Lecture 4
Natural response of first and second order systems

- first order systems
- second order systems
  - real distinct roots
  - real equal roots
  - complex roots
  - harmonic oscillator
  - stability
  - decay rate
  - critical damping
  - parallel & series RLC circuits

First order systems

$$ay' + by = 0 \quad \text{(with } a \neq 0\text{)}$$

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

$$Ty' + y = 0 \quad \text{or} \quad y' + ry = 0$$

where

- $T = a/b$ is a *time* (units: seconds)
- $r = b/a = 1/T$ is a *rate* (units: 1/sec)

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT + 1} = \frac{y(0)}{s + 1/T}$$

and so $y(t) = y(0)e^{-t/T}$

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, y *decays* exponentially

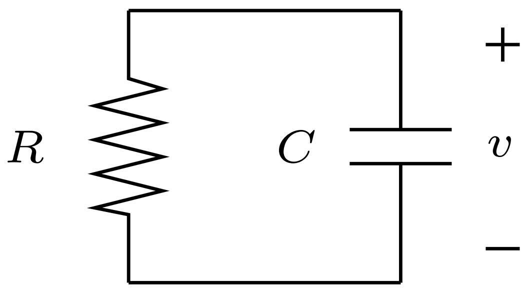
- T gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half ($0.693 = \log 2$)
- $4.6T$ gives time to decay by 0.01 ($4.6 = \log 100$)

if $T < 0$, y *grows* exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100

Examples

simple RC circuit:



circuit equation: $RCv' + v = 0$

solution: $v(t) = v(0)e^{-t/(RC)}$

population dynamics:

- $y(t)$ is population of some bacteria at time t
- growth (or decay if negative) rate is $y' = by - dy$ where b is birth rate, d is death rate
- $y(t) = y(0)e^{(b-d)t}$ (grows if $b > d$; decays if $b < d$)

thermal system:

- $y(t)$ is temperature of a body (above ambient) at t
- heat loss proportional to temp (above ambient): ay
- heat in body is cy, where c is thermal capacity, so $cy' = -ay$
- $y(t) = y(0)e^{-at/c}$; c/a is thermal time constant

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a(\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}(y'')}) + b(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

Roots of χ

(two) roots of characteristic polynomial χ are

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$i.e.$, we have $as^2 + bs + c = a(s - \lambda_1)(s - \lambda_2)$

three cases:

- roots are **real and distinct:** $b^2 > 4ac$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- roots are **real and equal:** $b^2 = 4ac$

$$\lambda_1 = \lambda_2 = -b/(2a)$$

- roots are **complex** (and conjugates): $b^2 < 4ac$

$$\lambda_1 = \sigma + j\omega, \quad \lambda_2 = \sigma - j\omega,$$

where $\sigma = -b/(2a)$ and

$$\omega = \frac{\sqrt{4ac - b^2}}{2a} = \sqrt{(c/a) - (b/2a)^2}$$

Real distinct roots $(b^2 > 4ac)$

$$\chi(s) = a(s - \lambda_1)(s - \lambda_2) \quad (\lambda_1, \lambda_2 \text{ real})$$

from page 4-6,

$$Y(s) = \frac{\alpha s + \beta}{a(s - \lambda_1)(s - \lambda_2)}$$

where α, β depend on initial conditions

express Y as

$$Y(s) = \frac{r_1}{s - \lambda_1} + \frac{r_2}{s - \lambda_2}$$

where r_1 and r_2 are found from

$$r_1 + r_2 = \alpha/a, \quad -\lambda_2 r_1 - \lambda_1 r_2 = \beta/a$$

which yields

$$r_1 = \frac{\lambda_1 \alpha + \beta}{\sqrt{b^2 - 4ac}}, \quad r_2 = \frac{-\lambda_2 \alpha - \beta}{\sqrt{b^2 - 4ac}}$$

now we can find the inverse Laplace tranform . . .

