Lecture 1
Signals

- notation and meaning
- common signals
- size of a signal
- qualitative properties of signals
- impulsive signals

Signals

a *signal* is a function of time, $e.g.$,

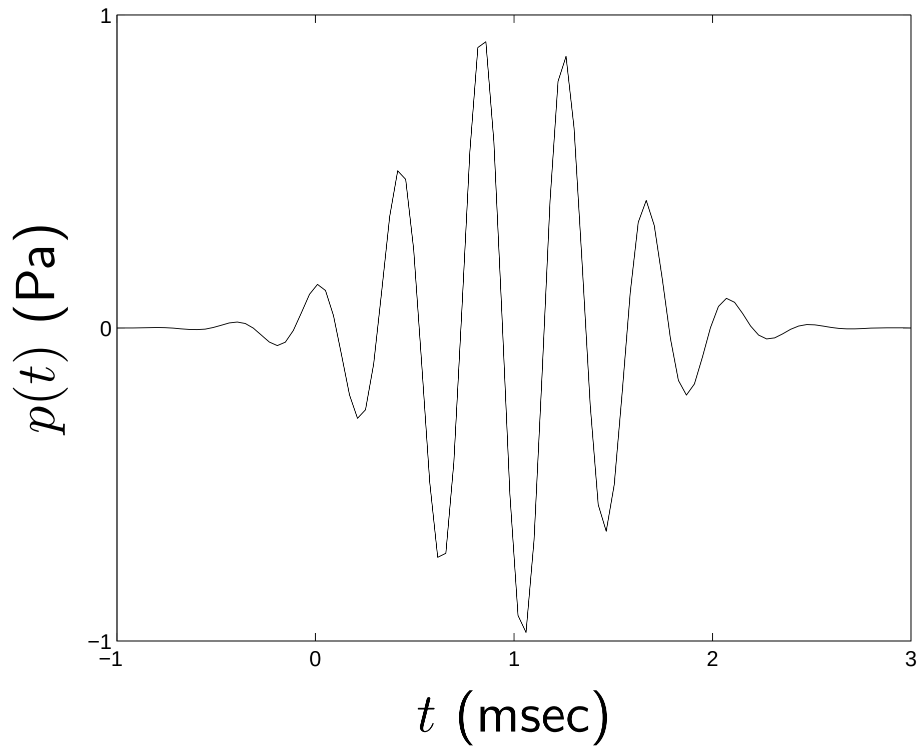
- f is the force on some mass
- $v_{\rm out}$ is the output voltage of some circuit
- p is the acoustic pressure at some point

notation:

- f, $v_{\rm out}$, p or $f(\cdot)$, $v_{\rm out}(\cdot)$, $p(\cdot)$ refer to the *whole signal* or *function*
- $f(t)$, $v_{\rm out}(1.2)$, $p(t+2)$ refer to the *value* of the signals at times t, 1.2, and $t+2$, respectively

for *times* we usually use symbols like t, τ, t_1, . . .

Example



Domain of a signal

domain of a signal: t's for which it is defined

some common domains:

- all t, *i.e.*, **R**
- nonnegative t: $t \geq 0$
  (here $t = 0$ just means some starting time of interest)
- t in some interval: $a \leq t \leq b$
- t at uniformly sampled points: $t = kh + t_0$, $k = 0, \pm 1, \pm 2, \ldots$
- *discrete-time signals* are defined for integer t, *i.e.*, $t = 0, \pm 1, \pm 2, \ldots$
  (here t means sample time or epoch, not real time in seconds)

we'll usually study signals defined on all reals, or for nonnegative reals

Dimension & units of a signal

dimension or **type** of a signal u, *e.g.*,

- *real-valued* or *scalar signal*: $u(t)$ is a real number (scalar)
- *vector signal*: $u(t)$ is a vector of some dimension
- *binary signal*: $u(t)$ is either 0 or 1

we'll usually encounter *scalar signals*

example: a vector-valued signal

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

might give the voltage at three places on an antenna

physical units of a signal, *e.g.*, V, mA, $\mathrm{m/sec}$

sometimes the physical units are 1 (*i.e.*, unitless) or unspecified

Common signals with names

- a constant (or static or DC) signal: $u(t) = a$, where a is some constant
- the *unit step* signal (sometimes denoted $1(t)$ or $U(t)$),

