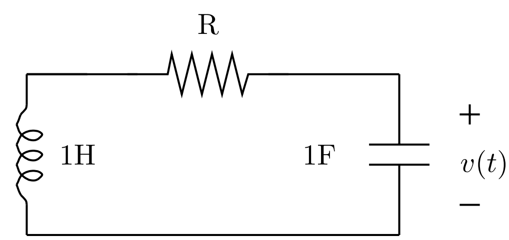
that describes this circuit.

(b) Find an expression for the rate of change of the total stored energy, *i.e.*,

$$\frac{d}{dt}\left(\frac{Li_{\mathrm{L}}(t)^2}{2} + \frac{Cv_{\mathrm{C}}(t)^2}{2}\right),$$

in terms of $i_{\mathrm{L}}(t)$ and $v_{\mathrm{C}}(t)$. Give one sentence interpreting your result.

33. *Transition from overdamped to critically damped to underdamped.* Consider three series RLC circuits with the same inductance and capacitance, $L = 1\text{H}$, $C = 1\text{F}$, and the same initial conditions: 1V across the capacitor and zero current in the circuit. The resistors in the three circuits differ slightly: in the first circuit we have $R = 1.99\Omega$; in the second circuit we have $R = 2\Omega$, and in the third circuit we have $R = 2.01\Omega$.



You know from lecture 4 that the *formulas* for the solution $v(t)$ (the voltage across the capacitor) of these three circuits are quite different: In the first case, $v(t)$ is an exponentially decaying sinusoid; in the second, it is the sum of an exponential and a term involving t times an exponential; in the last case, $v(t)$ is a sum of two decaying exponentials.

One student says:

> The voltage response is quite different in these three cases: in the first case the voltage crosses the value zero infinitely often; in the second and third cases, just once or maybe twice. So the solutions of these three circuits are indeed very different, even though the three resistor values are so close. The reason is that the value $R = 2\Omega$ is a "critical value" for this circuit, as seen in the formulas for lecture 4. It's not surprising that the solution of a circuit changes drastically as the resistance varies near a "critical value".

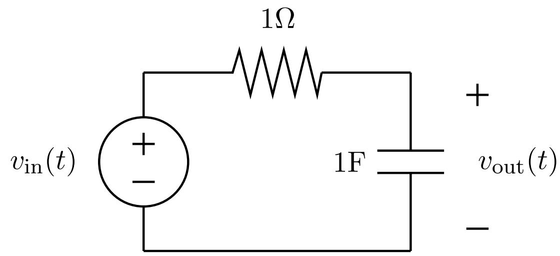
A second student then responds:

> Something is fishy here. I don't see how such a miniscule change in the resistor value can have such a great effect on the voltage across the capacitor. It just doesn't make physical sense to me. In fact, now that I think about it, exact critical damping can never be achieved in practice since it requires knowing the inductance, capacitance, and resistance with infinite precision, which is impossible.

(a) Who is right? Briefly discuss.

(b) Check your claims by plotting the three voltage waveforms using Matlab. (You might need to go back and change your answer to (a)!)

34. In the circuit shown below, $v_{\text{out}}(0) = 0$ and $v_{\text{in}}(t) = 1 - e^{-2t}$ for $t \geq 0$.



Find the output voltage, $v_{\text{out}}(t)$, for $t \geq 0$.

35. The vertical dynamics of a vehicle suspension system, when the vehicle is driving on level ground, are given by

$$(m_v + m_l)d''(t) + bd'(t) + kd(t) = 0.$$

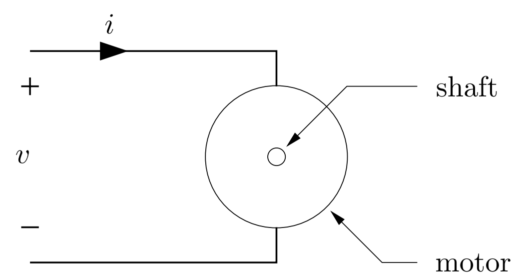
Here

- t is time (in seconds)
- $d(t)$ is the vertical displacement of the vehicle, with respect to its neutral position (in meters)
- $m_v = 10^3$kg is the vehicle mass
- $m_l \geq 0$ (also given in kg) is the mass of the vehicle load (passengers, cargo, etc.)
- $b = 2.2 \cdot 10^4$N/m/s is the suspension damping
- $k = 10^5$N/m is the suspension stiffness

The initial conditions are $d(0) = 0.1$m, $d'(0) = 0$m/s.

What is the *smallest* load mass m_l for which d is oscillatory? (By oscillatory, we mean that $d(t)$ passes through zero infinitely many times.)

36. *Stopping a DC motor.*



A DC motor is characterized by $v = k\omega + Ri + Li'$ where

- v is the voltage at its electrical terminals
- i is the current flowing into its electrical terminals
- ω is the rotational velocity of its shaft (in rad/sec)
- R is the resistance of the motor winding
- L is the inductance of the motor winding

- k is a constant called the *motor constant*

The resulting torque on the shaft is given by $\tau = ki$.

A common mechanical model for the shaft and its load is a mechanical resistance b and a rotational inertia J. Newton's law, which states that $J\omega'$ is equal to the total applied torque on the shaft, is then $J\omega' = -b\omega + \tau$.

To simplify your calculations, we'll assume the constants are

$$R = 1, \quad L = 1, \quad k = 1, \quad J = 1, \quad b = 1$$

(with, of course, the appropriate physical units).

For $t < 0$, the motor is used for some purpose (which doesn't concern us), so we have some (nonzero) initial current $i(0)$ and initial rotational velocity $\omega(0)$.

Our job is to *stop* the motor for $t \geq 0$, *i.e.*, cause $\omega(t)$ to converge to zero as $t \to \infty$. This is done by connecting its terminals to a resistance $R_{\text{ext}} \geq 0$ (the subscript 'ext' stands for 'external'), which results in $v = -R_{\text{ext}} i$.

What value of R_{ext} results in the motor velocity ω converging to zero fastest?

One engineer argues that the best thing to do is to set $R_{\text{ext}} = 0$, *i.e.*, just short circuit the motor. His argument is very simple: "applying a voltage to the motor is what makes it go, so to make it stop, set the voltage to zero by shorting the motor."

Do you agree? (I.e., is the best value $R_{\text{ext}} = 0$?)

37. For each of the following rational functions, find the poles and zeros (giving multiplicities of each), the real factored form, the partial fraction expansion, and inverse Laplace transform. (In some cases, the expression may already be in one of these forms.)

    (a) $\dfrac{1}{s+1} + \dfrac{1}{s+2} + \dfrac{1}{s+3}$

    (b) $\dfrac{s^2+1}{s^3-s}$

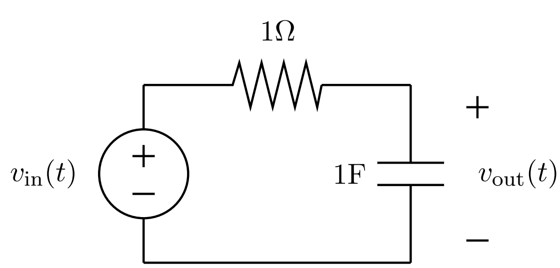
    (c) $\dfrac{(s-2)(s-3)(s-4)}{s^4-1}$

38. *Response of an RC circuit to a voltage ramp.* In the simple *RC* circuit shown below, the source voltage is given by

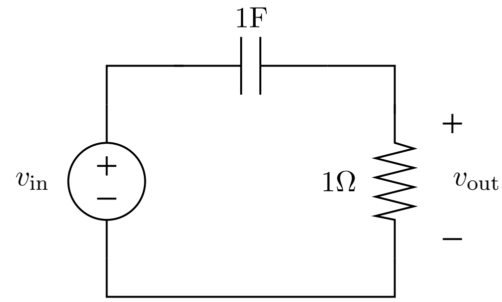
$$v_{\text{in}}(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

(which is called a 1V/sec *voltage ramp*). The capacitor is uncharged at $t = 0$.

Find $v_{\text{out}}(t)$ for $t \geq 0$.

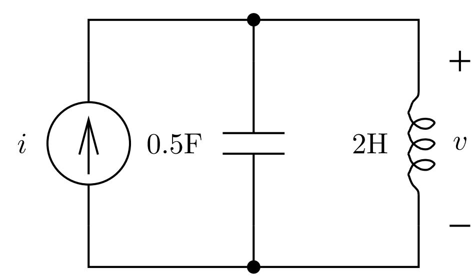


39. In the circuit below, $v_{\text{in}}(t) = 1 - e^{-2t}$ for $t \geq 0$, and the capacitor is uncharged at $t = 0$. Find $v_{\text{out}}(t)$ for $t \geq 0$.



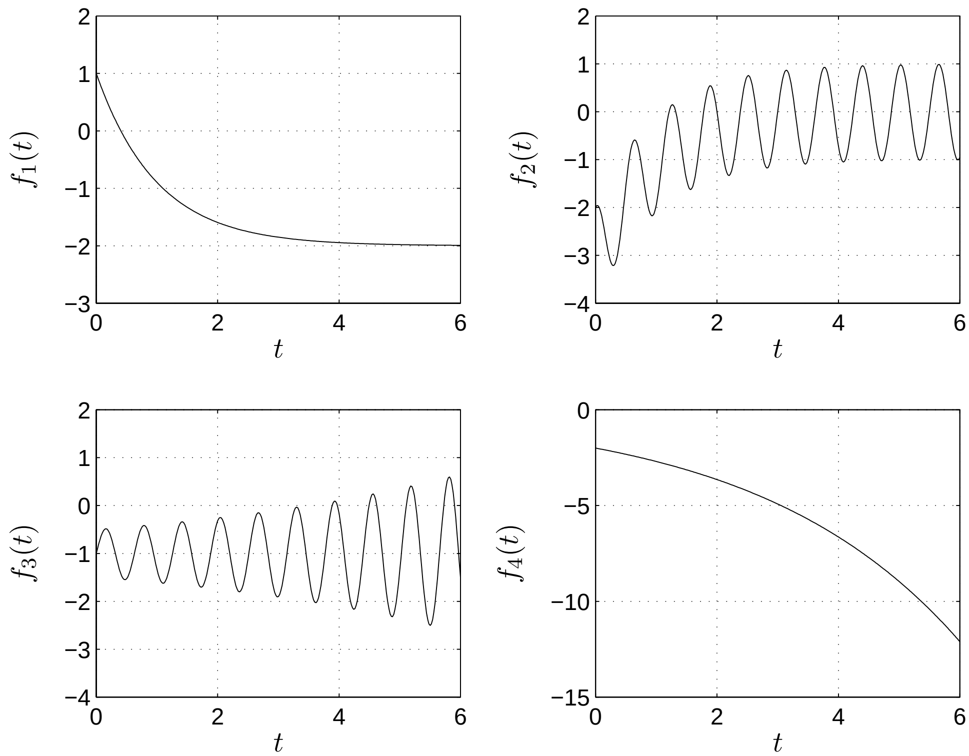
40. In the circuit below, the current i is a unit ramp at $t = 0$, *i.e.*, $i(t) = t$ for $t \geq 0$. The inductor current and capacitor voltage are both zero at $t = 0$.

Find $v(t)$ for $t \geq 0$.



41. Four functions f_1, f_2, f_3, and f_4, are shown below. Their Laplace transforms are F_1, F_2, F_3, and F_4, respectively. You can assume $F_1, \ldots, F_4$ are rational functions, with no more than three poles (counting multiplicities).

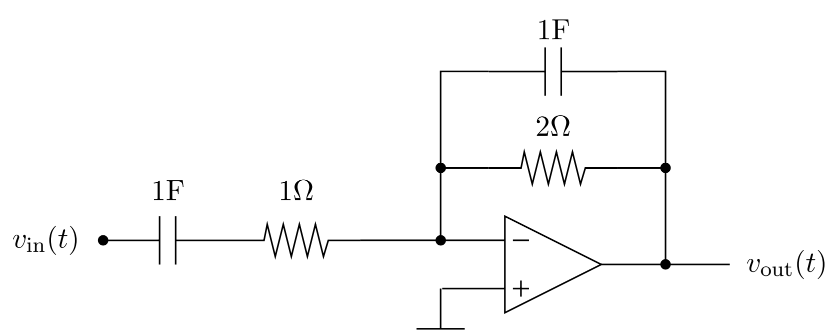
Please note carefully the vertical scales — they are *not* the same!



Estimate the poles of $F_1, \ldots, F_4$, using the smallest number needed to give a reasonable match. If you can get a reasonable match with one pole, then give just one pole; if two poles suffice then give just two. Give three poles only if three poles are required to match the given f_i.

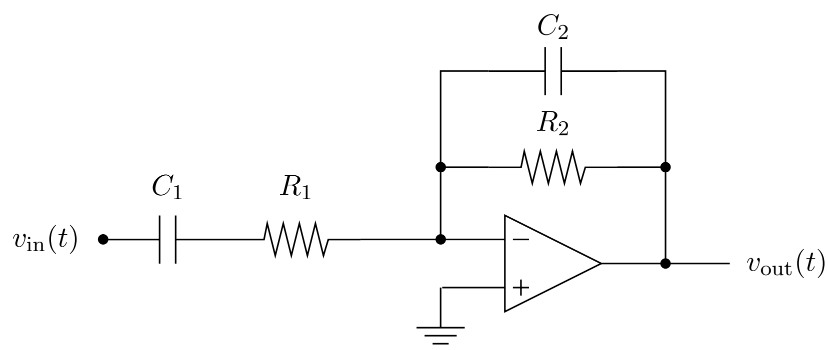
- We want *specific numbers*, not just qualitative answers such as 'one positive real, one complex pair' or $\sigma + j\omega$ (without specifying σ or ω). Make clear indications on the plots how you got the numbers.
- Give complex poles separately, as in '$1 + j$, $1 - j$'; we will *not* automatically supply conjugates of complex numbers.
- Give multiple poles by repeating them in your answers, *e.g.*, '-3, -1, -1' (meaning, one pole at $s = -3$, and another pole of multiplicity two at $s = -1$).

42. In the circuit below, the capacitors are uncharged at $t = 0$, the voltages v_{in} and v_{out} are referenced to ground, and the op-amp is ideal.



Find the DC gain, poles, and zeros of the transfer function from v_{in} to v_{out}. If the poles and/or zeros ar repeated, be sure to give the multiplicity.

43. In the circuit below, the capacitors are uncharged at $t = 0$, the voltages v_{in} and v_{out} are referenced to ground, and the op-amp is ideal.

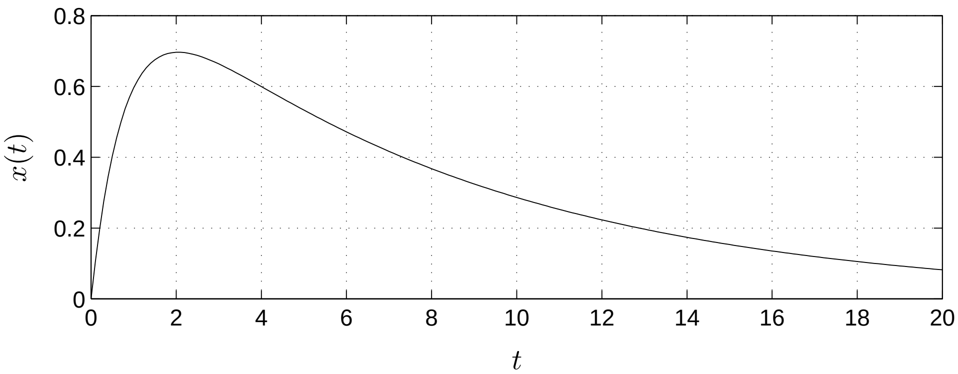


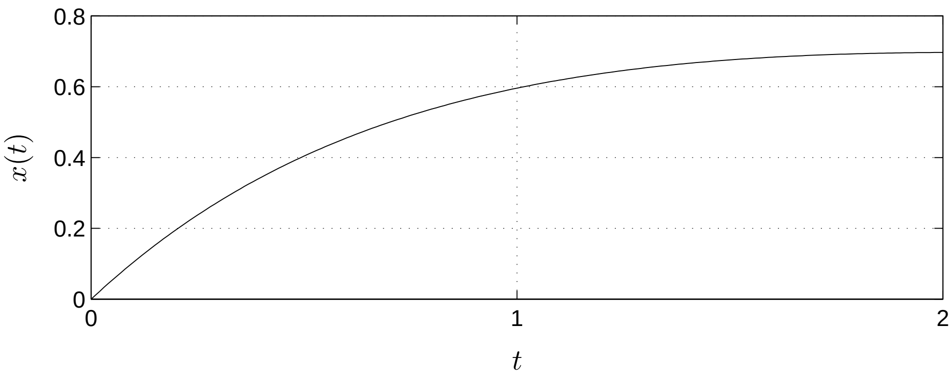
(a) Find the poles and zeros of the transfer function from v_{in} to v_{out}. If the poles and/or zeros are repeated, be sure to give the multiplicity.

(b) Now let $R_2 = 1\Omega$. Find numerical values for R_1, C_1, and C_2, so that the impulse response from v_{in} to v_{out} is $2e^{-2t} - 3e^{-3t}$. (There might be several solutions; we just want one.)

44. *Pharmacokinetics.* When a drug is administered to a patient, its concentration first rapidly increases, and then decays over a much longer period. The period of rapid increase is called the *uptake* phase, and the period of slower decay is called the *absorption* or *decay* phase.

The figures below shows a typical plot of concentration $x(t)$ versus time, with t measured in hours after the drug is administered. (The concentration might be measured in milligrams/cm^3, but it doesn't matter to us.) The first plot shows the concentration $x(t)$ over a time range $0 \leq t \leq 20$, which shows the decay phase well. The second plot shows the same concentration $x(t)$ over the shorter range $0 \leq t \leq 1$, which shows the uptake phase well.