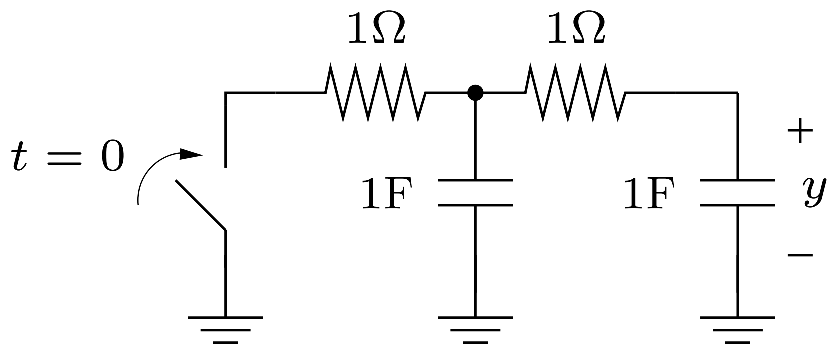
$$y(t) = r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t}$$

a sum of two (real) exponentials

- coefficients of exponentials, *i.e.*, λ_1, λ_2, depend only on a, b, c
- associated time constants $T_1 = 1/|\lambda_1|$, $T_2 = 1/|\lambda_2|$
- r_1, r_2 depend (linearly) on the initial conditions $y(0)$, $y'(0)$

- signs of λ_1, λ_2 determine whether solution grows or decays as $t \to \infty$
- magnitudes of λ_1, λ_2 determine growth rate (if positive) or decay rate (if negative)

Example: second-order RC circuit



at $t = 0$, the voltage across each capacitor is 1V

- for $t \geq 0$, y satisfies LCCODE (from page 2-18)

$$y'' + 3y' + y = 0$$

- initial conditions:

$$y(0) = 1, \quad y'(0) = 0$$

(at $t = 0$, voltage across righthand capacitor is one; current through righthand resistor is zero)

solution using Laplace transform

- characteristic polynomial: $\chi(s) = s^2 + 3s + 1$
- $b^2 = 9 > 4ac = 4$, so roots are real & distinct: $\lambda_1 = -2.62$, $\lambda_2 = -0.38$
- hence, solution has form

$$y(t) = r_1 e^{-2.62\,t} + r_2 e^{-0.38\,t}$$

- initial conditions determine r_1, r_2:

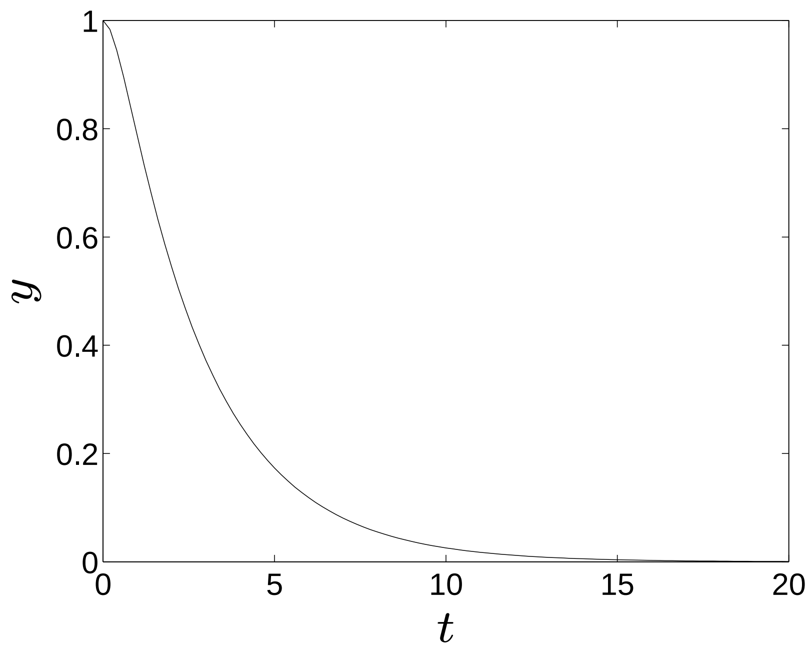
$$y(0) = r_1 + r_2 = 1, \quad y'(0) = -2.62 r_1 - 0.38 r_2 = 0$$

