Lecture 15
Applications of feedback

- Oscillators
- Phase-locked loop

Oscillators

feedback is widely used in *oscillators*, which generate sinusoidal signals

to generate sinusoidal signal at frequency ω_0,

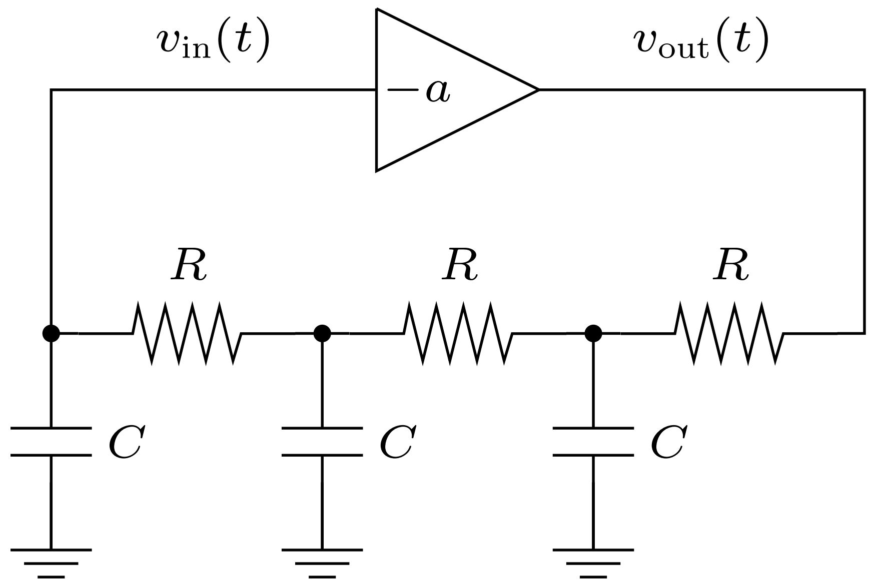
- closed-loop system should have poles at $\pm j\omega_0$
- other poles should have negative real part ($i.e.$, other terms decay)

closed-loop pole at $j\omega_0 \Leftrightarrow L(j\omega_0) = -1$, $i.e.$,

$$\angle L(j\omega_0) = 180^\circ + q360^\circ, \quad |L(j\omega_0)| = 1$$

intuition: gain around whole loop, including $-$ sign, is $q360^\circ$; the feedback ‘regenerates’ the signal (at frequency ω_0)