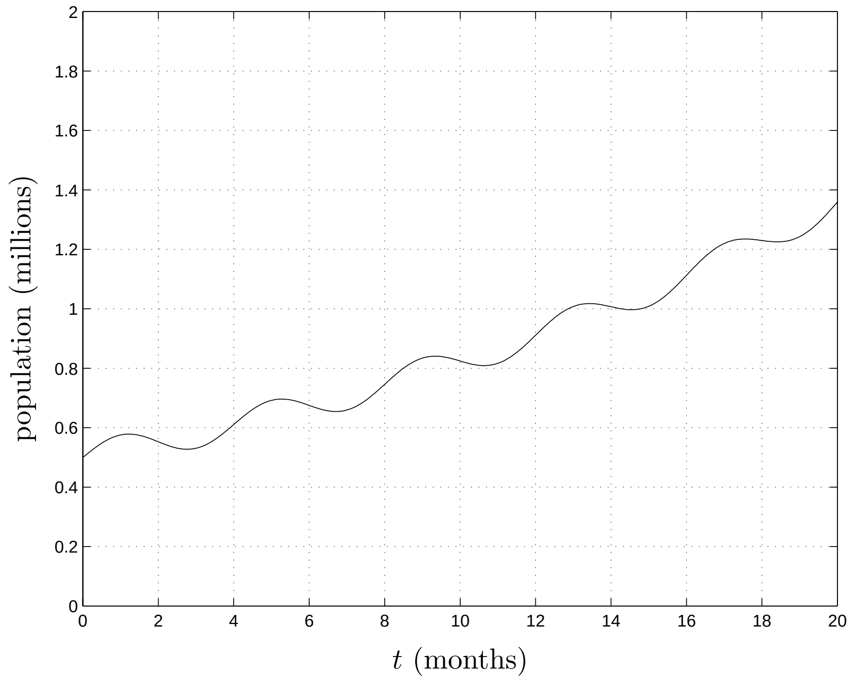
A common (approximate) model is that the concentration x satisfies a second order autonomous LCCODE:

$$x'' + bx' + cx = 0.$$

Estimate the coefficients b and c that (approximately) describe the $x(t)$ shown in the two plots above.

You **must** explain how arrive at your estimates.

45. *Modeling population dynamics.* The plot below shows the population (in millions) of some species over a period of 20 months. (The population is so large that we consider it to be a real number, ignoring the fact that it must be an integer.) As you can see, it is characterized by periodic oscillations above and below a steady growth.



A very simple way to model population dynamics is to give an autonomous linear constant coefficient differential equation that it (approximately) satisfies, of the form

$$a_0 y + a_1 y' + \cdots + a_{n-1} y^{(n-1)} + y^{(n)} = 0,$$

where $y(t)$ is the population (in millions) at time t (in months). Note that we have set the nth order coefficient to be 1, so the order of this equation is n.

Find $a_0, \ldots, a_{n-1}$ that model the plot shown above. Use the smallest degree (*i.e.*, n) that you can.

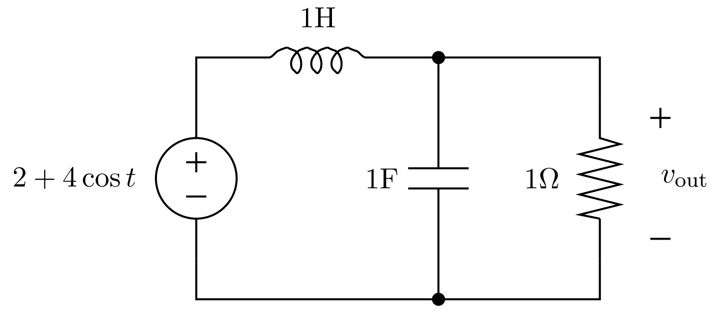
You *must* explain how you arrive at your coefficients (in the blank space on the facing page). Put your final differential equation in the box at the bottom of the facing page, in a form such as $-2y + 3.6y' + y'' - 3.1y^{(3)} + y^{(4)} = 0$ (which is *not* the answer, by the way).

46. An engineer is looking for a function v that satisfies

$$\frac{d^4 v}{dt^4} - v = 0, \qquad v(0) = 2, \qquad \lim_{t \to \infty} v(t) = 0.$$

What can you say about such a v? If you believe no such v exists, give your answer as "impossible". If you can give v expicitly, do so. If you can give a qualitative description of what such a v would look like, do so. Give the most specific answer you can.

47. Consider the circuit shown below.



(a) Suppose that the circuit is in periodic steady-state. Find $v_{\rm out}(t)$.

(b) Suppose that the inductor current and capacitor voltage at $t = 0$ are both zero. Find $v_{\rm out}(t)$. Does $v_{\rm out}(t)$ converge to the periodic steady-state solution you found in part (a), as $t \to \infty$? What are the poles of the Laplace transform of $v_{\rm out}$?

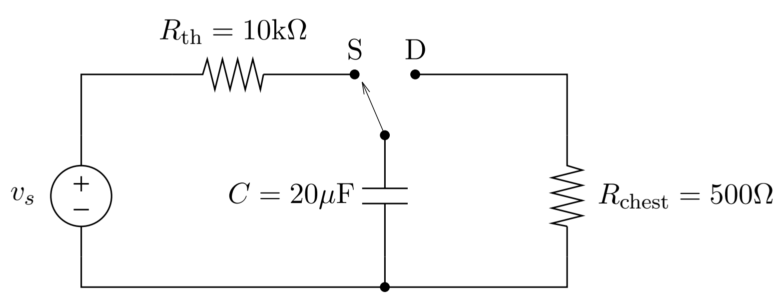
48. *How critical is critical damping?* We consider the second order LCCODE $ay'' + by' + cy = 0$, with $a, b, c > 0$. We assume that a and c are fixed, and consider the effect of the parameter b on the asymptotic decay rate of the system, defined as $D = -\max\{\Re\lambda_1, \Re\lambda_2\}$, where λ_i are roots of the characteristic polynomial. (Thus, D is positive; large D corresponds to a fast decay rate.) You know that the choice of b given by $b = b_{\rm crit} = 2\sqrt{ac}$ gives the maximum value of D, which is $D_{\rm crit} = \sqrt{c/a}$.

    Find the range of b for which the decay rate is at least 90% of the maximum possible decay rate, $D_{\rm crit}$. If you can express your answer as a percentage above and below $b_{\rm crit}$, do so.

49. Suppose that f satisfies $d^3f/dt^3 = f$, $f(0) = 1$, $df/dt(0) = d^2f/dt^2(0) = 0$. Find $f(t)$.

50. *Positive real zeros and sign changes in* f. Suppose that $F(z) = 0$ for some real, positive z. You may assume that z is such that the defining integral for the Laplace transform converges. Show that f must change sign, *i.e.*, assume both negative and positive values at various times. Another way to say this is, f cannot be nonnegative for all $t \ge 0$ or nonpositive for all $t \ge 0$.

51. *Defibrillators.* (This problem is from EE101, and is here because the next problem continues it.) A defibrillator is used to deliver a strong shock across the chest of a person in cardiac arrest or fibrillation. The shock contracts all the heart muscle, whereupon the normal beating can (hopefully) start again. The first defibrillators used the simple circuit shown below.



With the switch in the standby mode, indicated as ‘S’, the 20μF capacitor is charged up by a power supply represented by a Thevenin voltage v_s and Thevenin resistance $R_{\rm th} = 10{\rm k}\Omega$. When the switch is thrown to ‘D’ (for ‘defibrillate’), the capacitor discharges across the patient's chest, which we represent (pretty roughly) as a resistance of 500Ω. (The connections are made by two 'paddles' pushed against the sides of the chest.)

On most defibrillators you can select the 'dose,' *i.e.*, total energy of the shock, which is usually between 100J and 400J.

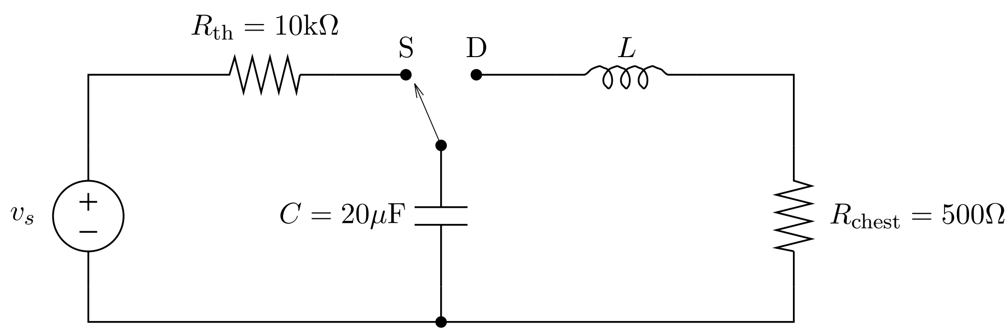
(a) Find v_s so that the dose is 100J. You can assume the capacitor is fully charged when the switch is thrown to 'D'. We'll use this value of v_s in parts 1b, 1c, and 1d.

(b) How long after the switch is thrown to 'D' does it take for the defibrillator to deliver 90% of its total dose, *i.e.*, 90J?

(c) What is the maximum power p_{max} dissipated in the patient's chest during defibrillation?

(d) Our model of the chest as a resistance of 500Ω is pretty crude. In fact the resistance varies considerably, depending on, *e.g.*, skin thickness. Suppose that the chest resistance is 1000Ω instead of 500Ω. What is the total energy E dissipated in the patient during defibrillation?

52. *An improved defibrillator.* One problem with the defibrillator described in problem 1 is that the maximum power p_{max} (which you found in part 1c) is large enough to sometimes cause tissue damage. An electrical engineer suggested the modified defibrillator circuit shown below. The inductor is meant to 'smooth out' the current through the chest during defibrillation, and yield a lower value of p_{max} for a given dose.

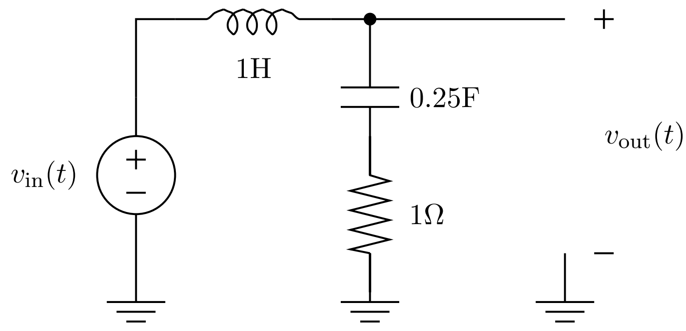


(a) Find the value of L that yields critical damping. We'll use this value of L in parts 2b and 2c.

(b) Find v_s so that the dose is 100J. You can assume the capacitor is fully charged when the switch is thrown to 'D'.

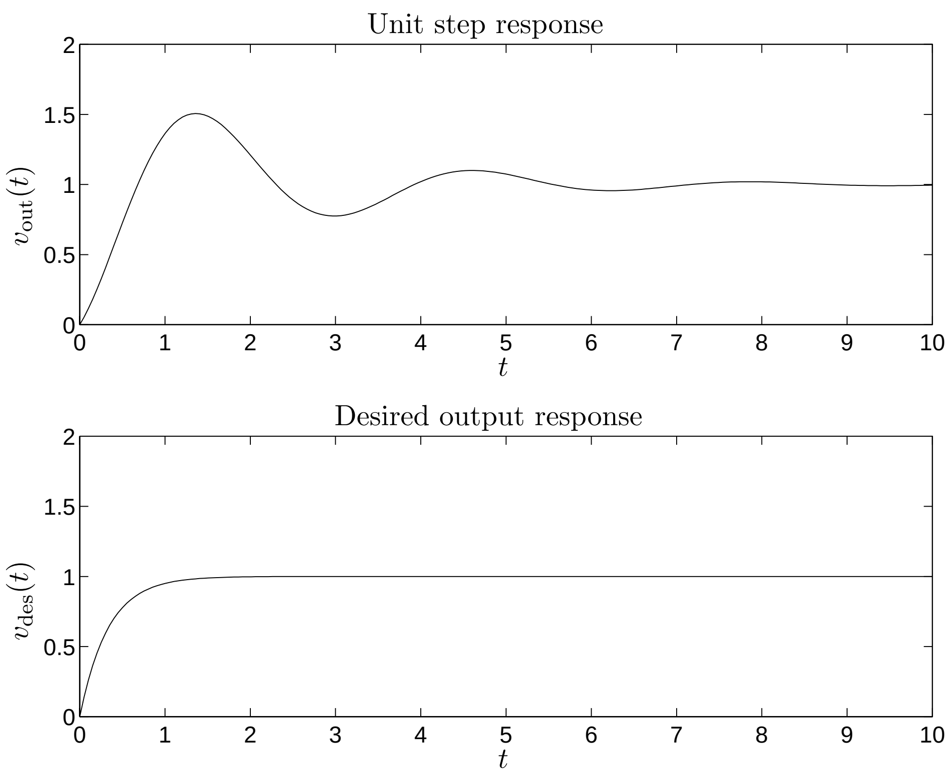
(c) Suppose v_s is equal to the value found in part 2b. What is the maximum power p_{max} dissipated in the patient's chest during defibrillation?

53. Consider the circuit shown below. You can assume the capacitor voltage and the inductor current are zero at $t = 0$.



Two plots are shown below. The top plot shows the unit step response from v_{in} to v_{out}, *i.e.*, $v_{\text{out}}(t)$ with $v_{\text{in}}(t) = 1$. Note that it exhibits some ringing, *i.e.*, oscillation, and settles (converges) in about 5sec or so.

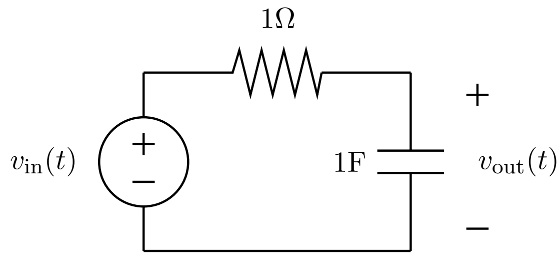
The bottom plot shows the *desired* output voltage, which is $v_{\text{des}}(t) = 1 - e^{-3t}$. Note that it exhibits no oscillation and settles quite a bit faster than the step response, *i.e.*, in about 1sec.



Finally, the problem: can you find an appropriate $v_{\text{in}}(t)$ such that we have $v_{\text{out}} = v_{\text{des}}$?

If there is no such v_{in}, give your answer as "impossible". Otherwise, give v_{in} that yields $v_{\text{out}} = v_{\text{des}}$.

54. In the circuit shown below, $v_{\text{out}}(0) = 0$ and $v_{\text{in}}(t) = 1 - e^{-2t}$ for $t \geq 0$. Find the Laplace transform $V_{\text{in}}(s)$ of $v_{\text{in}}(t)$. Find the output voltage, $v_{\text{out}}(t)$, for $t \geq 0$. (**Not** just its Laplace transform.)



55. *Reducing the rise-time of a signal.* In a certain digital system a voltage signal should ideally switch from 0V to 5V infinitely fast, *i.e.*, with zero rise-time. But due to the finite bandwidth of the electronics that generates the signal, it has the form

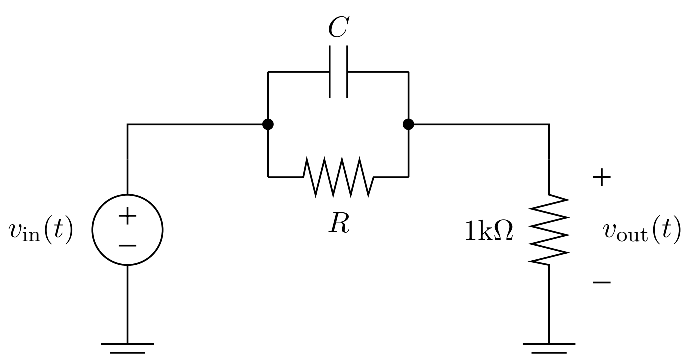
$$v_{\text{in}}(t) = 5\left(1 - e^{-t/T}\right) \quad \text{for } t \geq 0$$

where $T = 1\mu$sec. Thus, the signal has a rise-time around a few μsec.

An engineer claims that the circuit shown below can be used to reduce the rise-time of the signal, provided the component values R and C are chosen correctly. Specifically, the engineer claims that by choosing R and C correctly, we can have

$$v_{\text{out}}(t) = a\left(1 - e^{-10t/T}\right) \quad \text{for } t \geq 0$$

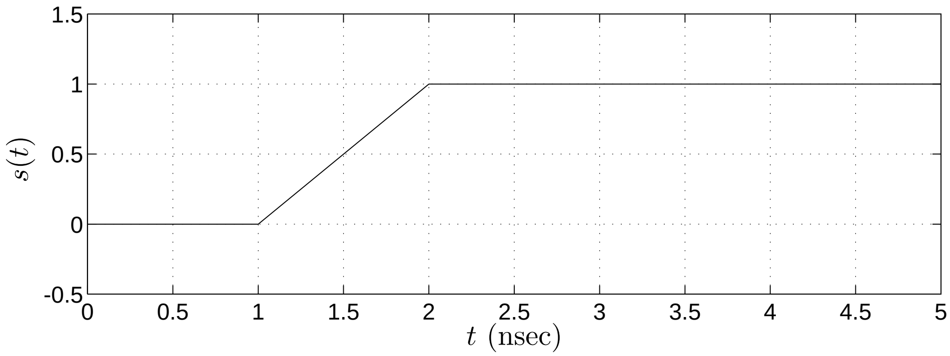
where a is some nonzero constant. Thus, the rise-time of v_{out} is a factor of 10 smaller than the rise-time of v_{in}, *i.e.*, a few hundred nsec.

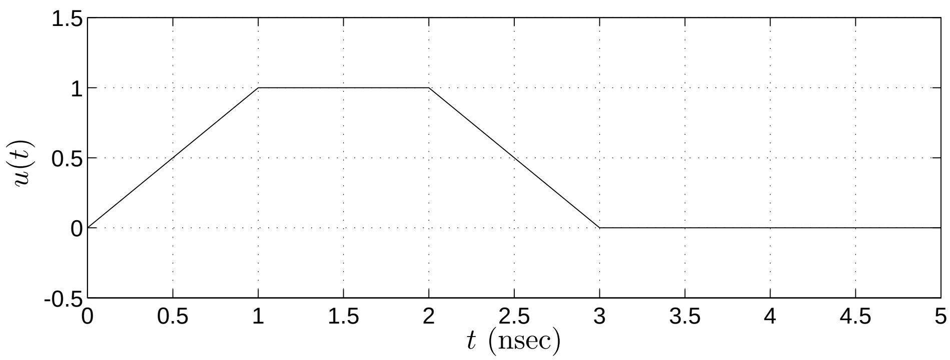


Here is the problem: determine whether the engineer's claim is true or false. If the claim is true, find specific, numerical values of R and C that validate the claim. If the claim is false, briefly explain why the engineer's idea will not work.