$$u(t) = 0 \text{ for } t < 0, \quad u(t) = 1 \text{ for } t \geq 0$$

- the *unit ramp* signal,

$$u(t) = 0 \text{ for } t < 0, \quad u(t) = t \text{ for } t \geq 0$$

- a *rectangular pulse* signal,

$$u(t) = 1 \text{ for } a \leq t \leq b, \quad u(t) = 0 \text{ otherwise}$$

- a *sinusoidal* signal:

$$u(t) = a\cos(\omega t + \phi)$$

a, b, ω, ϕ are called signal *parameters*

Real signals

most real signals, *e.g.*,

- AM radio signal
- FM radio signal
- cable TV signal
- audio signal
- NTSC video signal
- 10BT ethernet signal
- telephone signal

aren't given by mathematical formulas, but they do have defining characteristics

Measuring the size of a signal

size of a signal u is measured in many ways

for example, if $u(t)$ is defined for $t \geq 0$:

- *integral square* (or *total energy*): $\displaystyle\int_0^\infty u(t)^2 \, dt$
- squareroot of total energy
- *integral-absolute value*: $\displaystyle\int_0^\infty |u(t)| \, dt$
- *peak* or *maximum absolute value* of a signal: $\max_{t \geq 0} |u(t)|$
- *root-mean-square* (RMS) value: $\displaystyle\left(\lim_{T \to \infty} \frac{1}{T} \int_0^T u(t)^2 \, dt \right)^{1/2}$
- *average-absolute* (AA) value: $\displaystyle\lim_{T \to \infty} \frac{1}{T} \int_0^T |u(t)| \, dt$

for some signals these measures can be infinite, or undefined

example: for a sinusoid $u(t) = a\cos(\omega t + \phi)$ for $t \geq 0$

- the peak is $|a|$
- the RMS value is $|a|/\sqrt{2} \approx 0.707|a|$
- the AA value is $|a|2/\pi \approx 0.636|a|$
- the integral square and integral absolute values are ∞

the *deviation* between two signals u and v can be found as the size of the difference, $e.g.$, RMS$(u - v)$

Qualitative properties of signals

- u *decays* if $u(t) \to 0$ as $t \to \infty$
- u *converges* if $u(t) \to a$ as $t \to \infty$ (a is some constant)
- u is *bounded* if its peak is finite
- u is *unbounded* or *blows up* if its peak is infinite
- u is *periodic* if for some $T > 0$, $u(t + T) = u(t)$ holds for all t

in practice we are interested in more specific quantitative questions, $e.g.$,

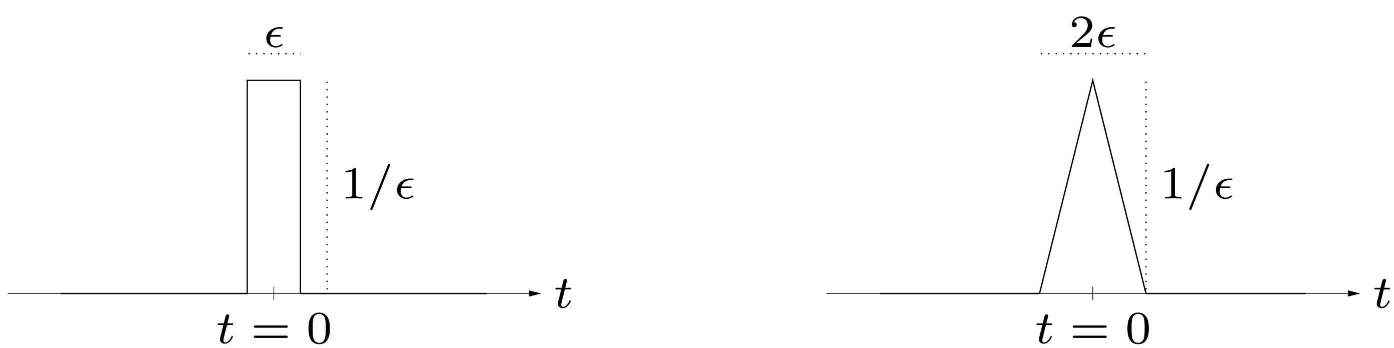
- how fast does u decay or converge?
- how large is the peak of u?