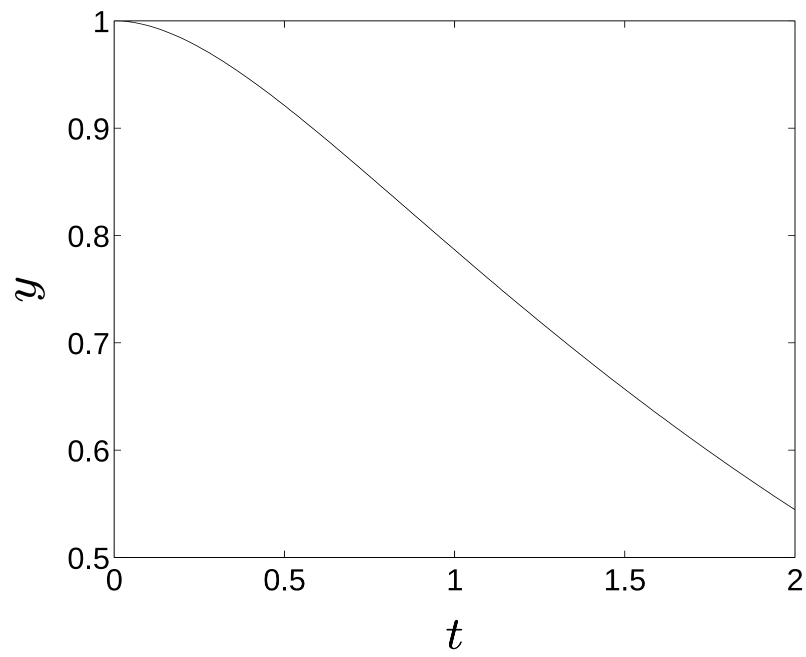
yields $r_1 = -0.17$, $r_2 = 1.17$,

$$y(t) = -0.17 e^{-2.62\,t} + 1.17 e^{-0.38\,t}$$

- first exponential decays fast, within 2sec ($T_1 = 1/|\lambda_1| = 0.38$)
- second exponential decays slower ($T_2 = 1/|\lambda_2| = 2.62$)

expanded scale, for $0 \le t \le 2$





Real equal roots $(b^2 = 4ac)$

$$\chi(s) = a(s - \lambda)^2 \quad \text{with } \lambda = -b/(2a)$$

from page 4-6,

$$Y(s) = \frac{\alpha s + \beta}{a(s - \lambda)^2}$$

express Y as

$$Y(s) = \frac{r_1}{s - \lambda} + \frac{r_2}{(s - \lambda)^2}$$

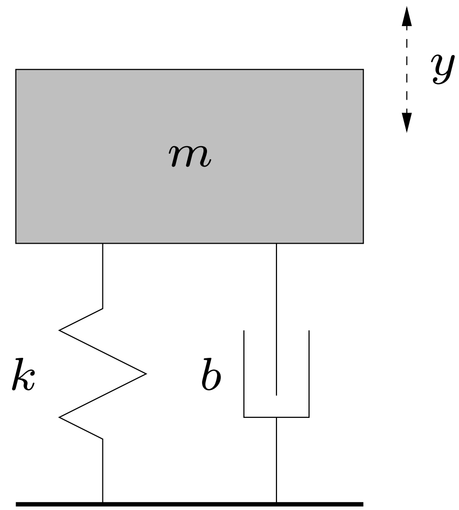
where r_1 and r_2 are found from $r_1 = \alpha/a$, $-\lambda r_1 + r_2 = \beta/a$, which yields

$$r_1 = \alpha/a, \quad r_2 = (\beta + \lambda\alpha)/a$$

inverse Laplace transform is

$$y(t) = r_1 e^{\lambda t} + r_2 t e^{\lambda t}$$

Example: mass-spring-damper



mass $m = 1$, stiffness $k = 1$, damping $b = 2$

- LCCODE (from page 2-19):