(You can assume the circuit starts in the relaxed state, *i.e.*, no charge on the capacitor. And no, you cannot use negative R or C.)

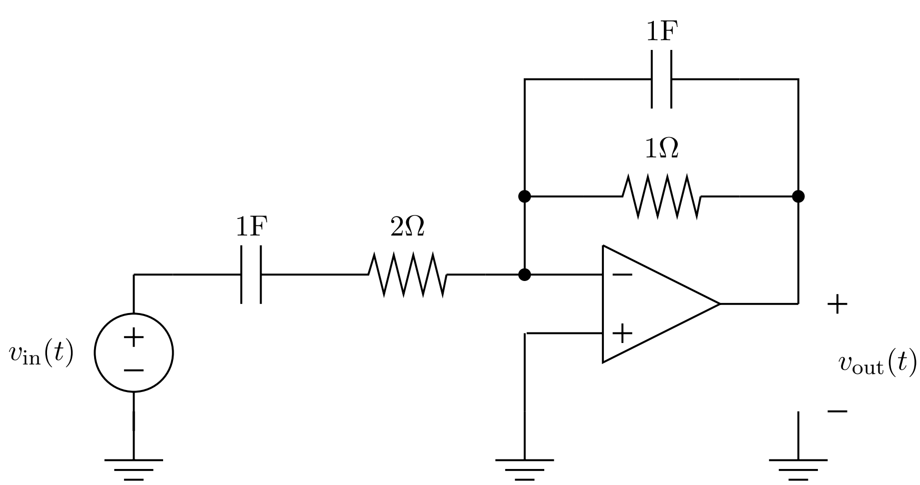
56. The top plot below shows the step response of a system described by a transfer function. Below that is a plot of an input $u(t)$ that we apply to this system. Sketch the response (output) $y(t)$.





57. Find the unit step response, from input current to shaft angle, of the DC motor described in lecture 8. What is the asymptotic angular velocity, *i.e.*, what does the angular velocity approach as t increases? Can you give a physical interpretation of your result?

58. In the circuit shown below you may assume the op-amp is ideal, and the voltage across each of the capacitors is zero at $t = 0$.

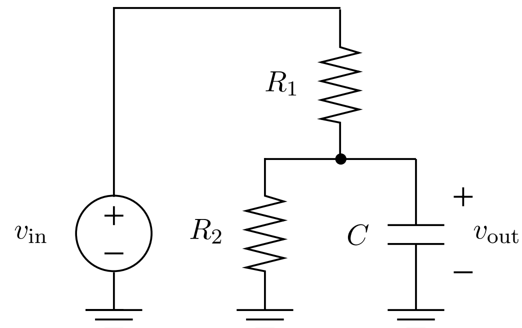


(a) Find the transfer function H from v_{in} to v_{out}. Try to express H in simple form.

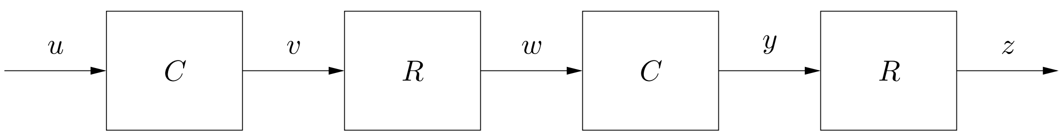
(b) Find the poles, zeros and DC gain of H.

(c) Suppose that $v_{\text{in}}(t) = 1$ for $t \geq 0$. Find $v_{\text{out}}(t)$.

59. In the circuit below, $v_{\text{in}}(t) = 2\sin(5t)$ for $t \geq 0$, $R_1 = R_2 = 1\Omega$, $C = 1F$, and the capacitor is initially uncharged at $t = 0$. Find $v_{\text{out}}(t)$ for $t \geq 0$.



60. *A cascade system with repeaters.* This problem concerns the block diagram shown below. The system consists of four cascaded subsystems: two identical channels, denoted C, and two identical repeaters, denoted R. The input signal u first propagates through a channel, then a repeater, then another channel, and finally a repeater.



Each channel C is described by a differential equation relating its input and output. If a is the input signal and b is the output signal of C, *i.e.*, $b = Ca$, we have

$$b' + b = a.$$

You can assume that the initial condition of each of the channels is zero, *i.e.*, $v(0) = y(0) = 0$.

The repeaters are described as follows. If a is the input signal to a repeater R and b is the output signal (*i.e.*, $b = Ra$), we have

$$b(t) = \begin{cases} 1 & \text{if } a(t) \geq 0.5 \\ 0 & \text{if } a(t) < 0.5 \end{cases}$$

for all $t \geq 0$. In other words, the repeater puts out 1 when the input signal is at least the threshold value 0.5, and puts out 0 when the input signal is less than the threshold.

Finally, the problem. The input signal u is a unit step at $t = 0$, *i.e.*, $u(t) = 1$ for $t \geq 0$. Find the output signal z.

61. *Pole location mix and match.* The plots shown at the end of this problem show five different signals (*i.e.*, functions of time), labeled a, b, c, d, and e.

We are interested in the poles of the Laplace transform of each signal. For each signal, identify the poles from the eight choices given below, which are labeled **I**, ..., **VIII**. For example, give your answer for b as **II**, if you believe that $B(s)$ has two poles, at $s = -2 + 0.3j$ and $s = -2 - 0.3j$.

Here are the choices for poles (which include multiplicities):

I: $-0.2 + j$, $-0.2 - j$

II: $-2 + 0.3j$, $-2 - 0.3j$

III: -4, -0.01

IV: 0, $-0.2 + j$, $-0.2 - j$

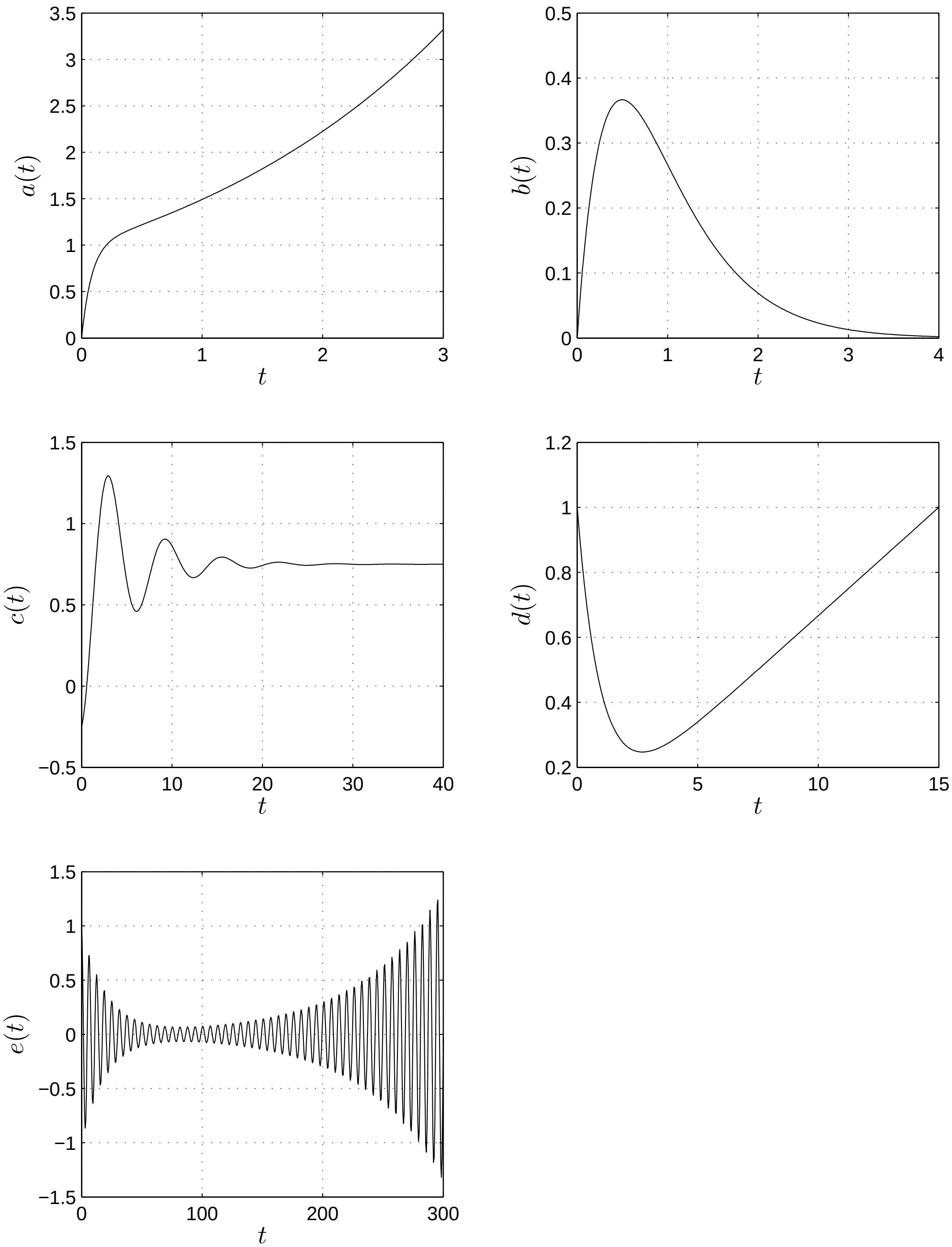
V: 0.4, -12

VI: $-0.05 + j$, $-0.05 - j$, $0.015 + j$, $0.015 - j$

VII: -0.05, $0.015 + j$, $0.015 - j$

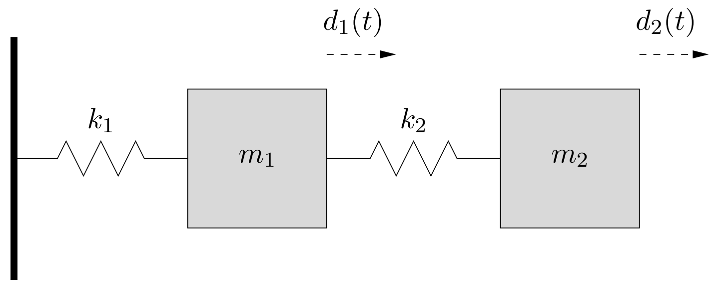
VIII: 0, 0, -1

Each of these choices of poles corresponds to at most one of the given signals, though, of course, some of the above choices correspond to none of the signals. Please note that the scales of the plots matter!



62. *Oscillation frequencies of a coupled mass-spring system.*

This problem concerns the mechanical system shown below. A mass m_1 is connected via a spring with stiffness k_1 to a rigid wall (shown as the dark bar at left) and also, via a spring with stiffness k_2, to a second mass m_2. The displacements of the masses (with respect to the equilibrium positions) are denoted $d_1(t)$ and $d_2(t)$, respectively.



This two mass system is governed by Newton's equations of motion, which are

$$m_1 d_1'' = -k_1 d_1 + k_2(d_2 - d_1), \qquad m_2 d_2'' = -k_2(d_2 - d_1).$$

For simplicity, we'll use the specific numbers

$$m_1 = m_2 = 1\text{kg}, \qquad k_1 = k_2 = 1\text{N/m},$$

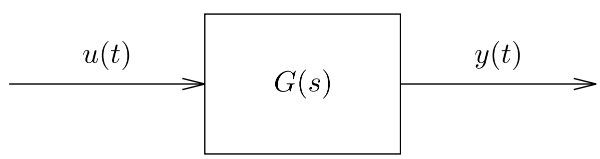
(and, of course, we measure t in seconds and the displacements in meters).

The general form of d_2 (and, for that matter, d_1) is given by

$$d_2(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2),$$

where the constants $A_1,\ A_2,\ \phi_1,\ \phi_2$ are determined by the initial conditions. Find the two frequencies ω_1 and ω_2. You can give your answer in an analytical form, or in numerical form (to three significant figures).

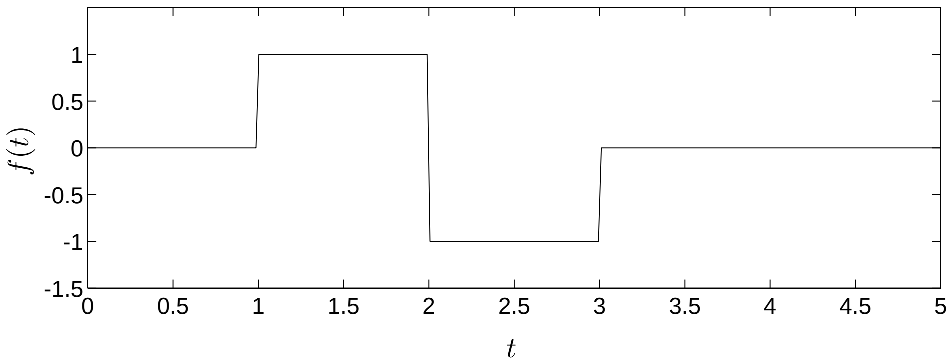
63. The system shown below is described by a transfer function G. The poles of G are at $s = -1$ and $s = -4$; G has only one zero, at $s = -2$. The DC gain of G is 1.

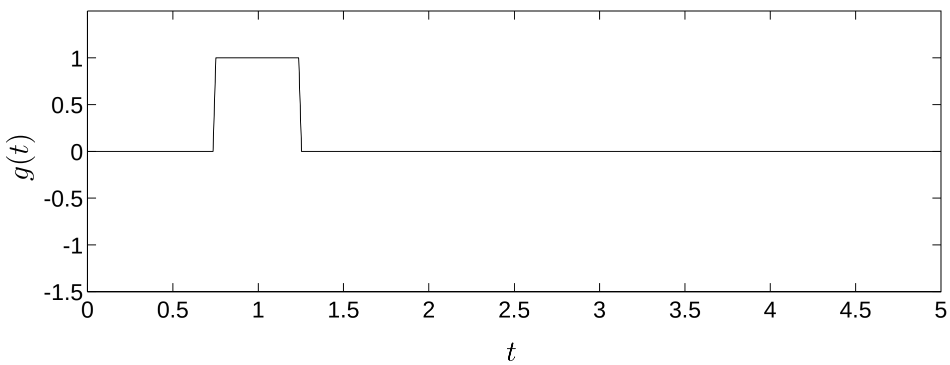


(a) Find the impulse response $g(t)$ of this system.

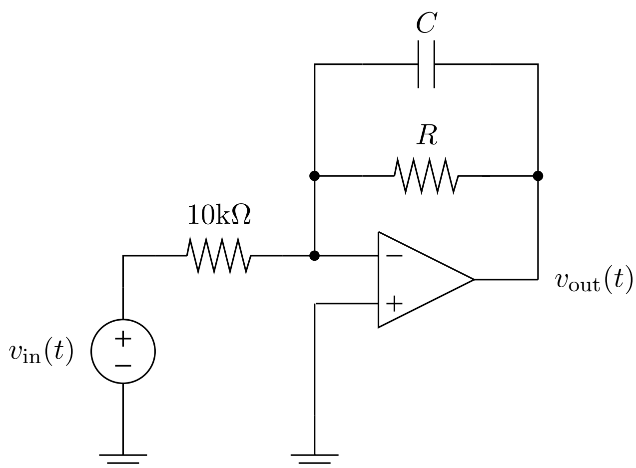
(b) Suppose that $u(t) = e^{-2t}$ for $t \geq 0$. Find $y(t)$.

64. The signals f and g are plotted below. Plot $f * g$.





65. The circuit below is a simple one-pole lowpass filter.



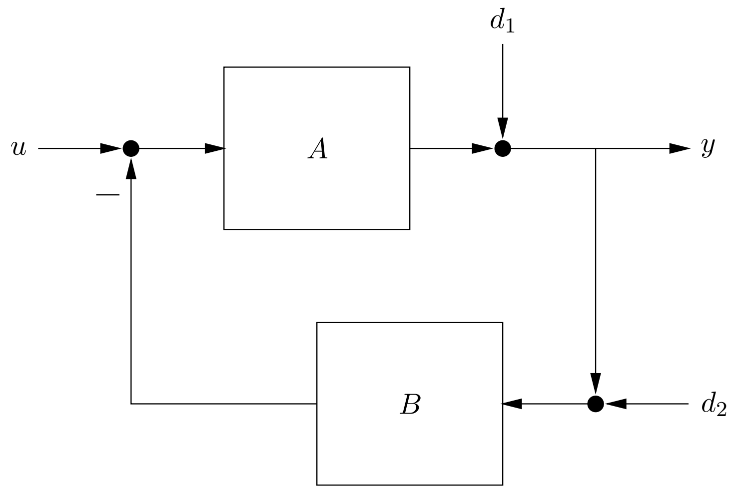
Find (positive) R and C such that:

- The (magnitude of the) DC gain is +12dB.
- The magnitude of the transfer function at the frequency 1kHz is 3dB less than the magnitude of the DC gain.

You can assume the op-amp is ideal. Give numerical values for R and C. An acurracy of 10% will suffice.

66. *A feedback system with two disturbances.* The system below shows a feedback system, with input u, output y, and two disturbance signals, d_1 and d_2 (that are also input signals). The

two systems A and B are scalar gains, with gains α and β, respectively. (In this context α is called the *forward gain* and β is called the *feedback gain*.) Note that on the left we have a difference junction; on the right and bottom we have summing junctions.



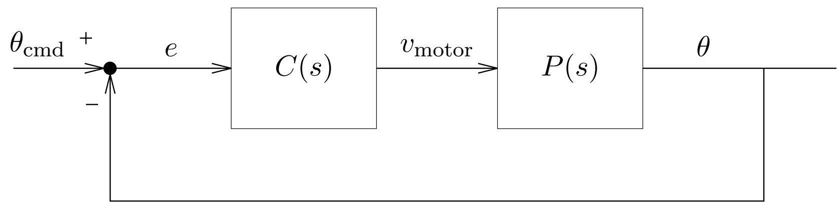
(a) Express y in terms of u, d_1, and d_2. You can assume that $\alpha\beta \neq -1$.