Example. voltage amplifier has gain $-a$, $a > 0$



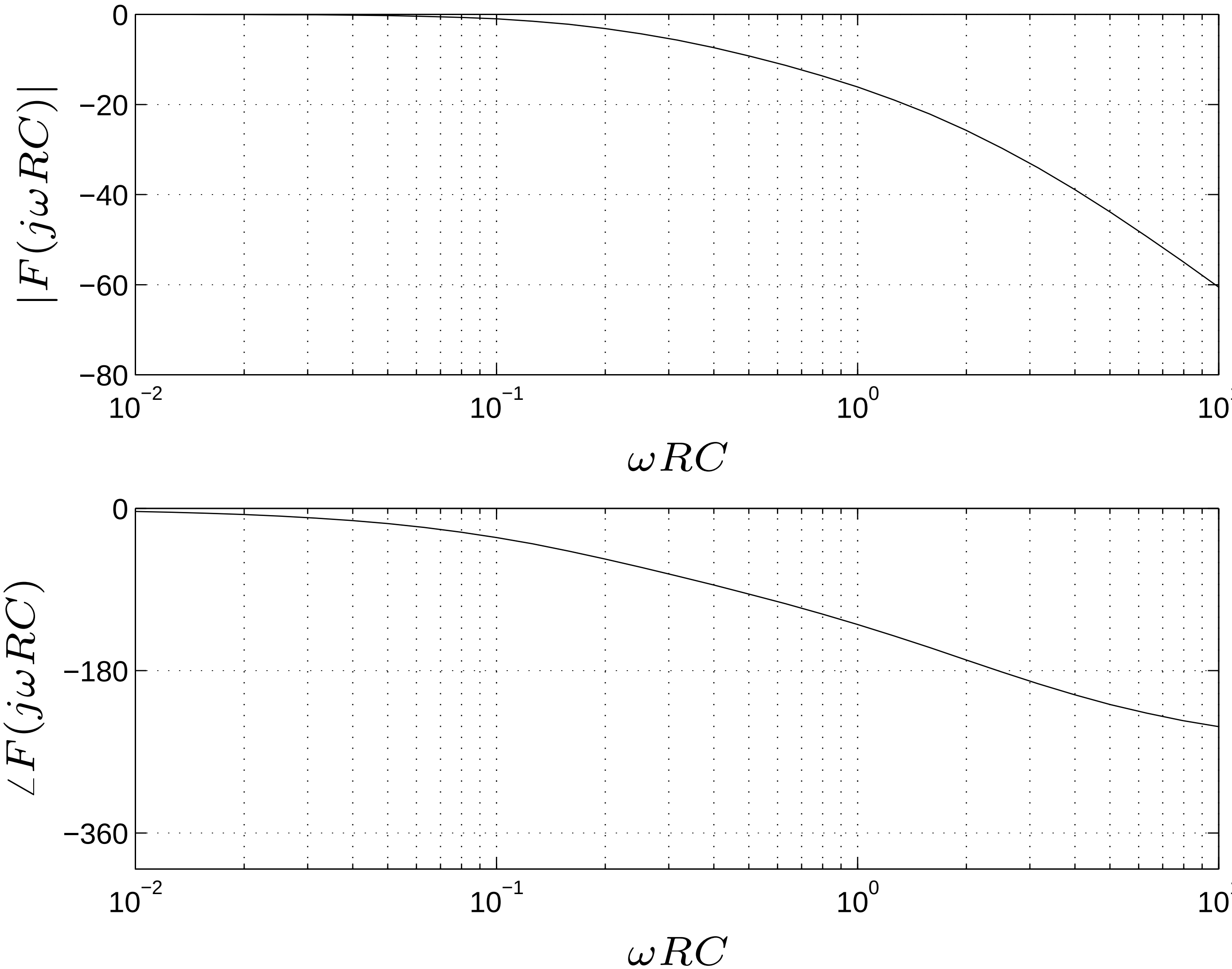
design variables: R, C, a

transfer function of RC feedback network is

$$F(s) = \frac{V_{\rm in}(s)}{V_{\rm out}(s)} = \frac{1}{(sRC)^3 + 5(sRC)^2 + 6(sRC) + 1}$$

loop transfer function is $L = aF$

Bode plot of F:



want loop transfer function $L = aF = -1$ at ω_0

$$\angle L(j\omega_0) = \angle aF(j\omega_0) = \angle F(j\omega_0) = 180^\circ$$

from Bode phase plot we see $\omega_0 \approx 2.5/(RC)$

from Bode magniture plot we see $F(\omega_0) \approx -30\text{dB}$, so we need $a \approx +30\text{dB}$

analytically: $\angle L(j\omega_0) = 180^\circ$ is same as

$$\Im 1/L(j\omega_0) = \frac{1}{a}\left((j\omega_0 RC)^3 + 6(j\omega_0 RC)\right) = 0$$

so $\omega_0 = \sqrt{6}/(RC)$

hence $L(j\omega_0) = \dfrac{a}{5(j\sqrt{6})^2 + 1} = -\dfrac{a}{29}$, so we need $a = 29$

summary: with gain $a = 29$, system oscillates at freq. $\omega_0 = \sqrt{6}/(RC)$

(can check third pole is real & negative)

in practice, gain needs to be a little larger; nonlinearities limit amplitude of oscillation (careful analysis is hard)

intuitive analysis of RC oscillator:

- at $\omega_0 = \sqrt{6}/(RC)$, RC network gives 180° loop phase shift
- amplifer gain $a = 29$ makes up for amplitude loss of RC network at ω_0