$$y'' + 2y' + y = 0$$

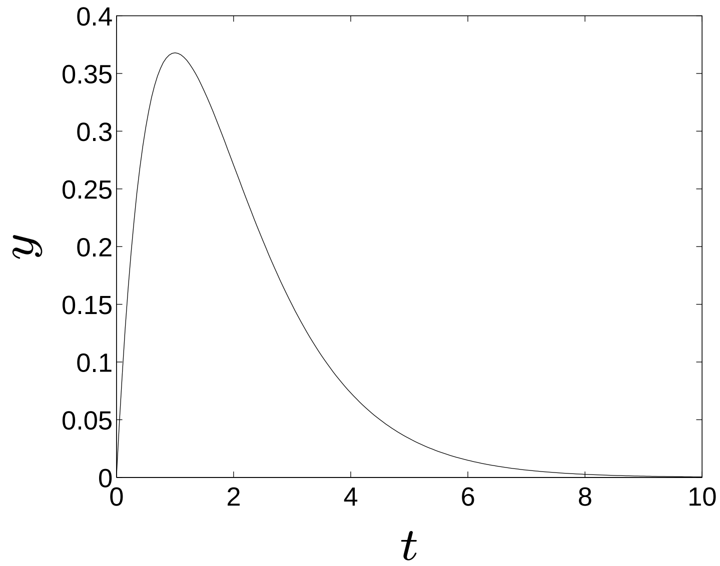
- initial conditions

$$y(0) = 0, \quad y'(0) = 1$$

solution using Laplace transform

- characteristic polynomial: $s^2 + 2s + 1 = (s+1)^2$
- solution has form $y(t) = r_1 e^{-t} + r_2 t e^{-t}$
- initial conditions determine r_1, r_2: $y(0) = r_1 = 0$, $y'(0) = -r_1 + r_2 = 1$ yields $r_1 = 0$, $r_2 = 1$, *i.e.*,

$$y(t) = te^{-t}$$



called *critically damped* system (more later)

Complex roots $(b^2 < 4ac)$

$$\chi(s) = a(s-\lambda)(s-\overline{\lambda}) \text{ with } \lambda = \sigma + j\omega,\ \overline{\lambda} = \sigma - j\omega$$

from page 4-6,

$$Y(s) = \frac{\alpha s + \beta}{a(s-\lambda)(s-\overline{\lambda})}$$

express Y as

$$Y(s) = \frac{r_1}{s-\lambda} + \frac{r_2}{s-\overline{\lambda}}$$

where r_1 and r_2 follow from $r_1 + r_2 = \alpha/a$, $-r_1\overline{\lambda} - r_2\lambda = \beta/a$:

$$r_1 = \frac{\alpha}{2a} + j\frac{\alpha b - 2a\beta}{4a^2\omega}, \quad r_2 = \overline{r}_1$$