(b) Now consider the (typical) values $\alpha = 10^5$, $\beta = 10^{-2}$. Which of the two disturbances has a greater effect on the output y, assuming the disturbances have the same size? Your answer must be one of:

- d_1 (*i.e.*, d_1 has a greater effect on y than d_2)
- d_2 (*i.e.*, d_2 has a greater effect on y than d_1)
- the same (*i.e.*, d_1 and d_2 have the same effect on y)
- can't determine (*i.e.*, we can't determine which disturbance has the greater effect on y, from the data given)

67. *A motor controller.* In the feedback system shown below, the plant is a DC motor. Its input is the applied motor voltage v_{motor}, and its output is the motor shaft angle θ. The controller is a simple proportional controller with a gain of 2, *i.e.*, $C(s) = 2$. The plant transfer function is

$$P_{\text{motor}}(s) = \frac{1}{s(s+3)}.$$



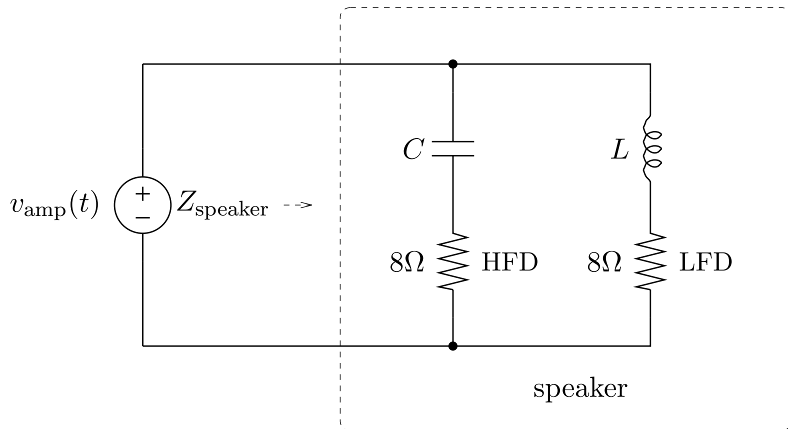
(a) Find the (closed-loop) transfer function T from the commanded shaft angle θ_{cmd} to the motor shaft angle θ.

(b) Suppose that θ_{cmd} is a unit step. Find the motor shaft angle $\theta(t)$, and the motor voltage $v_{\text{motor}}(t)$.

68. *A simple two-way crossover circuit.* A typical high-fidelity speaker has separate drivers for low and high frequencies. (The driver is the physical device that vibrates to create the sound you hear. The old terms for the low and high frequency drivers are *woofer* and *tweeter*, respectively.)

The circuit shown below, called a *speaker crossover network*, is used to divide the audio signal coming from the amplifier into a low frequency part for the low frequency driver (LFD) and a high frequency part for the high frequency driver (HFD). Since the audio spectrum is divided into two parts, this is called a two-way system (three-way are also common).

The amplifier is modeled as a voltage source (which is a very good model), and the low and high frequency drivers are modeled as 8Ω resistances (which is not a good model of real drivers, but we will use it for this problem).



The crossover network is designed so that the transfer function from the amplifier to each driver has magnitude −3dB at a frequency ω_c called the *crossover frequency* of the speaker.

(a) Choose C and L so that the crossover frequency is 2kHz. Do this carefully as you will need your answers in part b.

(b) Using the values for L and C found in 8a and 8b, find $Z_{\rm speaker}(s)$, the impedance of the two-way speaker seen by the amplifier (as indicated in the schematic).

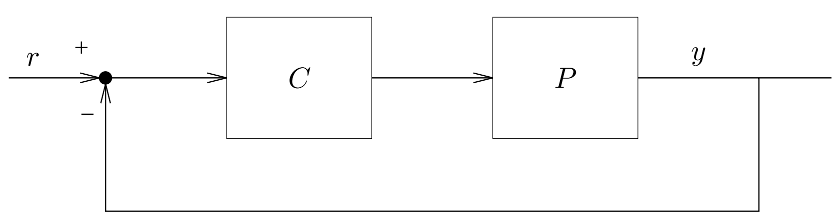
69. *Stability analysis of a PI controller.* Suppose that PI control,

$$C(s) = k_p + \frac{k_i}{s},$$

is used with the plant

$$P(s) = \frac{1}{s^2 + 2s + 2},$$

in the standard feedback control configuration:

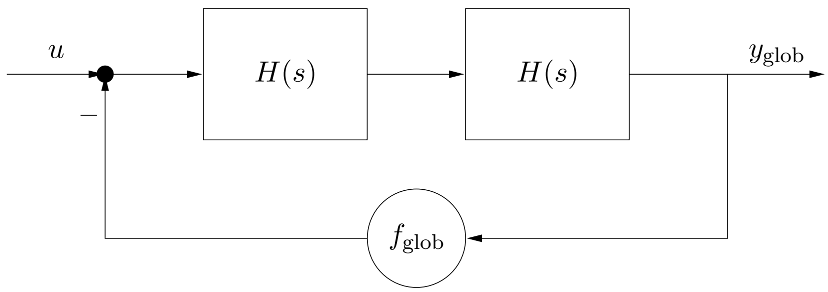


Find the conditions on k_p and k_i for which the closed-loop transfer function T from r to y is stable. You can assume that $k_p > 0$ and $k_i > 0$.

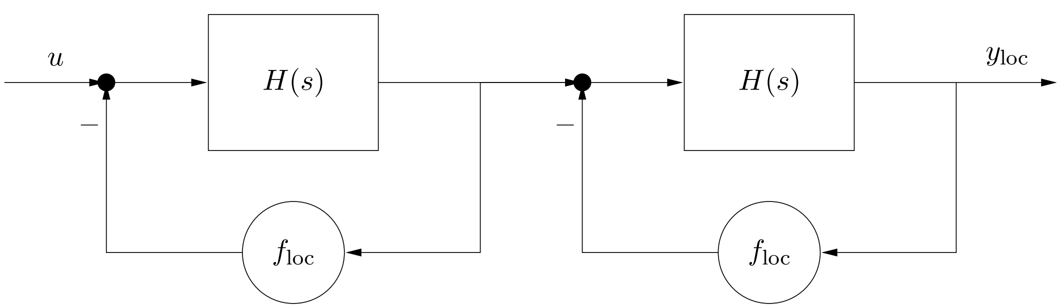
Express your conditions in the simplest form possible.

70. *Local versus global feedback: dynamic analysis.* We consider two amplifiers, each with a transfer function $H(s) = 100/(1 + s)$. We are going to use these amplifiers, together with feedback, to design a system with DC gain 40dB. (Roughly speaking, then, we have 40dB of 'extra' gain.)

Global feedback. In this arrangement we connect the two amplifiers in cascade, and then use feedback around the cascade connection, as shown below.



Local feedback. In this arrangement we use feedback around each amplifier, and then put the two closed-loop systems in cascade, as shown below. (To simplify things, we'll assume the two feedack gains are the same.)



In both arrangements, the feedback consists of a positive gain (independent of s).

(a) Find the value of $f_{\rm glob}$ that makes the closed-loop DC gain from u to $y_{\rm glob}$, in the global feedback arrangement, equal to 40dB.

(b) Find the value of $f_{\rm loc}$ that makes the closed-loop DC gain from u to $y_{\rm loc}$, in the local feedback arrangement, equal to 40dB.

We'll define the *settling time* of a system as the time it takes for the unit step response to settle to within about 10% of its final (asymptotic) value. If the system is unstable, we'll say the settling time is ∞.

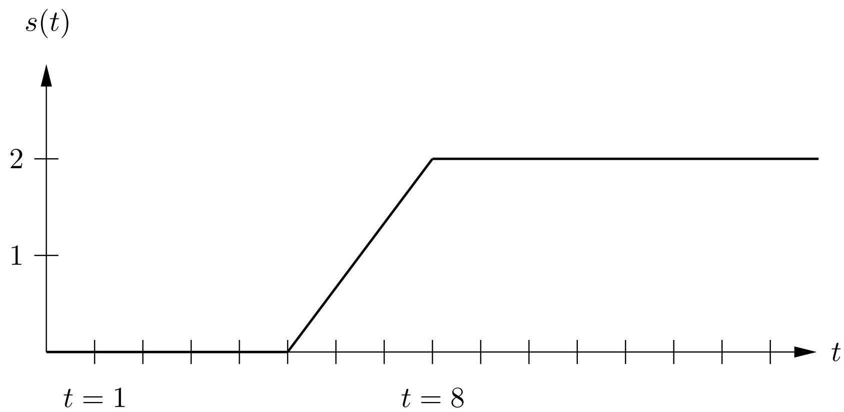
(c) Using the value for $f_{\rm glob}$ found in part 3a, find the settling time $T_{\rm glob}$ of the global feedback system.

(d) Using the value for $f_{\rm loc}$ found in part 3b, find the settling time $T_{\rm loc}$ of the local feedback system.

(e) Give a *one sentence intuitive explanation* of the results of 3c and 3d.

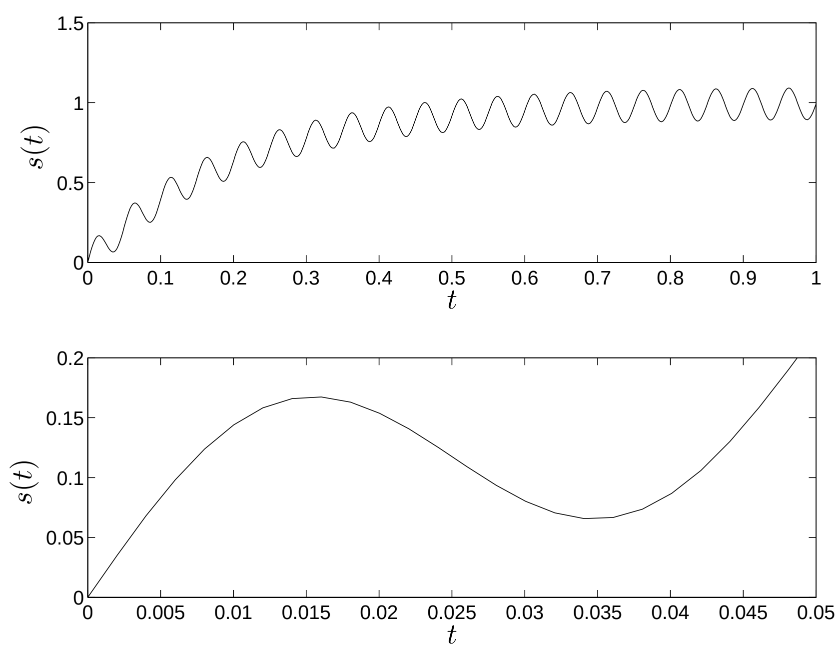
(f) You can estimate the settling times, provided you explain how you obtain the estimate. An accuracy of $\pm 30\%$ is fine.

71. *Step response of cascaded systems.* A system described by a transfer function has the step response shown below.



Now suppose that two such systems (*i.e.*, each with the step response shown above) are connected in cascade. At the bottom of this page sketch the step response of the resulting cascade connection. Make sure that all critical parts of your plot are clearly labeled. Be sure to make clear which portions of your plot are curved or straight.

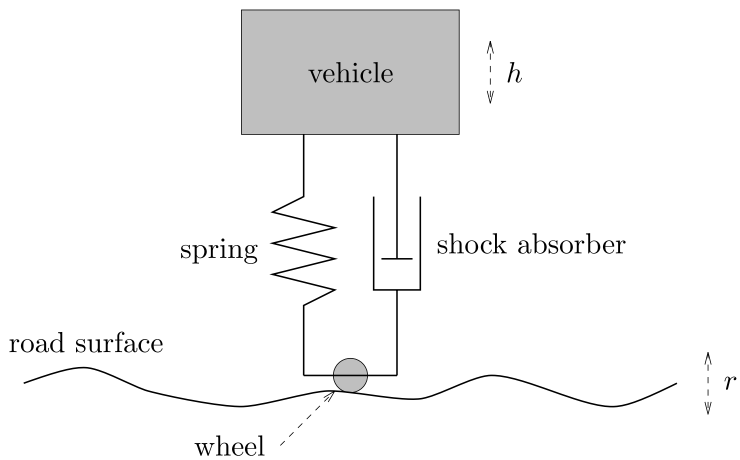
72. The unit step response $s(t)$ of a system described by a transfer function H, which has three poles, is shown in the two plots below. The two plots have different ranges; the second plot allows you to see details for small t.



(a) Estimate the poles. An accuracy of $\pm 20\%$ is acceptable.

(b) At high frequencies $|H(j\omega)|$ becomes small. From the data given, can you determine the rate at which it decreases for large frequency (*e.g.*, 12 db/octave)? Either give the rate (in dB/octave) or state "cannot determine" if the data given is not sufficient to determine the high-frequency rolloff rate.

73. *Analysis of a vehicle suspension.* A vehicle moves at a constant speed over a road surface. The suspension is modeled as a spring and a shock absorber between the vehicle and the wheel, which is assumed to remain in contact with the road. The height of the vehicle (above or below some reference level), at time t, is denoted $h(t)$. The height of the road surface at the wheel (above or below some reference level), at time t, is denoted $r(t)$. This is illustrated below.



Newton's equation for the (vertical) motion of the vehicle is

$$m\frac{d^2h}{dt^2} = -k(h(t) - r(t)) - b\frac{d}{dt}(h(t) - r(t))$$

where m is the vehicle mass, k is the spring stiffness, and b is the mechanical resistance of the shock absorber. In this problem you can use the values

$$m = 1, \quad k = 2, \quad b = 3.$$

Note: you do not need to know (remember?) any mechanics, how a suspension or shock absorber works, etc. All you need is the equation given above.

(a) *Running over a sinusoidal road surface.* Suppose the road surface is sinusoidal: $r(t) = \cos 2t$. You can assume the vehicle height h is also sinusoidal. Find the RMS value of the vehicle height h. (This RMS value can serve as a measure of how smooth the ride is.)

(b) Find the transfer function $G(s)$ from the road height r to the vehicle height h.

(c) *Running over a curb.* Suppose that the road height $r(t)$ is 0 for $t < 0$ and 1 for $t \geq 0$. In other words, the vehicle is running over smooth, level ground until it hits a unit height curb at $t = 0$. You can assume that for $t < 0$, $h(t) = 0$ (and hence, $dh/dt = 0$). Find the vehicle height $h(t)$ for $t \geq 0$ (*i.e.*, after it hits the curb).

74. *Transfer function from rainfall to river height.* The height of a certain river depends on the past rainfall in the region. Specifically, let $u(t)$ denote the rainfall rate, in inches-per-hour, in a region at time t, and let $y(t)$ denote the river height, in feet, above a reference (dry period) level, at time t. The time t is measured in hours; we'll only consider $t \geq 0$.

Analysis of past data shows that the relation between rainfall and river height can be accurately described by a transfer function:

$$Y(s) = H(s)U(s), \quad H(s) = \frac{10}{(3s+1)(30s+1)}$$

(You don't need to know any hydrology to do this problem, but you might be interested in the physical basis of this two-pole transfer function. The fast pole is due to runoff from surface water and small tributaries, which contribute a relatively small amount of water relatively quickly. The slow pole is due to flow from larger tributaries and deeper ground water, which contribute more water into the river, over a much longer time scale.)

A brief but intense downpour. (Parts a and b.) Suppose that after a long dry spell (*i.e.*, no rain) it rains intensely at 12 inches-per-hour, for 5 minutes. This causes the river height to rise for a while, and then later recede.

(a) How long does it take, after the beginning of the brief downpour, for the river to reach its maximum height? We'll denote this delay as t_{max} (in hours).

(b) What is the maximum height of the river? We'll denote this maximum height as y_{max} (in feet).

Note: you can make a reasonable approximation provided you say what you are doing.

A continual rain. (Parts c and d.) Suppose that after a long dry spell it starts raining continuously at a rate of 1 inch-per-hour (and doesn't stop). This causes the river height to rise.

(c) What is the ultimate height of the river, *i.e.*, $y_{\text{ult}} = \lim_{t\to\infty} y(t)$?

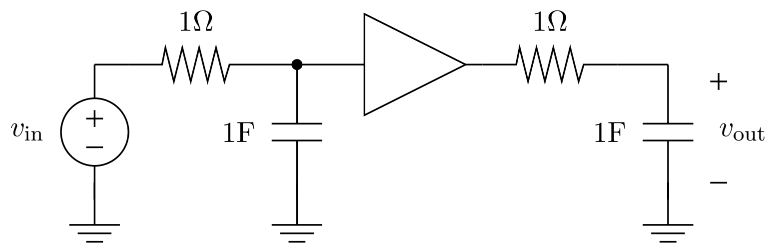
(d) A flood occurs when the river height $y(t)$ reaches 8 feet. How long will it take, after the onset of the steady rain, to reach flood condition? We'll denote this time as t_{flood}. If the river never reaches 8 feet, give your answer as 'never'.

Note: you can make a reasonable approximation provided you say what you are doing.

75. *Wire with repeater amplifier.* A voltage source drives a long cable, which has significant capacitance and series resistance (but negligible inductance). A repeater amplifier (shown as a triangle) is inserted halfway down the cable. The output voltage of the repeater amplifier (on its right) is exactly equal to its input voltage (on its left), and no current flows into its input terminal. A simple electrical model of this is shown below.

(Note that we have chosen simple, but unrealistic, numerical values for the wire series resistance and capacitance, so you don't have to worry about picofarads or nanoseconds here.)

You can assume both capacitors are uncharged at $t = 0$.



(a) Find the transfer function H from v_{in} to v_{out}.

(b) Suppose the voltage source v_{in} is a unit step. Find v_{out}.