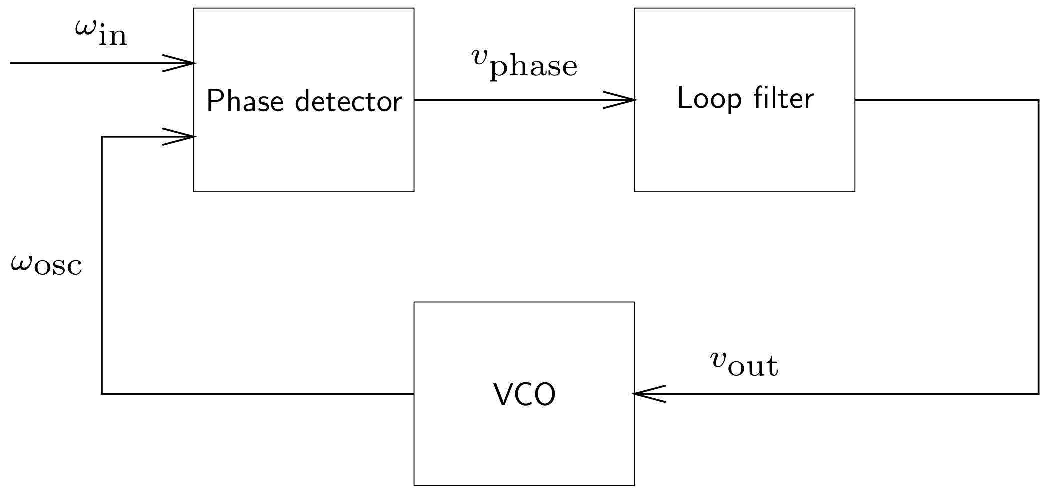
Phase-locked loop (PLL)

PLL is widely used to 'synchronize' one signal to another

applications:

- synchronize clocks, frequency multiplication/division
- video signal sync
- FM demodulation
- synchronize data communications transmitter, receiver
- track varying frequency source ($e.g.$, Doppler from space vehicle or aircraft)

basic PLL block diagram:



- signals on left (marked $\omega_{\rm in}$, $\omega_{\rm osc}$) are frequency-varying sinusoids, of form $\cos(\omega(t)t)$; *instantaneous frequency* $\omega(t)$ varies only a small amount around some fixed value
- signals on right ($v_{\rm phase}$, $v_{\rm out}$) are (usually) voltages

idea: feedback from phase detector adjusts voltage controlled oscillator (VCO) frequency to match input frequency

Phase detector

phase detector generates voltage proportional to phase difference of freqeuncy-varying sinusoids

$$\begin{aligned} v_{\mathrm{phase}}(t) &= k_{\mathrm{det}}\theta_{\mathrm{err}}(t) \\ \theta_{\mathrm{err}}(t) &= \int_0^t \left(\omega_{\mathrm{in}}(\tau) - \omega_{\mathrm{osc}}(\tau)\right)\, d\tau \end{aligned}$$

provided phase difference θ_{err} is less than $\pm 90^\circ$ or so

k_{det} is the detector gain (V/rad)

Voltage controlled oscillator

VCO generates frequency-varying sinusoid, with frequency depending on its input voltage $v_{\rm out}(t)$:

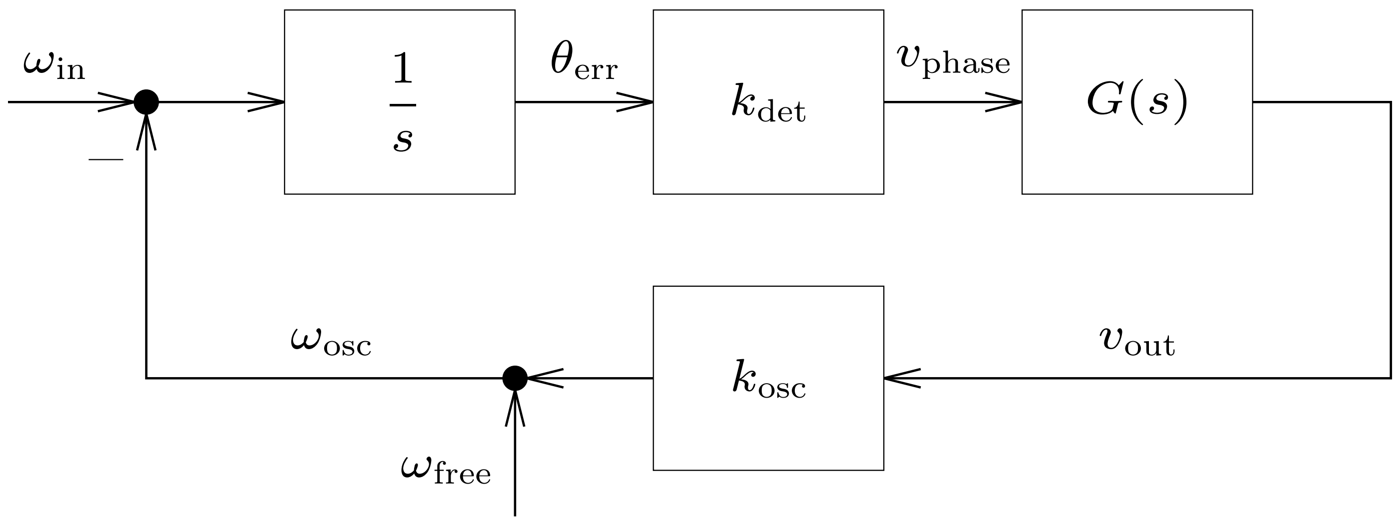
$$\omega_{\rm osc}(t) = \omega_{\rm free} + k_{\rm osc} v_{\rm out}(t)$$

- $\omega_{\rm free}$ is called the *free running frequency*
- $k_{\rm osc}$ is the VCO gain (in $(\text{rad/sec})/\text{V}$)

real VCOs have a minimum and maximum frequency

LTI analysis of PLL

loop filter is often LTI, $i.e.$, T.F. $G(s)$



this LTI model of PLL is good **provided**

- $|\theta_{\text{err}}| \leq 90^\circ$ (or so)
- ω_{osc} stays within limits

transfer function from $\omega_{\rm free}$ to $\omega_{\rm osc}$ is

$$\frac{s}{s + k_{\rm det}k_{\rm osc}G(s)}$$

which is zero at $s = 0$, so $\omega_{\rm free}$ does not affect $\omega_{\rm osc}$ (in steady-state)

transfer function H from $\omega_{\rm in}$ to $\omega_{\rm osc}$ is

$$H(s) = \frac{k_{\rm det}k_{\rm osc}G(s)/s}{1 + k_{\rm det}k_{\rm osc}G(s)/s} = \frac{k_{\rm det}k_{\rm osc}G(s)}{s + k_{\rm det}k_{\rm osc}G(s)}$$

phase detector gives integral action: $H(0) = 1$

for constant $\omega_{\rm in}$, $\omega_{\rm osc}(t) \rightarrow \omega_{\rm in}$ as $t \rightarrow \infty$,
$i.e.$, VCO frequency locks to input frequency

First order loop

simplest PLL uses $G(s) = 1$, which yields

$$H(s) = \frac{1}{1 + s/(k_{\rm det} k_{\rm osc})}$$

(hence the name 'first order')

- $k_{\rm osc} k_{\rm det}$ is called *loop bandwidth*
- $\omega_{\rm osc}(t)$ tracks $\omega_{\rm in}(t)$, with time constant $1/(k_{\rm det} k_{\rm osc})$

Second order loop

very common PLL uses lowpass loop filter

$$G(s) = \frac{1}{1 + s/\omega_{\rm loop}}$$

which yields

$$H(s) = \frac{1}{1 + s/(k_{\rm det}k_{\rm osc}) + s^2/(\omega_{\rm loop}k_{\rm det}k_{\rm osc})}$$

(hence the name 'second order')