inverse Laplace transform is

$$y(t) = r_1 e^{\lambda t} + \overline{r}_1 e^{\overline{\lambda} t}$$

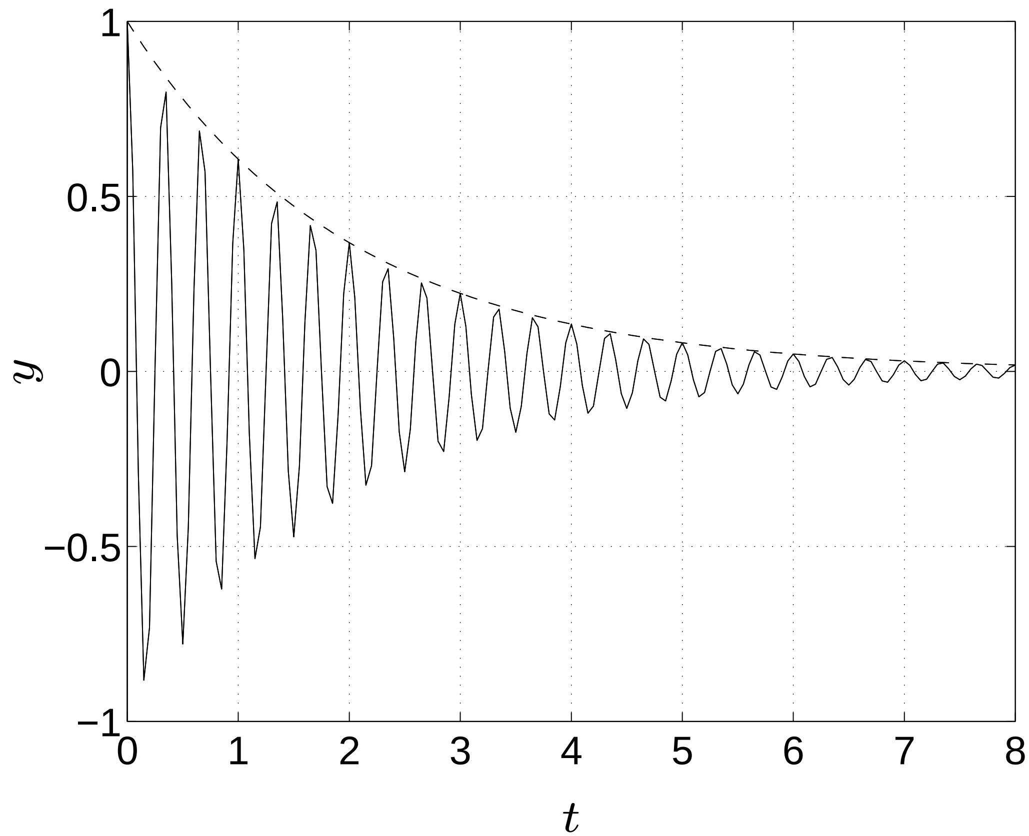
other useful forms:

$$\begin{aligned}
y(t) &= r_1 e^{\lambda t} + \overline{r}_1 e^{\overline{\lambda} t} \\
&= r_1 e^{\sigma t}(\cos \omega t + j \sin \omega t) + \overline{r}_1 e^{\sigma t}(\cos \omega t - j \sin \omega t) \\
&= (\Re(r_1) + j\Im(r_1))\; e^{\sigma t}(\cos \omega t + j \sin \omega t) \\
&\quad\; + (\Re(r_1) - j\Im(r_1)) e^{\sigma t}\; (\cos \omega t - j \sin \omega t) \\
&= 2e^{\sigma t}\; (\Re(r_1) \cos \omega t - \Im(r_1) \sin \omega t) \\
&= A e^{\sigma t} \cos(\omega t + \phi)
\end{aligned}$$

where $A = 2|r_1|$, $\phi = \arctan(\Im(r_1)/\Re(r_1))$

- if $\sigma > 0$, y is an exponentially *growing* sinusoid; if $\sigma < 0$, y is an exponentially *decaying* sinusoid; if $\sigma = 0$, y is a sinusoid
- $\Re\lambda = \sigma$ gives exponential rate of decay or growth; $\Im\lambda = \omega$ gives oscillation frequency
- amplitude A and phase ϕ determined by initial conditions
- $Ae^{\sigma t}$ is called the *envelope* of y

example



what are σ and ω here?

- oscillation period is $2\pi/\omega$
- envelope decays exponentially with time constant $-1/\sigma$

- envelope gives $|y|$ when sinusoid term is ± 1
- if $\sigma < 0$, envelope decays by $1/e$ in $-1/\sigma$ seconds
- if $\sigma > 0$, envelope doubles every $0.693/\sigma$ seconds
- growth/decay per period is $e^{2\pi(\sigma/\omega)}$
- if $\sigma < 0$, number of cycles to decay to 1% is