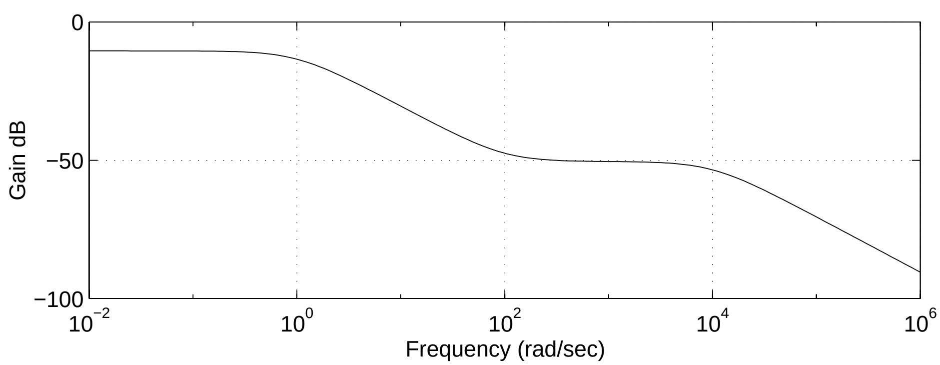
76. Sketch the Bode plot of the transfer function

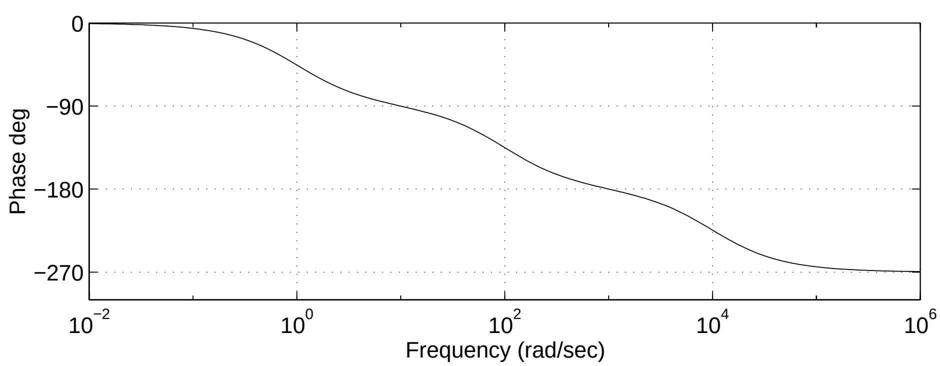
$$H(s) = \frac{s^2 - 0.1s + 4}{s^2 + 0.1s + 1}.$$

Use a magnitude range of −40dB to +40dB, and a phase range of −360° to +360°.

77. The Bode plot of a transfer function H is shown below. Estimate the DC gain and the poles and zeros of H.

Use the smallest number of poles and zeros that give a reasonable fit to the plot. Be sure to clearly indicate the multiplicity of any pole or zero that is repeated. If there are no zeros (or poles) then give your answer below as 'none'.

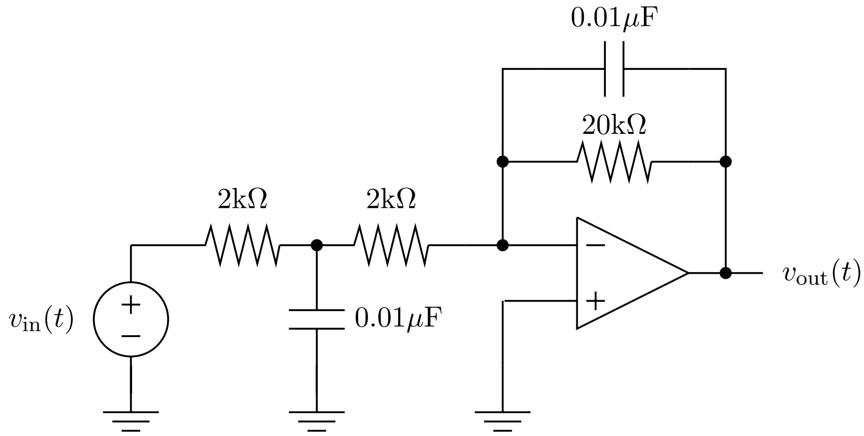




78. An amplifier with a transfer function H has a DC gain $H(0) = 10^3$, poles at $s = -100$ rad/sec and $s = -10^6$ rad/sec, and a zero at $s = +10^4$ rad/sec. (Note the signs of the poles and zeros!)

Sketch the Bode plot of H.

79. This problem concerns the circuit shown below. You can assume that the op-amp is ideal.



(a) Find the transfer function H from v_{in} to v_{out}.

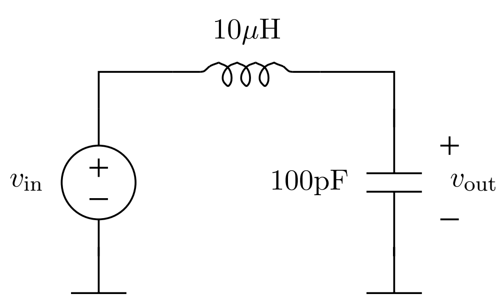
(b) Plot the poles and zeros in the complex plane. Verify that this circuit is stable.

(c) Sketch the Bode plot of H. Would you describe this system as low-pass, band-pass, high-pass, or none of these?

(d) Assume the capacitors are initially uncharged. Suppose that for $t \geq 0$, v_{in} is a sinusoid with amplitude 1V, frequency 3kHz, and phase 0°. You know that v_{out} will approach the sinusoidal steady-state response as $t \to \infty$. But how long will it take? Find a time T such that for $t \geq T$ the actual and steady-state responses are within about 1% of the amplitude of the steady-state response. Your number T does not have to be the smallest possible such T, just within a factor of two or three.

(e) Can you find appropriate initial capacitor voltages such that the system is in sinusoidal steady-state *immediately*, *i.e.*, from $t = 0$ on?

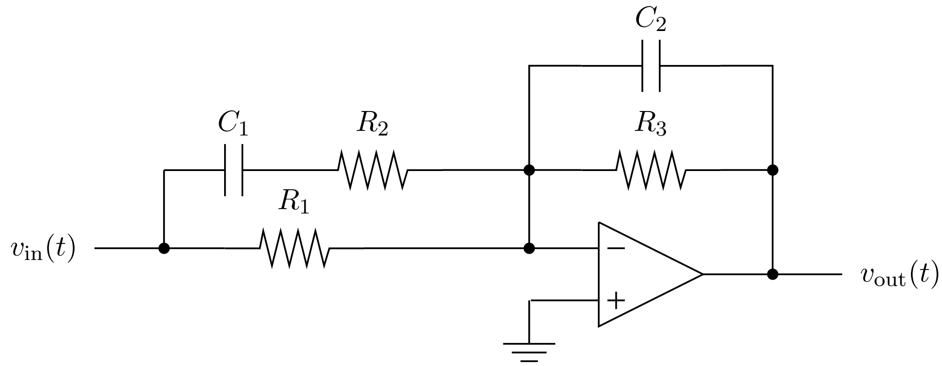
80. A voltage source drives a capacitive load through a wire that has significant inductance, as shown in the circuit below.



(a) Find the transfer function H from v_{in} to v_{out}. (You can assume, of course, that the initial conditions are zero.)

(b) Find the smallest frequency ω for which $|H(j\omega)|$ deviates 0.5dB or more from its DC value. (This can be intepreted as the frequency below which the inductance and capacitance can be ignored.)

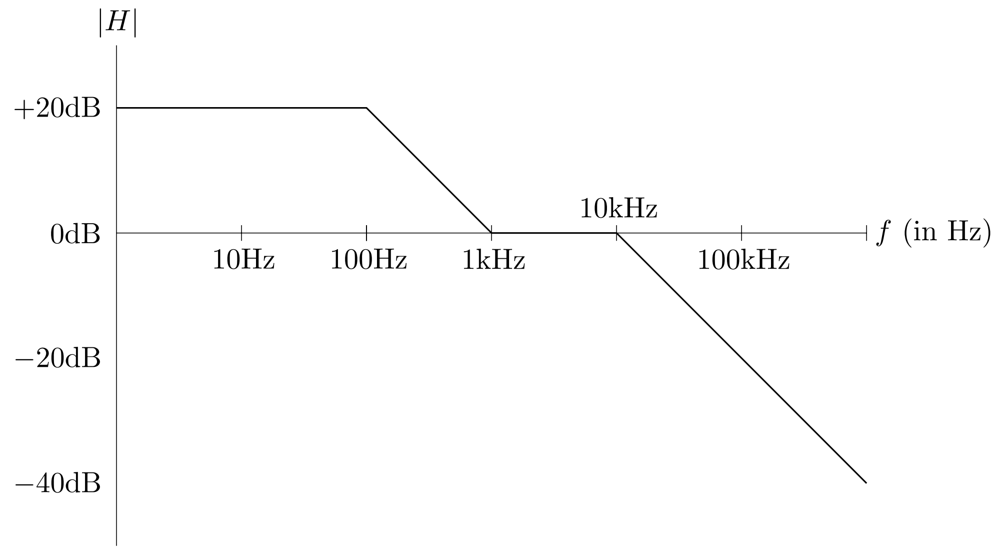
81. *An op-amp filter circuit.* This problem concerns the filter circuit shown below. The voltages v_{in} and v_{out} are with respect to ground, and the op-amp is ideal. The transfer function from v_{in} to v_{out} will be denoted H.



(a) Find the DC gain, poles, and zeros of H. (Express them in terms of the component values R_1, R_2, R_3, C_1, and C_2.) If there are no zeros (or poles), give your answer as 'none'. Express your answers in a simple form, and check them carefully, since you may want to use them in parts b and c.

(b) Suppose that $R_1 = R_2 = R_3 = 1\Omega$ and $C_1 = C_2 = 1\text{F}$. Find the unit step response $s(t)$ of the filter. (Assume zero initial voltage across C_1 and C_2.)

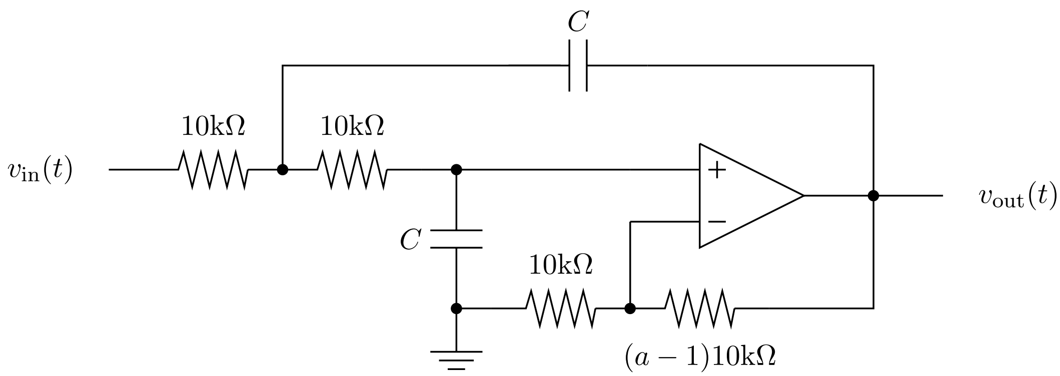
(c) For an audio application a filter is required with the magnitude Bode plot shown below:



For this application, the phase of H does not matter.

The resistor R_3 is fixed to be $10\text{k}\Omega$. Find (numerical, explicit values for) R_1, R_2, C_1, and C_2 so that the magnitude Bode plot of H matches (at least approximately) the required form shown above. (Needless to say, you cannot use negative values for R_1, R_2, C_1, and C_2.)

82. *Sallen-Key filter.* The circuit below, called a Sallen-Key filter section, is widely used. You can assume the op-amp is ideal, and both capacitors have zero initial voltage. Note that there are two free design parameters: the capacitance C (which of course must be positive) and the (gain) a, which is required to satisfy $a \geq 1$.



(a) Find the transfer function from v_{in} to v_{out}.

(b) Pick C and a to yield poles at $(-10^4 \pm j10^4)$rad/sec.

(c) Suppose we hook up two of the filters you designed in part (b) in cascade, *i.e.*, connect v_{out} of one to v_{in} of the other. Find the transfer function from the remaining input to the remaining output.

83. *A delay system.* Consider a system with input u and output y described by $y(t) = 0$ for $0 \leq t < 1$ and $y(t) = u(t-1)$ for $t \geq 1$. Thus the output is the same as the input but delayed one second. Find the transfer function H of this system. What is its DC gain? Sketch the Bode plot of H. Can you sketch its poles and zeros in the complex plane?

84. *A system with undershoot.* In this problem we consider a system described by the transfer function

$$H(s) = \frac{1-s}{(1+s)(1+2s)},$$

with input u and output y.

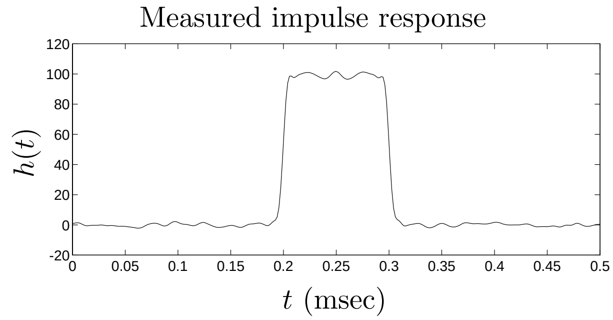
(a) Sketch the Bode plot of H. Be careful with the phase plot. Does the magnitude plot look like the magnitude plot of a simpler transfer function? Can you explain this?

(b) Sketch the step response. Make sure the final value and the slope at $t = 0+$ are correct. The interesting effect you see for small t is called *undershoot.*

(c) Suppose that at $t = 200$ the input switched from the value 3 to -1, *i.e.*, $u(t) = 3$ until $t = 200$; after that $u(t) = -1$. Sketch $y(t)$ for t near 200, say, several seconds before to several seconds after. Systems with undershoot are sometimes descibed this way: "when you change the input rapidly from one constant value to another, the output first moves in the wrong direction". Does this make sense?

(d) Can you find u such that $y(t) = 1 - e^{-t/2}$? Any comments about the u you found? Can you trace the interesting feature of u to some particular property of H, *e.g.*, its DC gain, pole locations, etc.?

85. The impulse response of a system described by a transfer function H is measured experimentally, and plotted below:



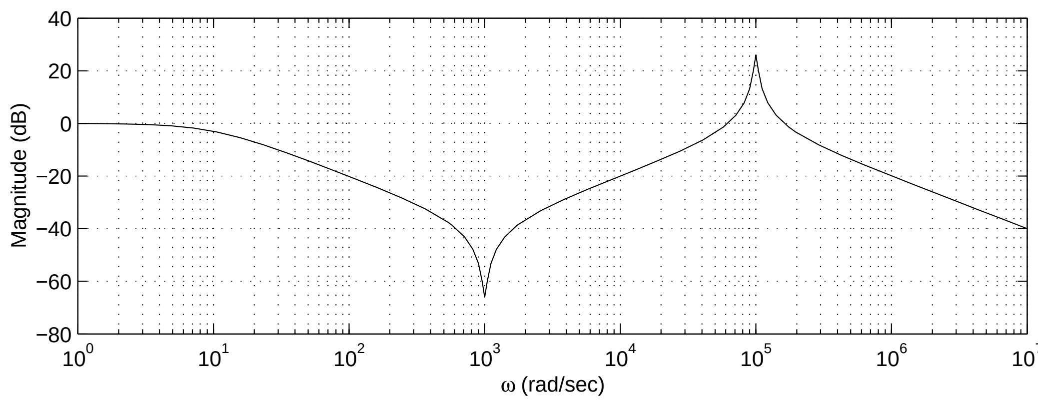
(a) Estimate $H(0)$, *i.e.*, the DC gain of this system.

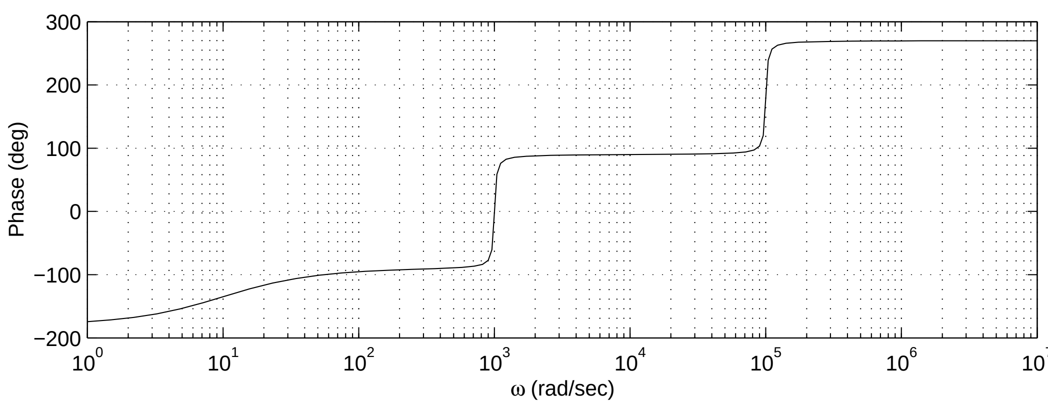
(b) Estimate $H(j\omega)$ for $\omega = 2\pi \cdot 500\text{Hz}$, $\omega = 2\pi \cdot 5\text{kHz}$, $\omega = 2\pi \cdot 10\text{kHz}$, and $\omega = 2\pi \cdot 1\text{MHz}$. Explain your approximations. An answer of the form "small" is OK provided you give some rough maximum as in "$H(j\omega)$ is small, probably less than 10^{-4} or so".

(c) Sketch the step response of this system.

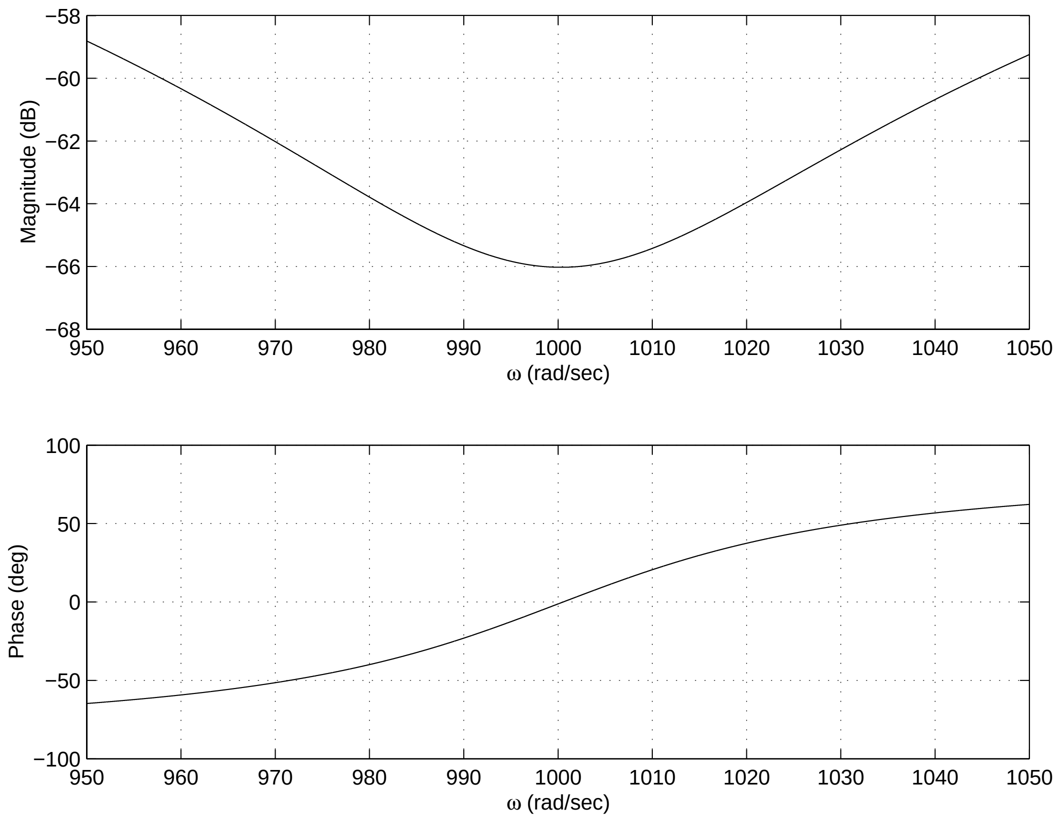
(d) The (10%-90%) *rise time* of a system is defined as the time elapsed between the first time the step response reaches 10% of its final value and the last time the step response equals 90% of its final value. Estimate the (10%-90%) rise time of this system.