Impulsive signals

(Dirac's) **delta function** or **impulse** δ is an *idealization* of a signal that

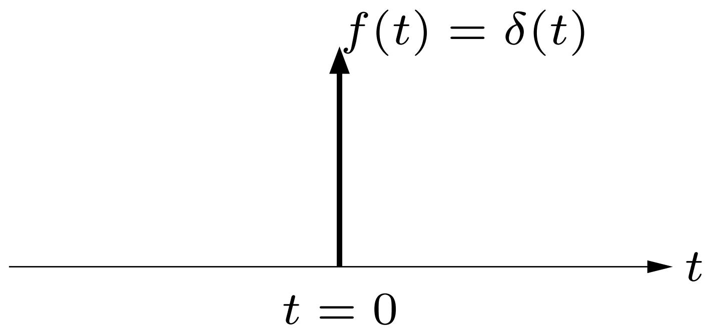
- is very large near $t = 0$
- is very small away from $t = 0$
- has integral 1

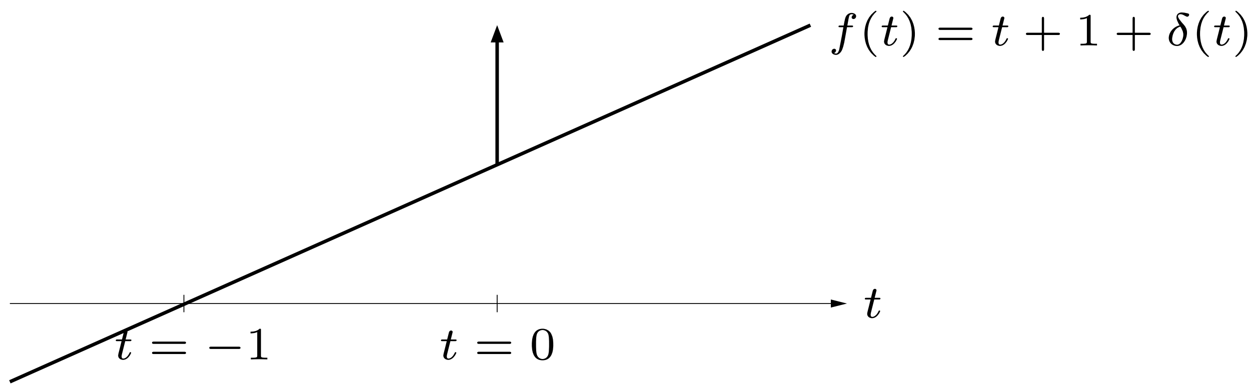
for example:



- the exact shape of the function doesn't matter
- ϵ is small (which depends on context)

on plots δ is shown as a solid arrow:





Formal properties

formally we **define** δ by the property that

$$\int_a^b f(t)\delta(t)\, dt = f(0)$$

provided $a < 0$, $b > 0$, and f is continuous at $t = 0$

idea: δ acts over a time interval very small, over which $f(t) \approx f(0)$

- $\delta(t) = 0$ for $t \neq 0$
- $\delta(0)$ isn't really defined
- $\displaystyle\int_a^b \delta(t)\, dt = 1$ if $a < 0$ and $b > 0$
- $\displaystyle\int_a^b \delta(t)\, dt = 0$ if $a > 0$ or $b < 0$

$\int_{a}^{b} \delta(t)\, dt = 0$ is **ambiguous** if $a = 0$ or $b = 0$

our convention: to avoid confusion we use limits such as $a-$ or $b+$ to denote whether we include the impulse or not

for example,

$$\int_{0+}^{1} \delta(t)\, dt = 0, \quad \int_{0-}^{1} \delta(t)\, dt = 1, \quad \int_{-1}^{0-} \delta(t)\, dt = 0, \quad \int_{-1}^{0+} \delta(t)\, dt = 1$$