$$(4.6/2\pi)(\omega/|\sigma|) = 0.73(\omega/|\sigma|)$$

The harmonic oscillator

system described by LCCODE

$$y'' + \omega^2 y = 0$$

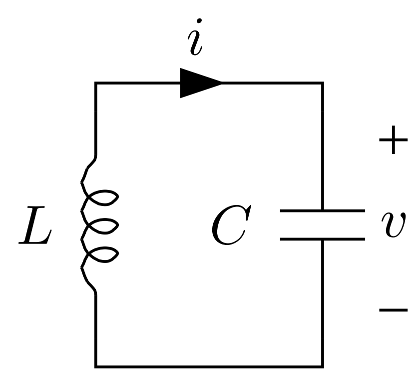
- characteristic polynomial is $s^2 + \omega^2$; roots are $\pm j\omega$
- solutions are sinusoidal: $y(t) = A\cos(\omega t + \phi)$, where A and ϕ come from initial conditions

LC circuit

- from $i = Cv'$, $v = -Li'$ we get

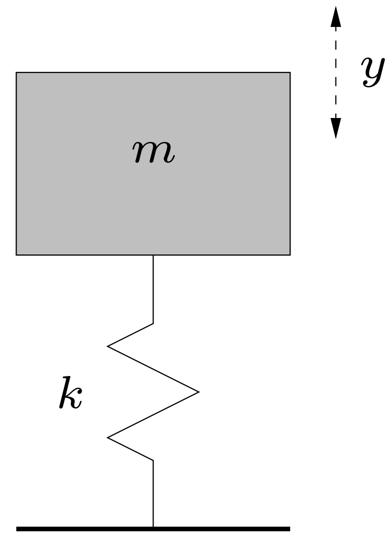
$$v'' + (1/LC)v = 0$$

- oscillation frequency is $\omega = 1/\sqrt{LC}$



mass-spring system

- $my'' + ky = 0$;
- oscillation frequency is $\omega = \sqrt{k/m}$



Stability of second order system

second order system

$$ay'' + by' + cy = 0$$

(recall assumption $a > 0$)

we say the system is **stable** if $y(t) \to 0$ as $t \to \infty$ no matter what the initial conditions are

when is a 2nd order system stable?

- for real distinct roots, solutions have the form $y(t) = r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t}$

  for stability, we need

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0,$$

  we must have $b > 0$ and $4ac > 0$, *i.e.*, $c > 0$

- for real equal roots, solutions have the form $y(t) = r_1 e^{\lambda t} + r_2 t e^{\lambda t}$

  for stability, we need
  $$\lambda = -b/2a < 0$$
  $i.e.$, $b > 0$; since $b^2 = 4ac$, we also have $c > 0$

- for complex roots, solutions have the form $y(t) = Ae^{\sigma t} \cos(\omega t + \phi)$

  for stability, we need
  $$\sigma = \Re\lambda = -b/2a < 0$$
  $i.e.$, $b > 0$; since $b^2 < 4ac$ we also have $c > 0$

summary: second order system with $a > 0$ is stable when

$$b > 0 \text{ and } c > 0$$

Decay rate

assume system $ay'' + by' + cy = 0$ is stable $(a, b, c > 0)$; how fast do the solutions decay?

- **real distinct roots** $(b^2 > 4ac)$

  since $\lambda_1 > \lambda_2$, for t large,

$$\left|r_1 e^{\lambda_1 t}\right| \gg \left|r_2 e^{\lambda_2 t}\right|$$

  (assuming r_1 is nonzero); hence asymptotic decay rate is given by

$$-\lambda_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a}$$

- **real equal roots** ($b^2 = 4ac$)

  solution is $r_1 e^{\lambda t} + r_2 t e^{\lambda t}$ which decays like $e^{\lambda t}$, so decay rate is

$$-\lambda = b/(2a) = \sqrt{c/a}$$

- **complex roots** ($b^2 < 4ac$)

  solution is $Ae^{\sigma t} \cos(\omega t + \phi)$, so decay rate is

$$-\sigma = -\Re(\lambda) = b/(2a)$$

Critical damping

question: given $a > 0$ and $c > 0$, what value of $b > 0$ gives maximum decay rate?