86. *Pole-zero identification from Bode plot.* The following Bode plots show the magnitude and phase of the frequency response of a rational transfer function H.

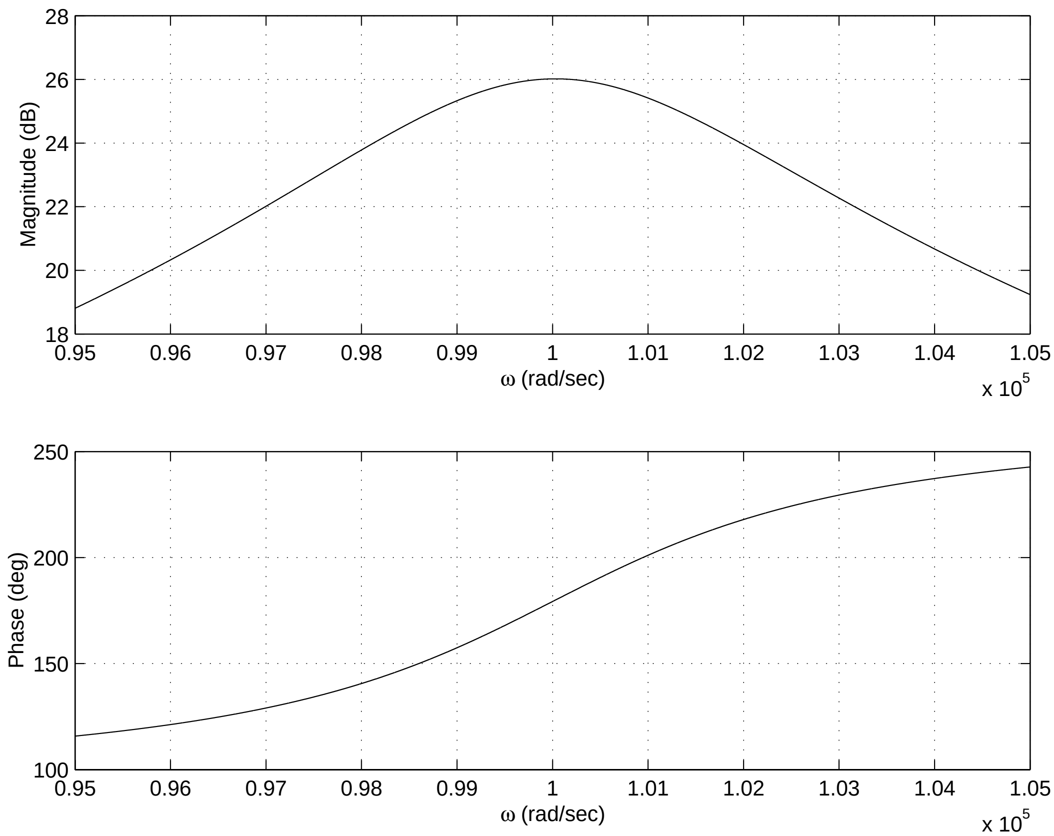




The following Bode plots show the same frequency response (on a linear frequency scale), zoomed in around $\omega = 10^3$.



The following Bode plots show the same frequency response (on a linear frequency scale), zoomed in around $\omega = 10^5$.



Estimate the poles, zeros, and DC gain of the transfer function H. Give the DC gain as a number, *not in dB*.

- You can make reasonable assumptions about the behavior of the frequency response outside the range plotted.
- Use the smallest number of poles and zeros required to explain the plots.
- Be sure to give the multiplicities of any repeated poles or zeros.
- Be careful about the signs of the poles, zeros, and DC gain.

87. *Deconvolution.* A signal w passes through a channel (which is a convolution system) to produce the signal z, *i.e.*, $z = h * w$. The impulse response of the channel is $h(t) = te^{-t}$ for $t \geq 0$. This means that z is a kind of 'smeared' or 'averaged' version of w.

    You can assume that w and z are smooth, *i.e.*, they have derivatives of all orders. You can also assume that $z'(0) = z(0) = 0$.

    Here's the question: is it possible to reconstruct the signal w knowing only the signal z?

    This process of undoing the effect of convolution is called *deconvolution*. As you can imagine, deconvolution (when it is possible) has many applications.

    Your answer should be one of the following:

    - *Yes, it's possible to recover w from z.* In this case, give an explicit formula that expresses w in terms of z. Give the simplest formula you can.

- *No, it's not possible to recover w from z.* In this case, you must explain why. For example, you could give two different signals w and $\tilde{w}$ that produce the same z, *i.e.*, for which $h * w = h * \tilde{w}$. In this case, give the simplest possible w and $\tilde{w}$ that make your point.

88. *Using Matlab.*

    (a) Use Matlab to plot the Bode plot of the transfer function of problems 1 and 2, to verify your sketches.

    (b) Consider the transfer function $H(s) = (s+1)/(s^2+s+1)$. Find the poles and zeros, and plot the impulse response, step response, and Bode plot using Matlab.

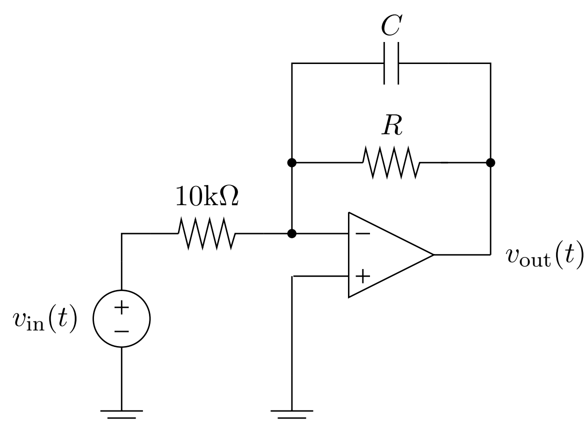
    (c) Now consider the transfer function

    $$G(s) = H(s)\frac{s+3}{s+3.1}.$$

    Intuition suggests that G is not much different from H since we have added a pole and a zero that *almost* cancel each other out. *Before* doing the next part, *guess* how the Bode plots of G and H will differ. Give a geometric explanation. Give the partial fraction expansion of H, and compare it to the partial fraction expansion of G.

    (d) Now use Matlab to plot the impulse response, step response, and Bode plot of G using Matlab. Compare with your prediction.

89. The circuit below is a simple one-pole lowpass filter.

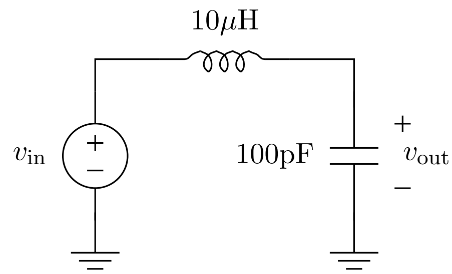


Find (positive) R and C such that:

- The (magnitude of the) DC gain is +12dB.
- The magnitude of the transfer function at the frequency 1kHz is 3dB less than the magnitude of the DC gain.

You can assume the op-amp is ideal. Give numerical values for R and C. An acurracy of 10% will suffice.

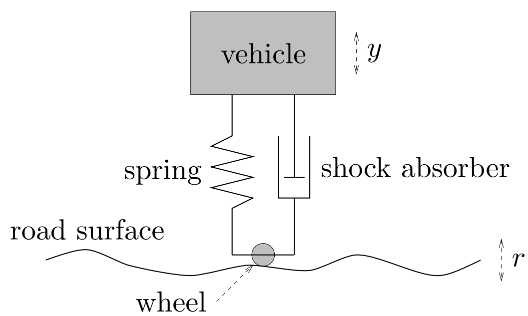
90. *Effects of wire inductance and capacitive load.* A voltage source drives a capacitive load through a wire that has significant inductance, as shown in the circuit below.



Find the transfer function H from v_{in} to v_{out}. (You can assume, of course, that the initial conditions are zero.)

Find the smallest frequency ω_{min} for which $|H(j\omega_{\text{min}})|$ deviates 0.5dB or more from the DC value. (This can be intepreted as the frequency below which the inductance and capacitance can be ignored.)

91. *Vehicle suspension: analysis of 'bottoming out'.* Once again we consider a vehicle suspension, modeled as a spring and a shock absorber between the vehicle and the wheel, which is assumed to remain in contact with the road. The height of the vehicle (above or below some reference level) at time t, is denoted $y(t)$. The height of the road surface at the wheel (above or below the reference level), at time t, is denoted $r(t)$. This is illustrated below.



Newton's equation for the (vertical) motion of the vehicle is

$$m\frac{d^2y}{dt^2} = -k(y(t) - r(t)) - b\frac{d}{dt}(y(t) - r(t))$$

where m is the vehicle mass (in kg), k is the spring stiffness (in N/m), and b is the mechanical resistance of the shock absorber (in N/m/s). For this problem we have the (nonrealistic, but simple) values

$$m = 1, \qquad k = 1, \qquad b = 2.$$

In this problem we focus on the *displacement of the vehicle relative to the road*, *i.e.*, the signal $d = y - r$. The value $d(t)$ tells you how much the car suspension is compressed (if $d(t) < 0$) or extended (if $d(t) > 0$) compared to its neutral position.

Any real suspension is limited in displacement. If $d(t)$ becomes too large, the suspension hits some hard rubber stops designed to prevent extreme damage. When this occurs, the ride is terrible, and also, the LCCODE model is invalid. When you hit the limits of your suspension system, it's often called 'bottoming out'.

In this problem, we'll use the simple limit $|d(t)| \leq 0.1$m to describe the limits of the suspension system. We'll use the descriptive slang expression 'bottoming out' to mean that '$|d(t)| \leq 0.1$m does not hold'.

(a) Find the transfer function H from r to d.

(b) Find the poles, zeros, and DC gain of H. If poles or zeros are repeated, be sure to give multiplicities.

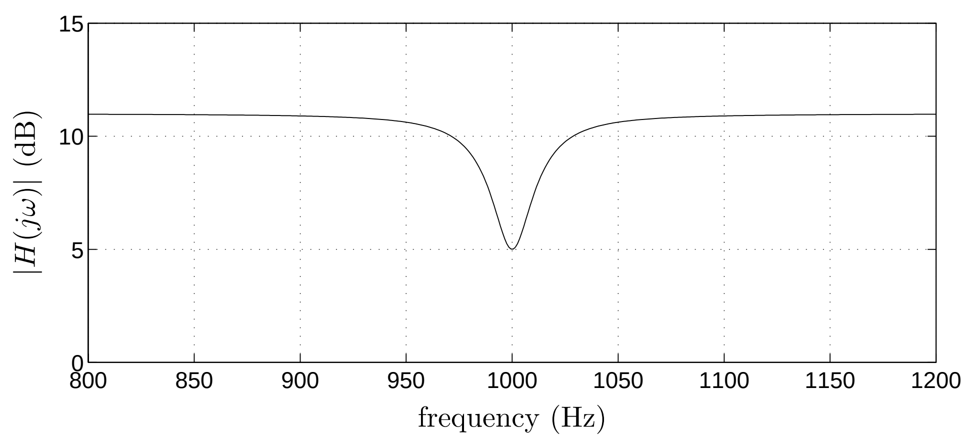
(c) The vehicle runs over a curb of (positive) height C (in m) at time $t = 0$, *i.e.*, the signal r is C times a unit step signal. What is the maximum curb height $C_{\max}$ the suspension can handle without bottoming out?

(d) Suppose the vehicle is in sinusoidal steady state while being driven at speed S (in m/s) over a sinusoidal test track, with one period/m and a variation ± 0.2m. (Thus, r is a sinusoidal signal with frequency $2\pi S$ rad/s and amplitude 0.2m.) What is the maximum speed $S_{\max}$ the vehicle can handle without bottoming out?

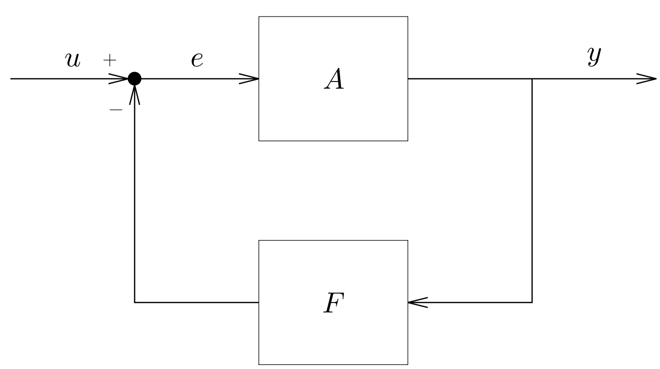
92. *Notch filter design.* Find a transfer function H that has the Bode magnitude plot shown below. Note that the vertical axis is given in dB and the horizontal axis, which is linear, is given in Hz.

Express your answer as the ratio of two unfactored polynomials. *Justify* your choice of poles and/or zeros. An accuracy of $\pm 10\%$ for the coefficients is acceptable.

Once the design is complete, use Matlab to create a full (magnitude and phase) Bode plot for the filter.



93. In the block diagram below, A and F are static gains.



The open-loop gain A varies over the range 60 ± 3dB. The closed-loop gain G varies over the range $40 \pm \Delta$dB.

Find the closed-loop gain variation Δ (in dB). An accuracy of 10% will suffice.

94. *Feedback amplifier design.* In this problem, you'll use the standard feedback amplifier configuration, described by the equations

$$y = Ae, \qquad e = u - Fy.$$

The raw amplifier, denoted A, is linear and time invariant, with transfer function given by

$$A(s) = \frac{A_{\mathrm{dc}}}{1 + sT},$$

and the feedback F is a simple gain (*i.e.*, a constant). The constant A_{dc} (which is positive) is the DC gain of the raw amplifier, and T (which is also positive) is the 63% rise time of the raw amplifier. The DC gain of the amplifier varies over a range (say, with temperature, manufacturing variations, etc.). For simplicity, we'll assume that the rise time T does not vary, and that the feedback F is implemented using high precision components, and so does not vary.

You can choose among four different raw amplifiers, with characteristics:

- *Raw amplifier 1:* DC gain is 40 ± 3dB; rise time is 0.5μsec.
- *Raw amplifier 2:* DC gain is 65 ± 5dB; rise time is 2μsec.
- *Raw amplifier 3:* DC gain is 80 ± 10dB; rise time is 15μsec.

Choose *one* of the raw amplifiers and an appropriate feedback gain F according to the following design rules:

- The (nominal) closed-loop DC gain is 30dB.
- The variation in closed-loop DC gain is no more than $\pm 5\%$.
- The closed-loop 63% rise time is as small as possible.

For calculating the closed-loop rise time you can use the nominal gain of the raw amplifier (*i.e.*, 50dB for raw amplifier 1, etc.), and ignore the gain variation.

95. *Sensitivity of closed-loop gain to feedback gain.* In the lecture we saw that a (small) change δA in the open-loop gain induces a change δG in the closed-loop gain, where

$$(\delta G/G) = S\frac{1}{1 + AF}(\delta A/A), \quad S = \frac{1}{1 + AF}.$$

Now suppose that the feedback gain undergoes a (small) change δF. Find S_F such that

$$(\delta G/G) = S_F(\delta F/F).$$

S_F is called the sensitivity of G w.r.t. the parameter F. (Note that $S = 1/(1 + AF)$ is the sensitivity of G w.r.t. A.)

Note that S_F can also be expressed as

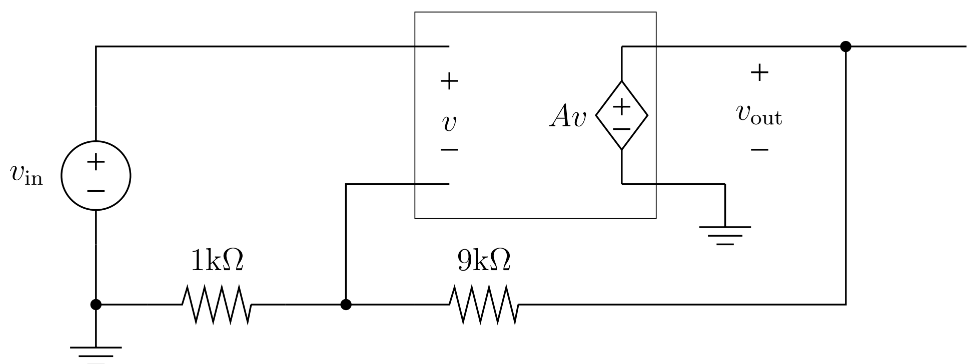
$$S_F = \frac{\partial G}{\partial F}\frac{F}{G}.$$

We can interpret S_F as giving (approximate) dB change in G per (small) dB change in F.

How are S and S_F related? Is it possible to design a feedback system in which the closed-loop gain G is fairly insensitive to changes in both open-loop gain A and feedback gain F?