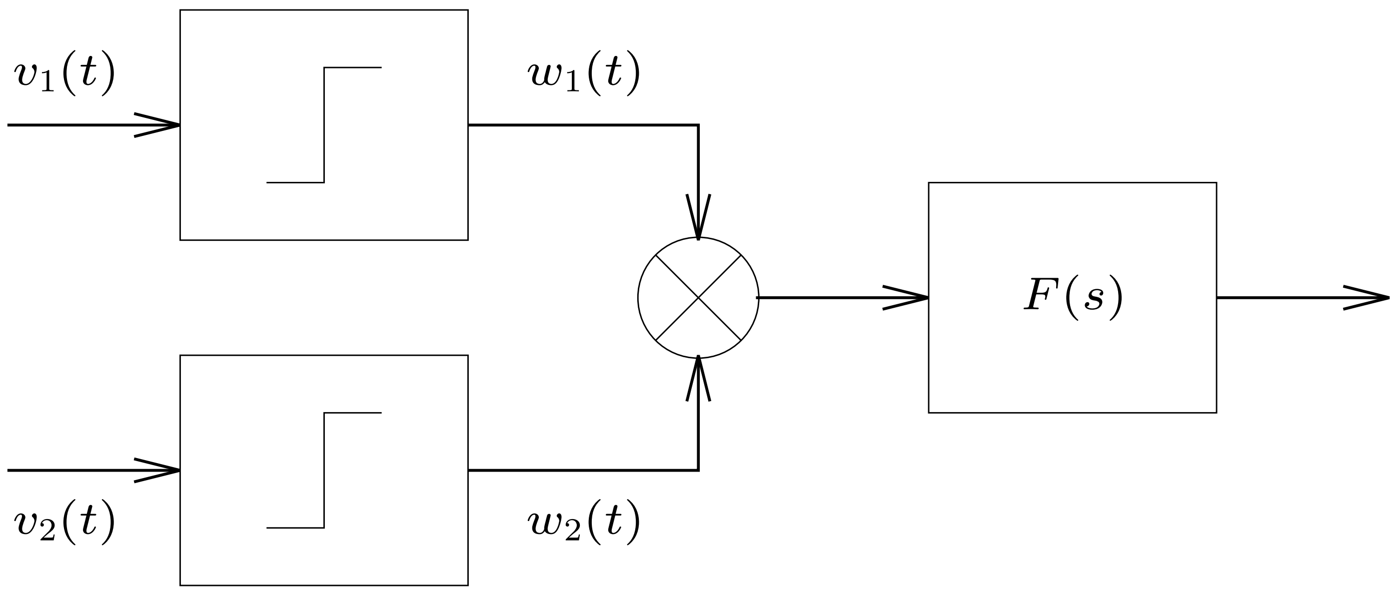
common choices for $\omega_{\rm loop}$:

- $\omega_{\rm loop} = 4k_{\rm det}k_{\rm osc}$ (sets both poles of H to $-\omega_{\rm loop}/2 = -2k_{\rm det}k_{\rm osc}$)
- $\omega_{\rm loop} = 2k_{\rm det}k_{\rm osc}$
  (sets poles of H to $(-1 \pm j)\omega_{\rm loop}/2 = (-1 \pm j)k_{\rm det}k_{\rm osc}$)