answer:

$$b = 2\sqrt{ac}$$

which corresponds to equal roots, and decay rate $\sqrt{c/a}$

- $b = 2\sqrt{ac}$ is called *critically damped* (real, equal roots)
- $b > 2\sqrt{ac}$ is called *overdamped* (real, distinct roots)
- $b < 2\sqrt{ac}$ is called *underdamped* (complex roots)

justification:

- if the system is underdamped, the decay rate is worse than $\sqrt{c/a}$ because

  $$b/(2a) < \sqrt{c/a},$$

  if $b < 2\sqrt{ac}$

- if the system is overdamped, the decay rate is worse than $\sqrt{c/a}$ because

$$\frac{b - \sqrt{b^2 - 4ac}}{2a} < \sqrt{c/a}$$

to prove this, multiply by $2a$ and re-arrange to get

$$b - 2\sqrt{ac} \stackrel{?}{<} \sqrt{b^2 - 4ac}$$

rewrite as

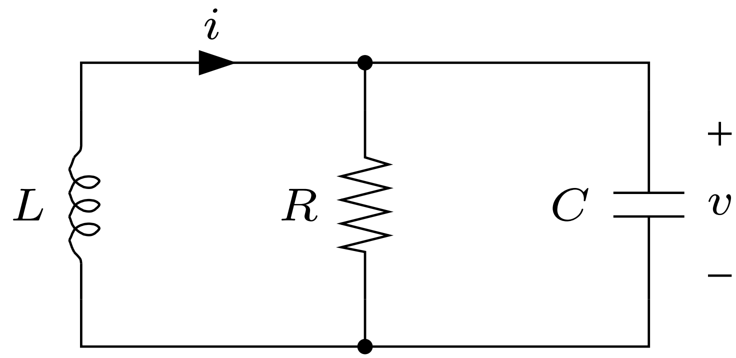
$$b - 2\sqrt{ac} \stackrel{?}{<} \sqrt{(b - 2\sqrt{ac})(b + 2\sqrt{ac})}$$

divide by $b - 2\sqrt{ac}$ to get

$$1 \stackrel{?}{<} \frac{\sqrt{b + 2\sqrt{ac}}}{\sqrt{b - \sqrt{ac}}}$$

which is true . . .

Parallel RLC circuit



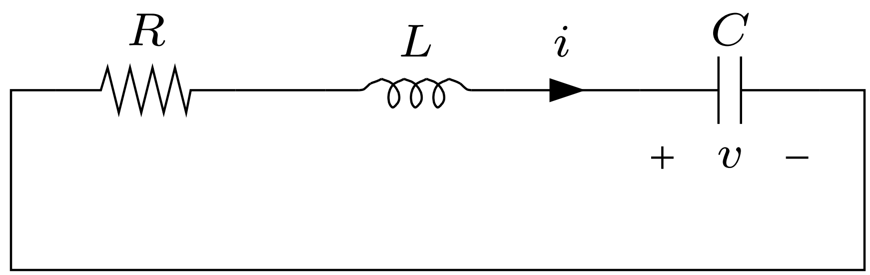
we have $v = -Li'$ and $Cv' = i - v/R$, so

$$v'' + \frac{1}{RC}v' + \frac{1}{LC}v = 0$$

- stable (assuming $L, R, C > 0$)
- overdamped if $R < \sqrt{L/(4C)}$
- critically damped if $R = \sqrt{L/(4C)}$
- underdamped if $R > \sqrt{L/4C}$; oscillation frequency is

$$\omega = \sqrt{1/LC - (1/2RC)^2}$$

Series RLC circuit



by KVL, $Ri + Li' + v = 0$; also, $i = Cv'$, so

$$v'' + \frac{R}{L}v' + \frac{1}{LC}v = 0$$

- stable (assuming $L, R, C > 0$)
- overdamped if $R > 2\sqrt{L/C}$
- critically damped if $R = 2\sqrt{L/C}$
- underdamped if $R < 2\sqrt{L/C}$; oscillation frequency is

$$\omega = \sqrt{1/LC - (R/2L)^2}$$