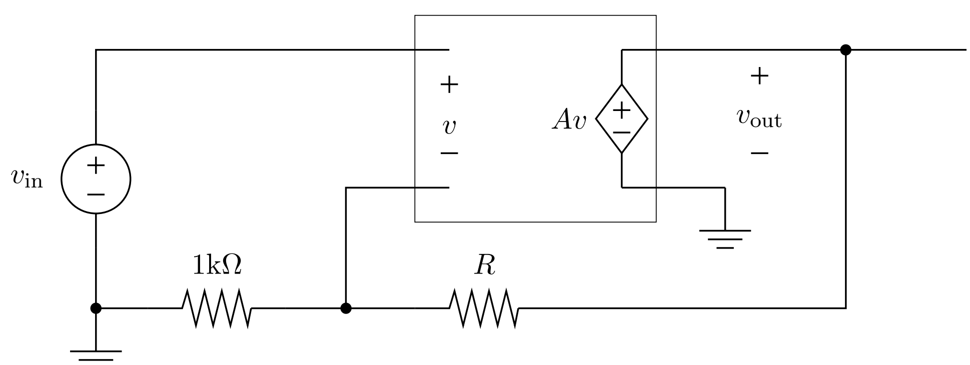
What happens to S_F when the loop gain AF is large? What practical implications does this have?

96. In the circuit shown below, the gain A is 45 ± 5dB (and is positive). The resistors have a tolerance of $\pm 1\%$, *i.e.*, they can vary up to 1% from the values shown.



How much can the closed-loop gain (from v_{in} to v_{out}) vary, taking into account both the variation in op-amp gain and the resistor variation? Which causes more variation in closed-loop gain, the variation in op-amp gain or the resistor values?

97. G is the gain from v_{in} to v_{out} in the circuit shown below.



(a) Assume that the gain A is 60dB (and is positive). Find R so that G is 20dB.

(b) Now suppose that A can vary ± 2dB, *i.e.*, from 58dB to 62dB, and R can vary $\pm T\%$ from the value found in part 1, where T is the so-called *tolerance*. There is a requirement that despite these variations, the closed-loop gain G must not vary more than ± 1dB (from 20dB).

You can specify 20% tolerance ($T = 20$), 5%, 1%, or 0.1%. What is the largest tolerance you can specify and still meet the requirement? Circle one below.

(Lower tolerance resistors cost more, so we are asking: what is the cheapest resistor that can meet our requirement?)

98. *Local or global feedback?* You have two amplifiers, with gains $a_1 \approx 10$ and $a_2 \approx 10$, each of which varies with temperature, component aging, etc. You need to design a feedback amplifier circuit with overall gain $g \approx 30$. You are going to use feedback to trade your 'excess gain' (about 10dB) for lower sensitivity of the overall gain with respect to the amplifier gains. You can neglect loading effects, *i.e.*, you can assume the input resistance of the amplifiers is infinite and and the output resistance is zero.

    There are two basic approaches: local and global feedback. You can cascade the two amplifiers and wrap feedback around the cascade connection. This approach is called *global feedback* since there is one feedback loop around both amplifiers. Another approach is called *local feedback*. Here you wrap some feedback around each of the amplifiers, and form a cascade system of the two (closed-loop) amplifier circuits.

    The question: which arrangement yields lower sensitivity to the amplifier gains?

    First, find an appropriate global feedback gain and appropriate local feedback gains. For the local feedback case you can apply the same feedback around each amplifier.

    Compute the sensitivity of G with respect to A_1 and A_2 for the local and global feedback designs. That is, find the quantities $\frac{\partial G}{\partial A_i}\frac{A_i}{G}$, for $i = 1, 2$.

    Remarks:

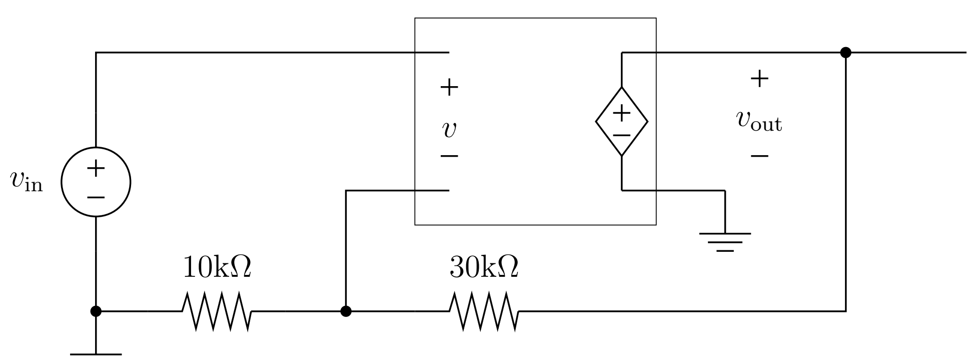
    - When we study dynamic feedback, we'll see that the global feedback arrangement can be much worse than local. But here, we are only interested in static properties.
    - In most real applications, some combination of local and global feedback is used.
    - It is not hard to generalize this problem: the same conclusion holds for many amplifiers, even with different gains.

99. *Computer-analysis of linearization effect of feedback.* In this problem you will study a static nonlinear feedback amplifier. It will also serve as an introduction to (or review of) Matlab.

    The amplifier in the circuit shown below can be modeled as

$$v_{\text{out}} = \tanh(v/(2V_{\text{T}})),$$

    where $V_{\text{T}} \approx 26\text{mV}$ at room temperature.



Remark: This characteristic is very common. It comes from a junction transistor differential input pair, which you will learn about in EE113. For an FET-input op-amp, *i.e.*, and op-amp with first stage made from FETs, the characteristic is different but similar in shape.

You can neglect loading effects, *i.e.*, you can assume the input resistance is infinite and output resistance is zero.

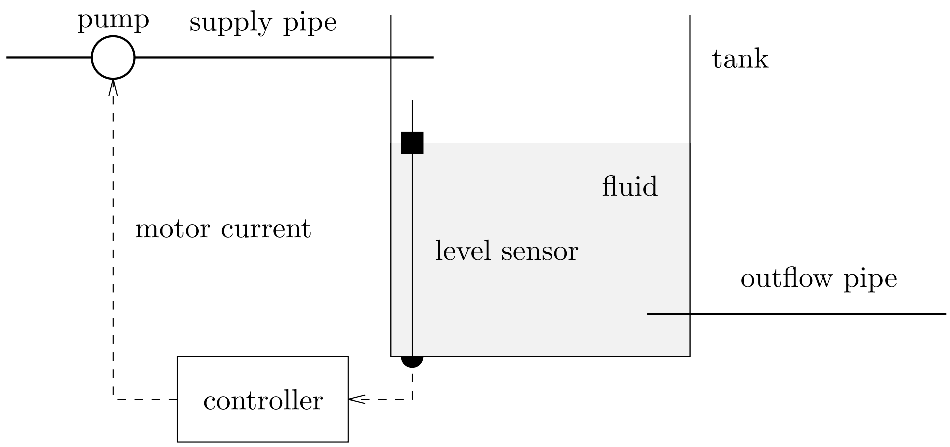
- Use Matlab to plot the open-loop voltage transfer characteristic (*i.e.*, from v to v_{out}) over some appropriate range. Be sure to label your axes with units. Over what input voltage range would you say the (open-loop) amplifier acts (approximately) linearly? Over what output voltage range?
- Use Matlab to plot the closed-loop voltage transfer characteristic (*i.e.*, from v_{in} to v_{out}). To find the closed-loop characteristic, you can use the tracing method described in the lecture notes. Be sure to label your axes with units. Over what input voltage range would you say the (closed-loop) amplifier acts (approximately) linearly? Over what output voltage range?
- Use Newton's method to (try to) solve for v_{out}, given $v_{\text{in}} = 0.2\text{V}$. Try several different starting points v_0, *e.g.*, 0, ± 0.05, ± 0.1. To show the convergence (or divergence) of Newton's method, plot the error $|v_{\text{out}\,k} - \tanh(v_k/(2V_{\text{T}}))|$ versus iteration number k. It's more instructive to plot the error on a log scale. this can be done using the Matlab command `semilogy()`. Briefly describe what you observe.

For this problem you must submit not only the plots requested, but also a listing of the associated Matlab code. So please make an attempt at good programming style (*i.e.*, mnemonic variable names, comments, etc.).

100. *Reduction of offsets via feedback.* The forward path in a feedback system is characterized by $y = Ae + \alpha$, where α is a constant called (for obvious reasons) the *output offset.* The feedback path is linear with gain F. Find the output offset α_{cl} for the closed-loop system. (That is, find α_{cl} such that $y = Gu + \alpha_{\text{cl}}$.) What is the ratio of the closed-loop output offset to the open-loop output offset?

    Offsets can also be expressed in terms of the input. We can write $y = Ae + \alpha$ in the form $y = A(e + \beta)$, where $\beta = \alpha/A$ is called the *input-referred* offset of the open-loop system. Similarly we can express the closed-loop offset in the form $y = G(u + \beta_{\text{cl}})$, where β_{cl} is the input-referred offset of the closed-loop system. Find the input-referred offset for the closed-loop system, and compare it to the input-referred offset of the open-loop system.

101. *Liquid level control.* In an industrial process, a tank is used to collect a fluid, as shown in the figure below. The tank is fed through a supply pipe attached to a pump; fluid is removed from an outflow pipe. A liquid level sensor is used to measure the level of the liquid in the tank. The outflow is not measured. It is usually near a fixed known value Q_0, but can vary a bit above and below that value. The supply flow rate can be varied around the value Q_0 by varying the pump motor current around its normal operating value. The goal is to keep the tank nearly half filled, despite variations in outflow (around its normal value Q_0). We'll assume the tank starts out half filled.



Note that when the outflow is equal to the known value Q_0, and the pump current is equal to its normal operating value, the supply flow is also equal to Q_0, so the net fluid flow into the tank is zero. This means the tank level remains constant, *i.e.*, half full.

If the actual outflow is larger than Q_0, the tank level will start to drop; hopefully our feedback control system will counteract this by increasing the supply flow rate.

We will be concerned with deviations of the various quantities from their "ideal" or "normal" values, so let's define some variables that represent the deviations. $q_{\rm in}$ will denote the (deviation) in supply flow rate, *i.e.*, the actual (total) supply flow rate, minus Q_0. Thus, $q_{\rm in} < 0$ means that the actual supply flow rate is less than Q_0. Similarly, $q_{\rm out}$ will denote the deviation of the outflow rate from Q_0. The deviation of the liquid level from its ideal height (*i.e.*, half full) will be denoted l, so $l > 0$ means the liquid level is too high and $l < 0$ means the liquid level is too low. The sensor signal will be denoted $v_{\rm sens}$; we'll assume the sensor electronics provides the proper offset so that $v_{\rm sens} = 0$ when $l = 0$. The deviation of the pump motor current from its normal operating value will be denoted $i_{\rm pump}$.

(We'll henceforth deal only with these deviations from normal values, but we won't keep saying "deviation from". For example, we'll call $q_{\rm in}$ the supply flow rate. It's too cumbersome to always say "the deviation in supply flow rate".)

The liquid level height l is proportional to the integral of the net flow into the tank, *i.e.*,

$$l(t) = \alpha \int_0^t (q_{\rm in}(\tau) - q_{\rm out}(\tau))\, d\tau,$$

where α is the reciprocal of the area of the tank (horizontal cross section). We'll assume $\alpha = 1$. We can also express this via transfer functions as

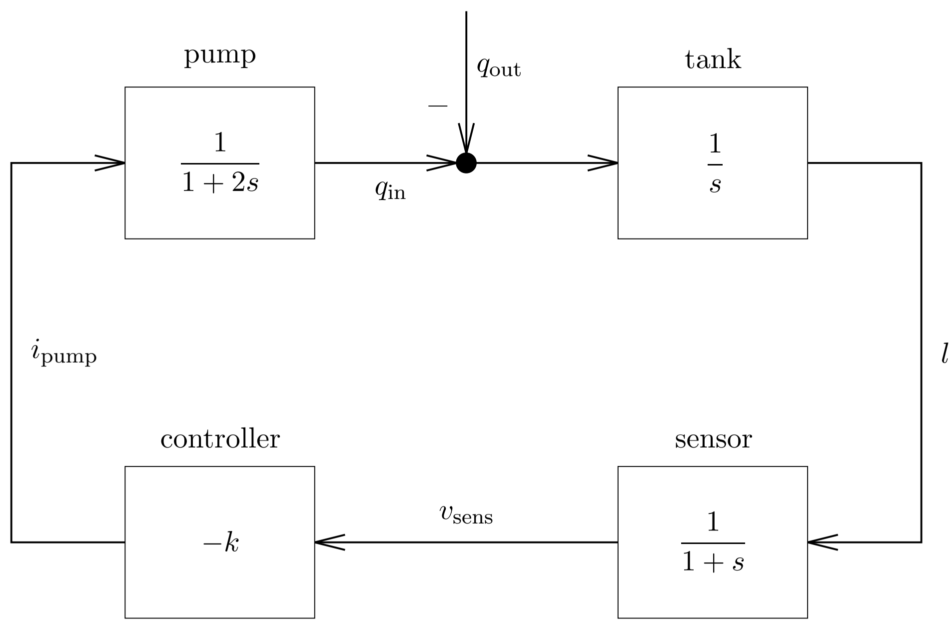
$$L(s) = \frac{1}{s}(Q_{\rm in}(s) - Q_{\rm out}(s)).$$

The electronics in the liquid level sensor includes a smoothing filter with a time constant of 1, so the relation between the actual liquid level l and the liquid level sensor output voltage $v_{\rm sens}$ can be described by the transfer function $1/(1+s)$ (which has unity DC gain).

Similarly the supply flow rate does immediately change to a new value when the pump motor current is changed; instead the relation between pump motor current and supply flow rate is given by the transfer function $1/(1+2s)$ (*i.e.*, the supply flow rate is a smoothed version of the current, with a time constant of 2).

We will use basic proportional control, given by $i_{\text{pump}} = -kv_{\text{sens}}$, where $k \geq 0$ is the proportional feedback gain. Note that this controller "does the right thing" — if the liquid level is low, it increases the motor current, which increases the supply flow rate, which increases the liquid level ...

Putting all of this together we get the following block diagram:



(a) Find the transfer function G from q_{out} to l (which depends on the gain k). What is the DC gain of G for $k > 0$? What is the DC gain for $k = 0$, *i.e.*, open loop? What does your answer mean?

(b) Find the poles of the transfer function G for $k = 0$ (*i.e.*, open loop), $k = 0.01$, $k = 0.2$, $k = 1$, and $k = 2$.

(c) Plot the step response of G for $k = 0$ (*i.e.*, open loop), $k = 0.01$, $k = 0.2$, $k = 1$, and $k = 2$. Give a brief intuitive explanation, in words, why the plots look the way they do. Briefly comment on what they mean.

(d) Suppose that someone dumps some liquid directly into the tank at $t = 0$. This can be modeled as an outflow of the form $q_{\text{out}} = -\beta\delta(t)$, where β is the quantity of liquid and δ is Dirac's delta function. (Why?) Thus, the impulse response of the transfer function G tells us the response of the system after a unit volume of liquid is removed (very quickly) at $t = 0$. Plot the impulse response for $k = 0$ (*i.e.*, open loop), $k = 0.01$, $k = 0.2$, $k = 1$, and $k = 2$. Give some brief comments on what you see.

102. Consider the control system from the previous problem.

(a) What proportional gains k yield a stable transfer function (from q_{out} to l)?

(b) You decide to use a proportional controller with gain $k = 0.2$. The company that makes the sensor introduces a longer averaging time in the level sensor, so its transfer function becomes $1/(1 + T_{\rm sens}s)$, where $T_{\rm sens} > 1$. You might guess that this longer averaging could induce instability. How large can $T_{\rm sens}$ be, before the closed-loop transfer function from $q_{\rm out}$ to l becomes unstable?

(When you complain to the sensor manufacturer, they explain that the longer averaging time is a feature, not a bug. They say the longer averaging time yields a 'smoother' display, which doesn't 'jump around' as much as before.)

(c) Now suppose we decide to use integral control, replacing the control law $i_{\rm pump} = -kv_{\rm sens}$ with

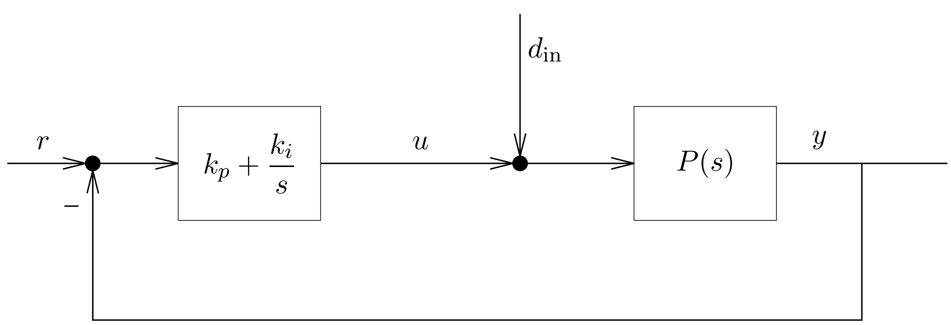
$$i_{\rm pump}(t) = -k_p v_{\rm sens}(t) - k_i \int_0^t v_{\rm sens}(\tau)\, d\tau.$$

(As a result of problem 1, you have switched level sensor manufacturers, so $T_{\rm sens}$ is back to 1.)

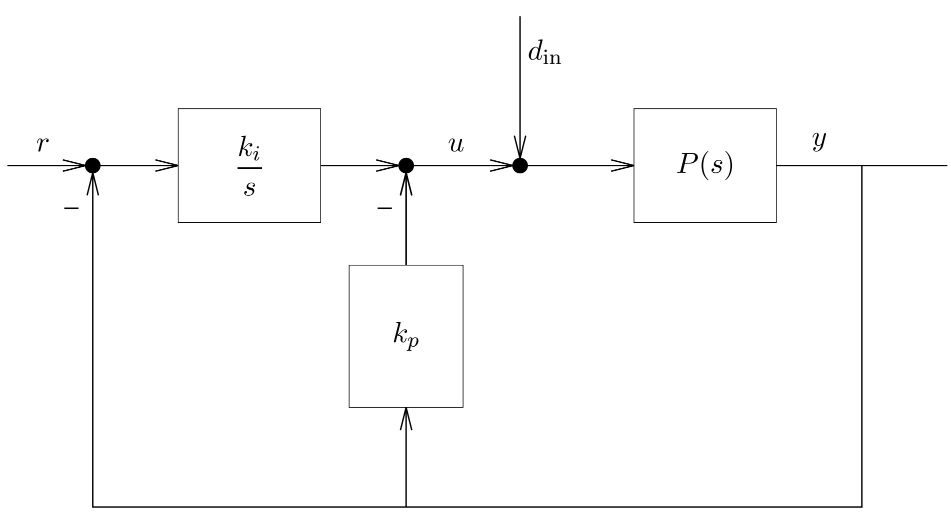
With $k_p = 0.2$, find the range of k_i over which the closed-loop transfer function from $q_{\rm out}$ to l is stable.