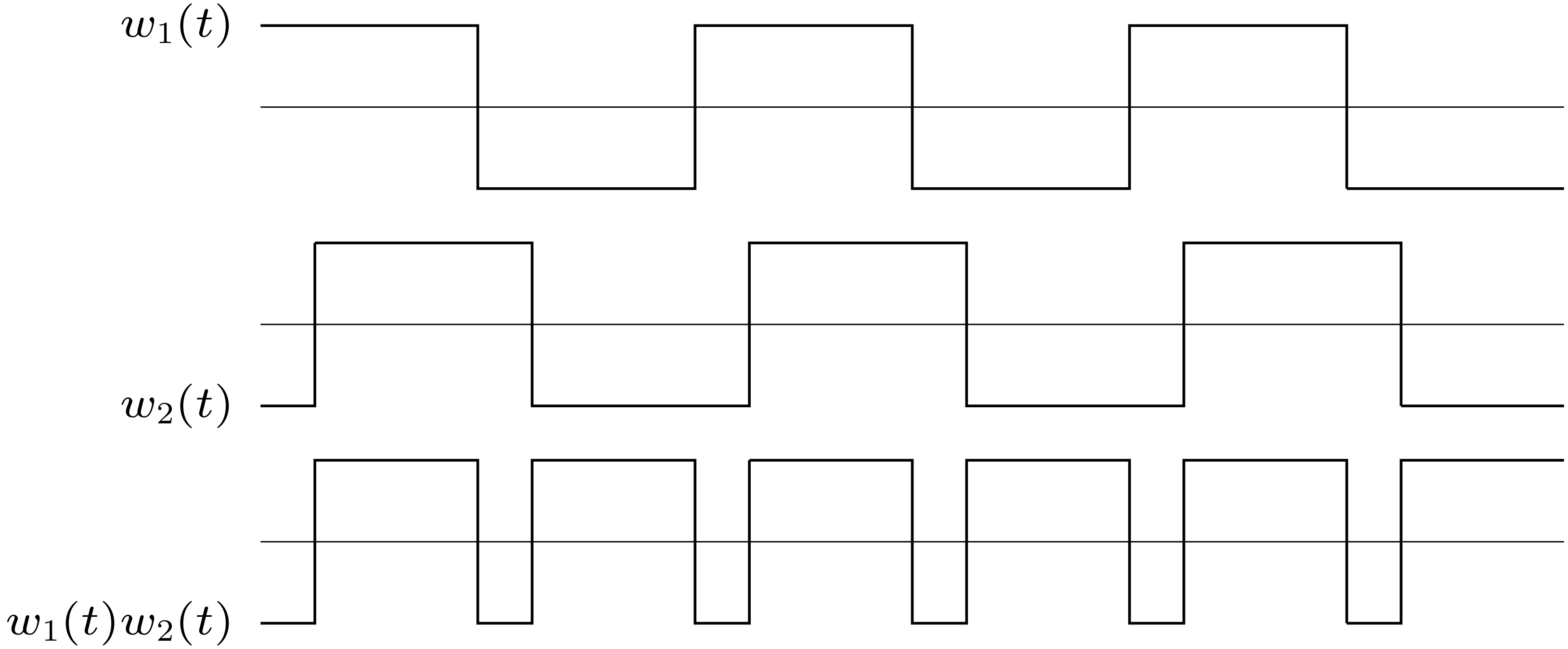
$\omega_{\rm osc}$ is 2nd order lowpass filtered version of $\omega_{\rm in}$

Phase detectors

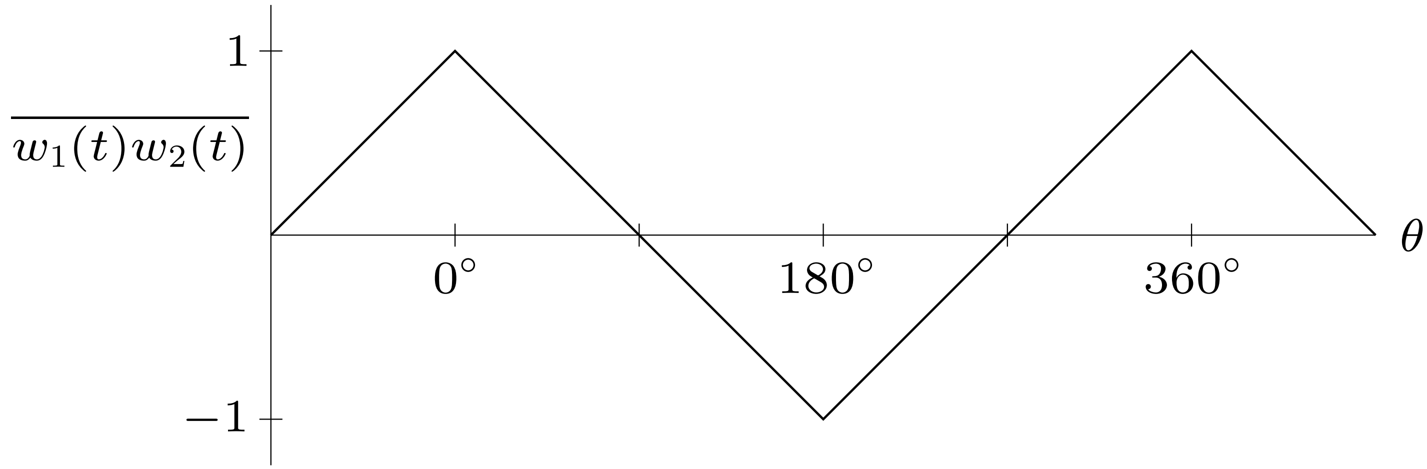
basic idea of common phase detector:



- input signals are converted to squarewaves, then multiplied
- lowpass filtered (averaged) by $F(s)$


$w_1(t)$
$w_2(t)$
$w_1(t)w_2(t)$

average value of $w_1(t)w_2(t)$ depends on phase difference θ:



- this phase detector operates between 0° and 180° instead of $\pm 90^\circ$; doesn't affect PLL
- if bandwidth of lowpass filter is $\ll$ input frequencies, its output $\approx \overline{w_1(t)w_2(t)}$
- phase detector is linear (plus constant) for $0 \leq \theta \leq 180^\circ$

Lock range

lock range: range of $\omega_{\rm in}$ over which can the PLL can maintain 'lock'

recall that we must have $|\theta_{\rm err}| < \pi/2$ or so for linear phase detector operation

transfer function from $\omega_{\rm in} - \omega_{\rm free}$ to $\theta_{\rm err}$:

$$F(s) = \frac{1}{s + k_{\rm det} k_{\rm osc} G(s)}$$

for $\omega_{\rm in}$ constant, we have in steady-state

$$\theta_{\rm err} = F(0)(\omega_{\rm in} - \omega_{\rm free}) = \frac{\omega_{\rm in} - \omega_{\rm free}}{k_{\rm det} k_{\rm osc} G(0)}$$

so lock range for slowly varying $\omega_{\rm in}$ is

$$|\omega_{\rm in} - \omega_{\rm free}| \leq (\pi/2) k_{\rm det} k_{\rm osc} G(0)$$

lock range analysis for rapidly varying $\omega_{\rm in}$:

assume $|\omega_{\rm in}(t) - \omega_{\rm free}(t)| \le \Omega$ for all t

$$\theta_{\rm err} = f * (\omega_{\rm in} - \omega_{\rm free})$$

where f is impulse response of F

by peak-gain analysis, peak of $\theta_{\rm err}(t)$ is no more than

$$\Omega \int_0^\infty |f(\tau)|\, d\tau$$

so lock occurs for

$$\Omega \int_0^\infty |f(\tau)|\, d\tau \le \pi/2$$

first order loop filter ($G(s) = 1$):

$$F(s) = \frac{1}{s + k_{\mathrm{det}} k_{\mathrm{osc}}}, \quad f(t) = e^{-k_{\mathrm{det}} k_{\mathrm{osc}} t}$$

so

$$\int_0^\infty |f(\tau)| \, d\tau = F(0) = \frac{1}{k_{\mathrm{det}} k_{\mathrm{osc}}}$$

therefore lock range, for rapidly varying ω_{in}, is

$$\Omega \leq (\pi/2) k_{\mathrm{det}} k_{\mathrm{osc}}$$

(same as slowly-varying lock range . . .)

PLL: nonlinear analysis

our linear analysis holds as long as

- phase difference stays in the linear range (say, $\pm 90^\circ$)
- frequency stays within VCO limits (usually not the problem)

when phase difference moves out of linear range, dynamics of PLL are highly nonlinear, very complex

- analysis of losing and acquiring lock is very difficult
- there is no complete theory or analysis, but lots of practical experience