Scaled impulses

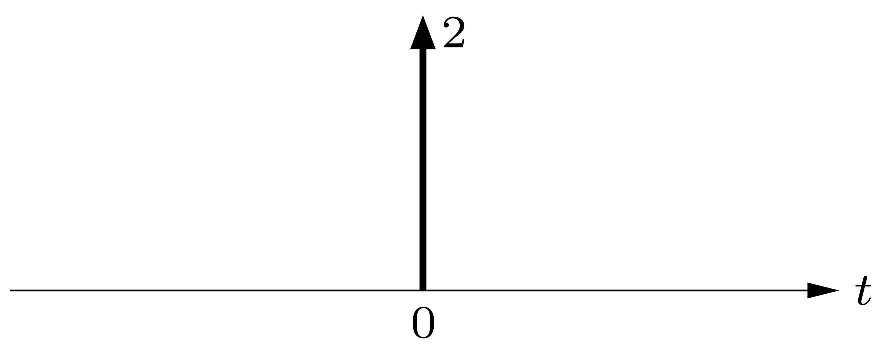
$\alpha\delta(t-T)$ is sometimes called an impulse at time T, with *magnitude* α

we have

$$\int_a^b \alpha\delta(t-T)f(t)\,dt = \alpha f(T)$$

provided $a < T < b$ and f is continuous at T

on plots: write magnitude next to the arrow, *e.g.*, for 2δ,



Sifting property

the signal $u(t) = \delta(t - T)$ is an impulse function with impulse at $t = T$

for $a < T < b$, and f continuous at $t = T$, we have

$$\int_a^b f(t)\delta(t - T)\, dt = f(T)$$

example:

$$\begin{aligned}
&\int_{-2}^{3} f(t)(2 + \delta(t+1) - 3\delta(t-1) + 2\delta(t+3))\, dt \\
&\quad = \quad 2\int_{-2}^{3} f(t)\, dt + \int_{-2}^{3} f(t)\delta(t+1)\, dt - 3\int_{-2}^{3} f(t)\delta(t-1)\, dt \\
&\qquad\quad + 2\int_{-2}^{3} f(t)\delta(t+3))\, dt \\
&\quad = \quad 2\int_{-2}^{3} f(t)\, dt + f(-1) - 3f(1)
\end{aligned}$$

Physical interpretation

impulse functions are used to model physical signals

- that act over short time intervals
- whose effect depends on integral of signal

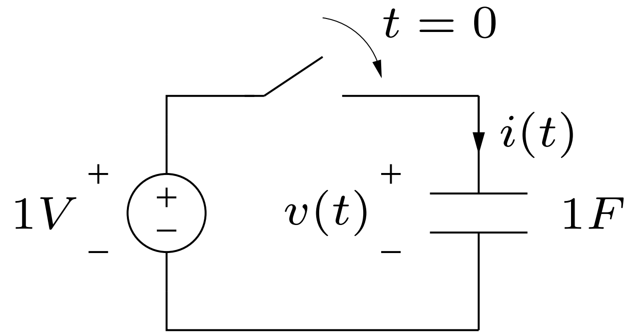
example: hammer blow, or bat hitting ball, at $t = 2$

- force f acts on mass m between $t = 1.999$ sec and $t = 2.001$ sec
- $\displaystyle\int_{1.999}^{2.001} f(t)\, dt = I$ (mechanical impulse, $\mathrm{N} \cdot \mathrm{sec}$)
- blow induces change in velocity of

$$v(2.001) - v(1.999) = \frac{1}{m} \int_{1.999}^{2.001} f(\tau)\, d\tau = I/m$$

for (most) applications we can model force as an impulse, at $t = 2$, with magnitude I