(d) Plot the step response from $q_{\rm out}$ to l for the proportional control with $k_p = 0.2$, and integral control with $k_p = 0.2$ and k_i taking on various values. Briefly discuss what happens if k_i is too small or too large. Choose a reasonable value, and briefly defend your choice.

103. *A modified PI controller.* A block diagram of the standard PI control system is shown below (along with an input disturbance $d_{\rm in}$ we'll refer to later in the problem).



Some industrial PI controllers use the modified PI controller shown below:



One company that makes this modified PI controller claims that it "has the same disturbance rejection abilities as the standard PI controller, but, for step changes in r, has a smoother input u".

(a) Find the closed-loop transfer function, H_{std}, from d to y, with the standard PI controller. Also find the closed-loop transfer function, H_{mod}, from d to y, with the modified PI controller.

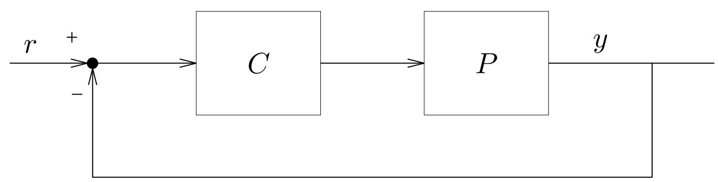
(b) Now consider the following specific case:

$$P(s) = \frac{1}{s+1}, \quad k_p = 4, \quad k_i = 6.$$

Suppose that $d = 0$ and a unit step is applied to r. Find the input u_{std} that occurs in the closed-loop system with the standard PI controller. Also find the input u_{mod} that occurs in the closed-loop system with the modified PI controller.

(c) Make a very brief comment about the company's claim.

104. Suppose that PI control, $C(s) = k_p + \frac{k_i}{s}$, is used with the plant $P(s) = \frac{1}{s^2+s+1}$, in the standard feedback control configuration:

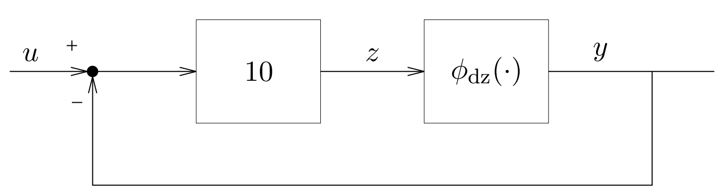


Find the conditions on k_p and k_i for which the closed-loop transfer function T from r to y is stable. You can assume that $k_p > 0$ and $k_i > 0$.

Express your conditions in the simplest form possible. (We will take points off for correct answers left in messy form.)

105. *Static analysis of feedback around amplifier with dead-zone.* In this problem we consider only static conditions, *i.e.*, all signals are scalars.

The block diagram below shows a feedback amplifier with input u, output y, and unity feedback. The amplifier is modeled as a (linear) gain of 10, followed by a nonlinear function ϕ_{dz} which is called a *deadzone*.

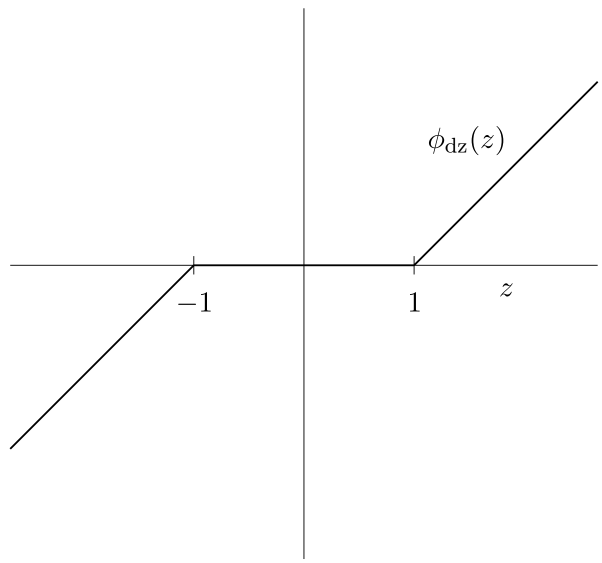


The gain of 10 represents the (linear) 'gain stage' of the amplifier; the dead-zone represents a (nonlinear) push-pull output stage. (This is just for background — you don't need to know about amplifiers to solve this problem!)

The deadzone characteristic is given by

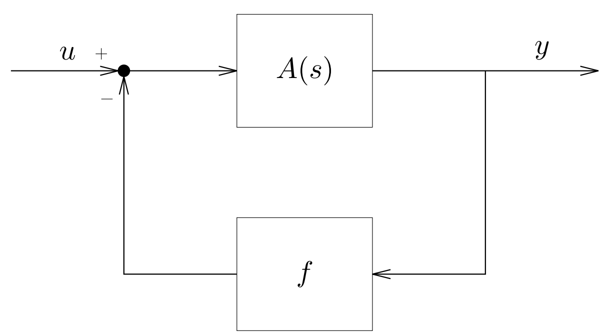
$$y = \phi_{\rm dz}(z) = \begin{cases} z - \alpha & z \geq \alpha, \\ z + \alpha & z \leq -\alpha, \\ 0 & |z| < \alpha, \end{cases}$$

where $\alpha \geq 0$ is a parameter. In this problem you can take $\alpha = 1$ for simplicity. (In real amplifiers a typical value is about 0.5V.) This characteristic is plotted below.



Sketch the static closed-loop characteristic relating u to y. Make sure you clearly label the axis scales, critical points on the characteristic, relevant slopes, etc.

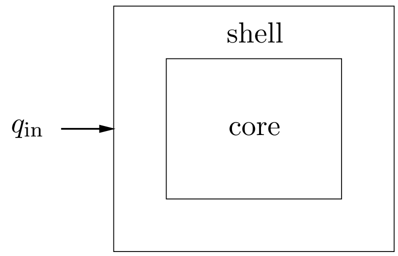
106. *Feedback amplifier stability via minimum closed-loop gain.* Consider an amplifier (op-amp) with T.F. $A(s)$, in the standard feedback configuration (shown below), with feedback gain f which is constant and positive. As usual, G will denote the closed-loop transfer function from u to y.



One common method for specifying what values of $f > 0$ make the closed-loop system stable is to give a *minimum* value $G_{\rm min}$ for the closed-loop DC gain $G(0)$. In other words, the closed-loop system is stable if and only if $G(0) > G_{\rm min}$. (Recall that we consider f constant and positive only).

Consider a specific op-amp with $A(s) = \dfrac{100}{(s+1)(s^2+s+1)}$. Find $G_{\rm min}$ for this op-amp.

107. *A thermal system.* A thermal model of the system shown below consists of outer shell, at temperature T_{shell}, which surrounds an inner core, at temperature T_{core}. (Each of these temperatures is measured with respect to the ambient temperature.)



An external heat flow q_{in} (in W) flows into the shell, and in addition there in a heat flow $k_{\text{shell}}T_{\text{shell}}$ from the shell into the surroundings, where k_{shell} is the thermal conductance between the shell and the surroundings.

The heat flow from the shell into the core is denoted q_{core}, which is given by

$$q_{\text{core}} = k_{\text{core}}(T_{\text{shell}} - T_{\text{core}}),$$

where k_{core} is the thermal conductance between the shell and the core.

The temperature of the shell satisfies

$$c_{\text{shell}}\frac{dT_{\text{shell}}}{dt} = q_{\text{in}} - k_{\text{shell}}T_{\text{shell}} - q_{\text{core}}$$

where c_{shell} is the heat capacity of the shell, and the right hand side is the net heat flow into the shell.

The temperature of the core satisfies

$$c_{\text{core}}\frac{dT_{\text{core}}}{dt} = q_{\text{core}},$$

where c_{core} is the heat capacity of the core.

For the remainder of this problem you can use the specific numerical values

$$c_{\text{core}} = c_{\text{shell}} = 1, \quad k_{\text{core}} = k_{\text{shell}} = 1.$$

1a. Find the transfer function H from q_{in} to T_{core}. (Of course you can assume here that the initial temperatures are both zero.)

1b. Suppose that the system starts with $T_{\text{shell}}(0) = T_{\text{core}}(0) = 0$, $q_{\text{in}}(t) = 2000$ for $0 \le t \le 0.001$, and $q_{\text{in}}(t) = 0$ for $t > 0.001$.

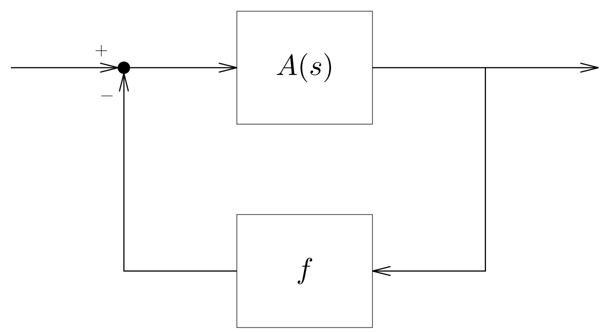
Find the maximum value of $T_{\text{core}}(t)$ for $t \ge 0$.

You can make an approximation, but you must explain what you're doing.

108. *Stability range for feedback amplifier.* An amplifier with transfer function A has a DC gain of 80dB, and three real, negative poles and no zeros. The dominant pole is at $s = -100$, and the two other poles are at $s = -10^4$ and $s = -10^6$, respectively. (This is a very typical amplifier transfer function.)

2a. Find the *unity-gain bandwidth* of the amplifier, defined as the frequency for which the magnitude of the frequency response is equal to one. A good estimate is acceptable.

2b. The amplifier is connected in a standard feedback arrangement with feedback gain $f \geq 0$, as shown below. Find the maximum value of f for which the closed-loop transfer function is stable.



2c. Suppose the feedback gain f is chosen to be equal to the maximum value found in part b. At what frequency does the feedback amplifier oscillate? A good estimate is acceptable. You must explain your answer.

109. *Driven guard.* The figure below shows a common setup for an instrumentation system. The voltage source and resistance at left are the sensor; the resistance at right is the amplifier input resistance. (We assume the amplifier output resistance is very low.) The voltage-controlled voltage amplifier has gain a, which is positive.

The capacitance represents the capacitance from the wire connecting the sensor to the amplifier to a *shield* or *guard* conductor around it. In a conventional arrangement, this shield or guard conductor would just be grounded (and therefore, the bottom lead of the capacitor would be grounded).

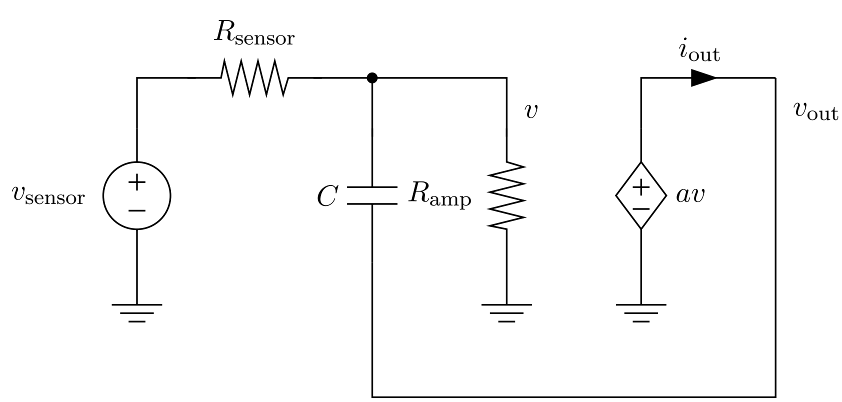
In this arrangement, the amplifier output drives the guard, and so it is called a *driven guard* system.

For this problem you can use the values

$$R_{\text{sensor}} = 100\Omega, \quad R_{\text{amp}} = 900\Omega, \quad C = 5\text{nF}$$

(which are quite typical).

For now, we'll leave the gain a unspecified.



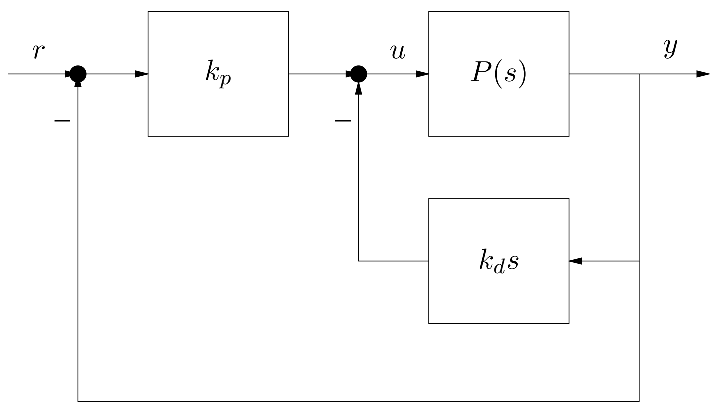
3a. Find the transfer function H from v_{sensor} to v_{out}. Please do this carefully as you may need to use it in the other parts of this problem.

3b. For what values of a is the transfer function H stable?

3c. Assume now $a = 0.9$. We define the *response time* of the system as the time it takes the step response from v_{sensor} to v_{out} to settle to within 90% of its ultimate value. Find the response time.

3d. Still using the value $a = 0.9$, suppose the sensor voltage jumps from 0V to 0.1V. What is the maximum magnitude of the amplifier output current i_{out}? (This value might be used, for example, to specify the output current capabilities of the amplifier.)

110. *PD control.* In lecture we studied proportional plus integral (PI) control. In this problem we study proportional plus derivative (PD) control. PD control is also used in practice (but not as often as PI control), often when a separate sensor is available that senses the derivative of the output, for example, a tachometer for a motor. A PD control system is usually implemented using the block diagram shown below. The numbers k_p and k_d are design parameters (which are usually, but not always, positive).



We'll study PD control applied to a DC motor, with transfer function

$$P(s) = \frac{1}{s(s+1)}.$$

4a. For what values of (k_p, k_d) is the closed-loop transfer function T (from r to y) stable?

4b. For what values of (k_p, k_d) are the poles of the transfer function T critically damped (and stable)? If no values of (k_p, k_d) yield critical damping, give your answer as 'none'. If only one choice works, give the specific values. If there are many choices, describe them all.

4c. Using the values $k_p = k_d = 1$, find the step response from r to u.

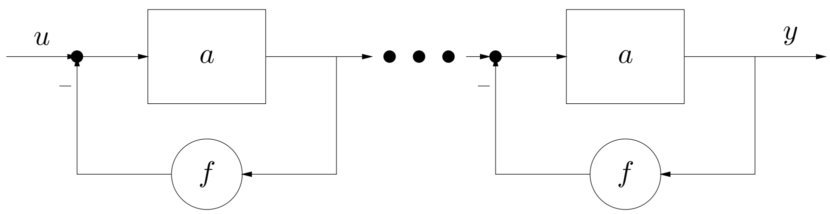
111. *Transfer function synthesis from specifications.* In this problem you will design a rational transfer function H from the followoing specifications:

- All the poles & zeros of H are real (to keep things simple).
- H has zero gain at DC and infinite frequency, *i.e.*, $H(0) = 0$, and $H(s) \to 0$ as $s \to \infty$.
- H is stable.

- For ω between 10^1 and 10^3, $|H(j\omega)|$ is 30 ± 3dB.
- For $\omega < 10^{-1}$, $|H(j\omega)|$ is smaller than -6dB.
- For $\omega > 10^5$, $|H(j\omega)|$ is smaller than -6dB.

Use as few poles & zeros as possible to meet these specifications.

112. *Design of multi-stage feedback amplifier.* In this problem you will design a multi-stage amplifier consisting of a number of cascaded stages, each of which is an amplifier of gain a with a feedback gain of f, as shown below.



The amplifier gain a varies with temperature (say), over the range 40 ± 3dB (but you can assume that the gains in each of the stages are the same). The feedback gain f can be assumed certain; it does not vary with temperature.

You are to design the multistage amplifier subject to the specifications and objectives described below. Your final design will consist of the number of stages you use, along with the feedback gain f used for each stage.

- Use as few stages as you can to meet all the specifications.
- The overall system gain (*i.e.*, from u to y) must be 60dB within ± 1dB, despite the variation in the amplifier gain.

If you cannot meet the specifications using any number of stages, give your answer as 'infeasible'. You must justify your design.

113. *Elmore delay of a convolution system.* Consider a convolution system with impulse response h and transfer function H. When $h(t) \geq 0$ for all t, a useful measure of the delay induced by the system is the *Elmore delay*, defined as

$$E(h) = \frac{\int_0^\infty t h(t)\, dt}{\int_0^\infty h(t)\, dt}$$

(provided the denominator isn't zero).

(a) Express the Elmore delay E directly in terms of the transfer function H. Try to express your answer in as simple a form as possible. (Your answer cannot contain the impulse response h.)

(b) Find the Elmore delay of the following systems:
   - a T-second delay
   - a first-order lowpass system with time constant T, *i.e.*, transfer function $1/(1 + sT)$
   - a T-second averager, *i.e.*, $y(t) = (1/T) \int_0^T u(t - \tau)\, d\tau$

(c) *Elmore delay of a cascade connection.* Suppose h and g are impulse responses for which the Elmore delay makes sense (*i.e.*, $h(t) \geq 0$ and $g(t) \geq 0$ for all t, and the denominator in the definition of Elmore delay doesn't vanish). Let $f = g * h$, *i.e.*, the impulse response of the cascade of the two systems. Are the following statements always true (*i.e.*, true for any g and h satisfying the conditions above)?

- $E(f) \geq E(g) + E(h)$, *i.e.*, the Elmore delay of a cascade of two systems is always at least as big as the sum of the Elmore delays of the systems.
- $E(f) \leq E(g) + E(h)$, *i.e.*, the Elmore delay of a cascade of two systems is always less than or equal to the sum of the Elmore delays of the systems.

If neither is always true give your answer as 'neither'. You must justify your answer. We do not want conditions such as '$E(f) \geq E(g) + E(h)$ is true provided ...'.