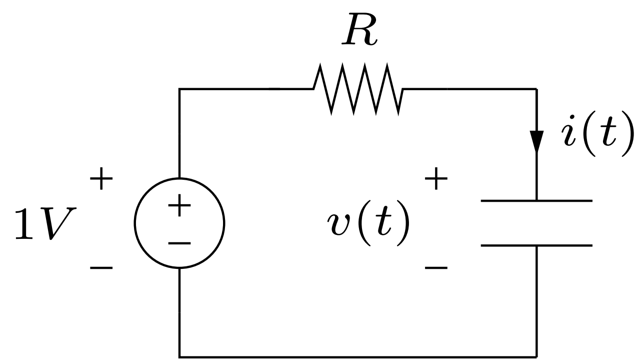
example: rapid charging of capacitor



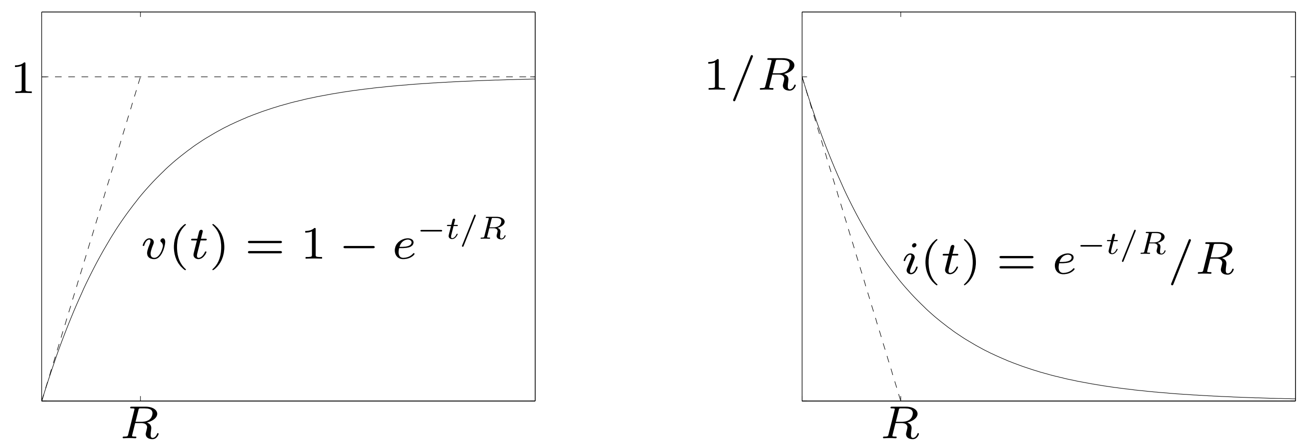
assuming $v(0) = 0$, what is $v(t)$, $i(t)$ for $t > 0$?

- $i(t)$ is very large, for a very short time
- a unit charge is transferred to the capacitor 'almost instantaneously'
- $v(t)$ increases to $v(t) = 1$ 'almost instantaneously'

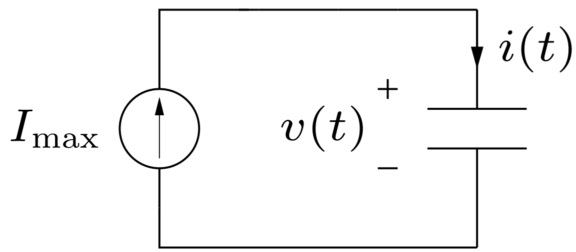
to calculate i, v, we need a more detailed model for example, include small resistance



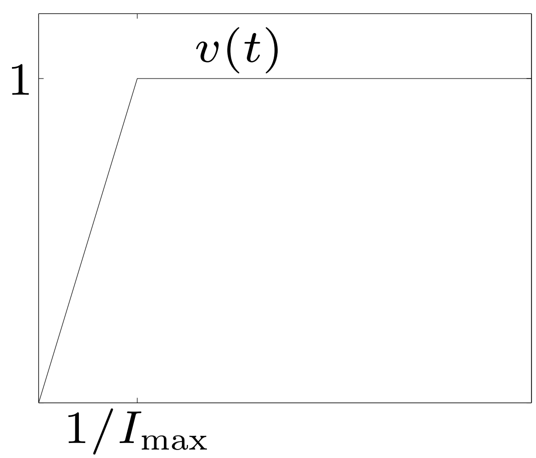
$$i(t) = \frac{dv(t)}{dt} = \frac{1 - v(t)}{R}, \quad v(0) = 0$$

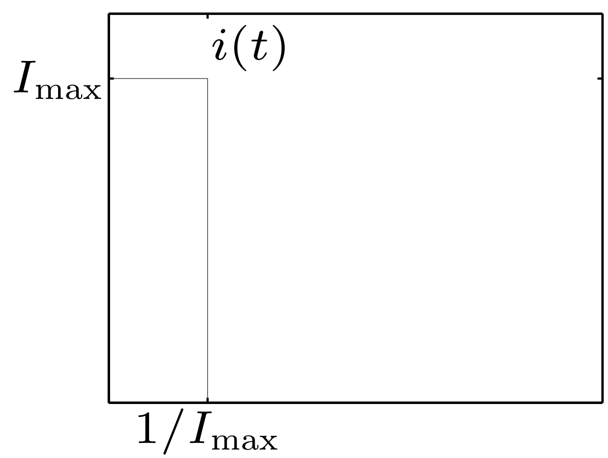


as $R \to 0$, i approaches an impulse, v approaches a unit step as another example, assume the current delivered by the source is limited: if $v(t) < 1$, the source acts as a current source $i(t) = I_{\max}$



$$i(t) = \frac{dv(t)}{dt} = I_{\max}, \quad v(0) = 0$$





as $I_{\max} \to \infty$, i approaches an impulse, v approaches a unit step in conclusion,

- large current i acts over very short time between $t = 0$ and ϵ
- total charge transfer is $\int_0^{\epsilon} i(t)\,dt = 1$
- resulting change in $v(t)$ is $v(\epsilon) - v(0) = 1$
- can approximate i as impulse at $t = 0$ with magnitude 1

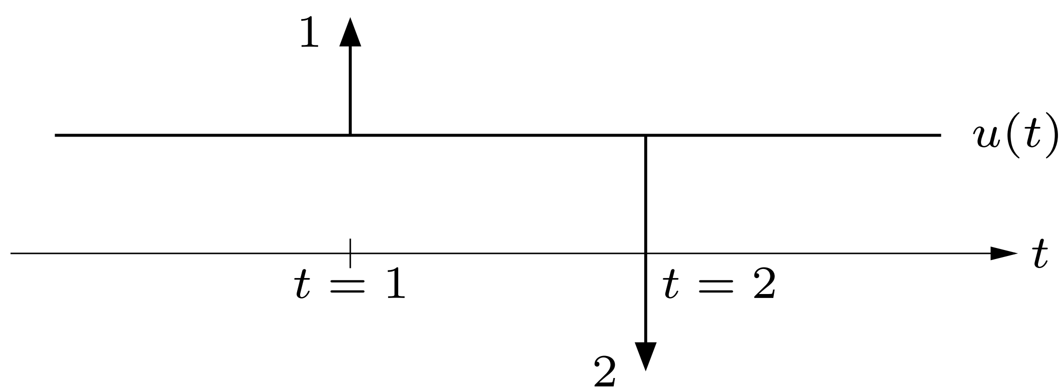
modeling current as impulse

- obscures details of current signal
- obscures details of voltage change during the rapid charging
- preserves total change in charge, voltage
- is reasonable model for time scales $\gg \epsilon$

Integrals of impulsive functions

integral of a function with impulses has jump at each impulse, equal to the magnitude of impulse

example: $u(t) = 1 + \delta(t-1) - 2\delta(t-2)$; define $f(t) = \int_0^t u(\tau)\, d\tau$



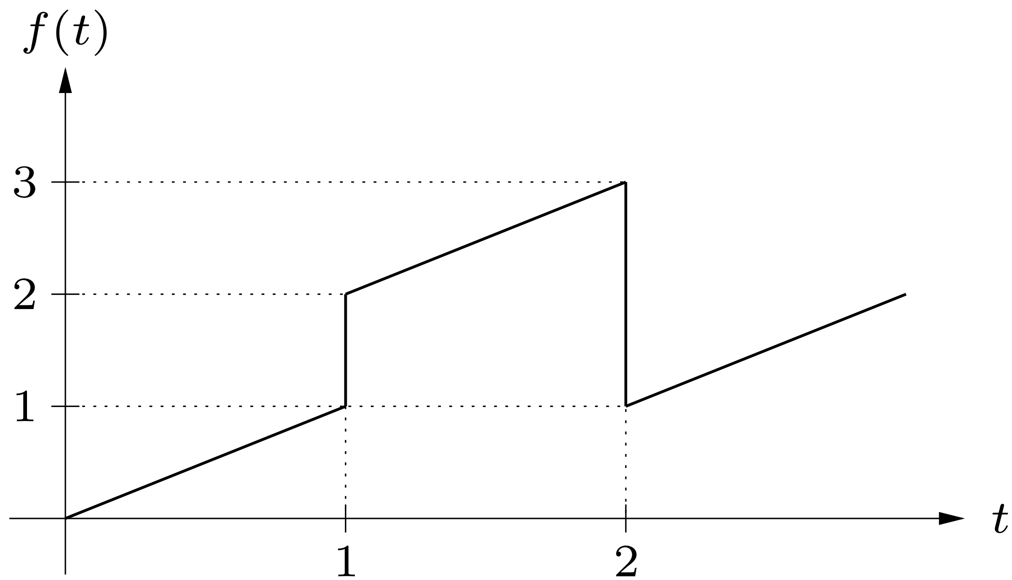
$f(t) = t$ for $0 \le t < 1$, $\quad f(t) = t+1$ for $1 < t < 2$, $\quad f(t) = t-1$ for $t > 2$

($f(1)$ and $f(2)$ are undefined)

Derivatives of discontinuous functions

conversely, derivative of function with discontinuities has impulse at each jump in function

- derivative of unit step function (see page 1–6) is $\delta(t)$
- signal f of previous page



$$f'(t) = 1 + \delta(t-1) - 2\delta(t-2)$$

Derivatives of impulse functions

integration by parts suggests we define

$$\int_a^b \delta'(t) f(t)\, dt = \delta(t) f(t)\Big|_a^b - \int_a^b \delta(t) f'(t)\, dt = -f'(0)$$

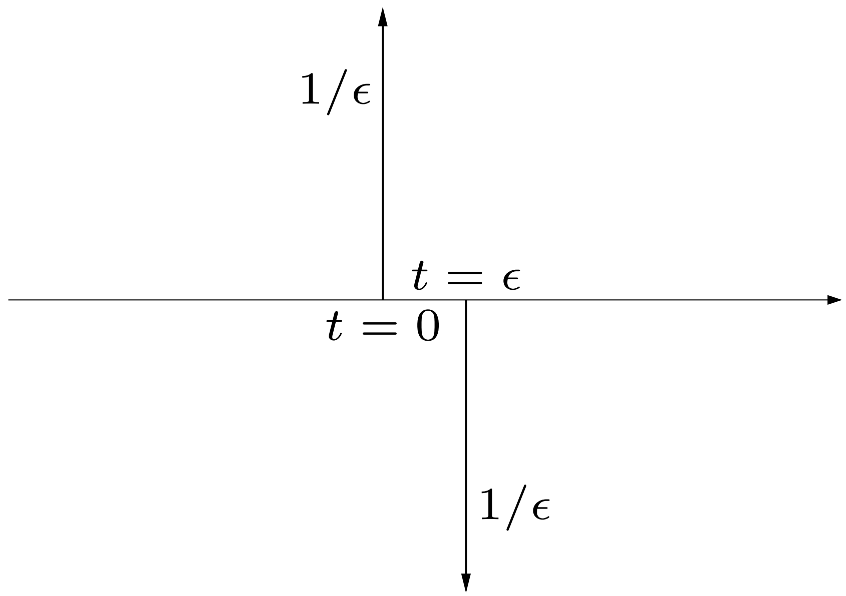
provided $a < 0$, $b > 0$, and f' continuous at $t = 0$

- δ' is called *doublet*
- δ', δ'', etc. are called *higher-order impulses*
- similar rules for higher-order impulses:

$$\int_a^b \delta^{(k)}(t) f(t)\, dt = (-1)^k f^{(k)}(0)$$

if $f^{(k)}$ continuous at $t = 0$

interpretation of doublet δ': take two impulses with magnitude $\pm 1/\epsilon$, a distance ϵ apart, and let $\epsilon \to 0$



for $a < 0$, $b > 0$,

$$\int_a^b f(t) \left(\frac{\delta(t)}{\epsilon} - \frac{\delta(t-\epsilon)}{\epsilon} \right) \, dt = \frac{f(0) - f(\epsilon)}{\epsilon}$$

converges to $-f'(0)$ if $\epsilon \to 0$

Caveat

there is in fact **no such function** (Dirac's δ is what is called a *distribution*)

- we manipulate impulsive functions as if they were real functions, which they aren't
- it is safe to use impulsive functions in expressions like

$$\int_a^b f(t)\delta(t-T)\,dt, \quad \int_a^b f(t)\delta'(t-T)\,dt$$

provided f (resp, f') is continuous at $t = T$, and $a \neq T$, $b \neq T$

- some innocent looking expressions don't make any sense at all ($e.g.$, $\delta(t)^2$ or $\delta(t^2)$)