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Lecture 9
Time-domain properties of convolution systems

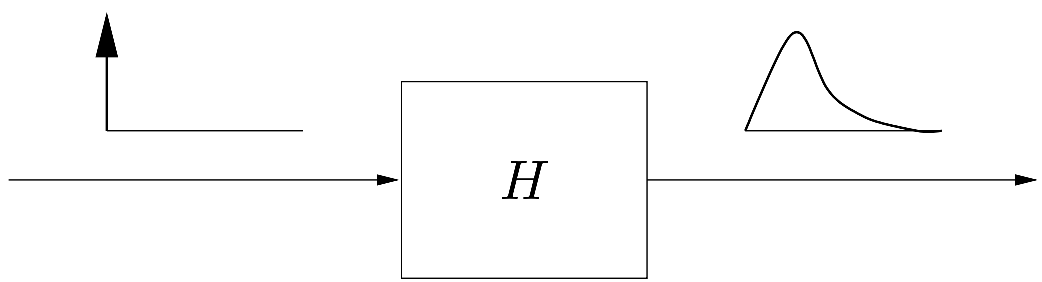
- impulse response
- step response
- fading memory
- DC gain
- peak gain
- stability

Impulse response

if $u = \delta$ we have

$$y(t) = \int_{0-}^{t} h(t-\tau)u(\tau)\, d\tau = h(t)$$

so h is the output (response) when $u = \delta$ (hence the name *impulse response*)



impulse response testing:

- apply impulse input and record resulting output (h)
- now you can predict output for *any* input signal
- practical problem: linear model often fails for very large input signals

Step response

the (unit) *step response* is the output when the input is a unit step:

$$s(t) = \int_0^t h(\tau)\, d\tau$$

(symbol s clashes with frequency variable, but usually this doesn't cause any harm)

relation with impulse response: $s(t)$ is the integral of h, so

$$h(t) = s'(t)$$

step response testing:

- apply unit step to input and record output (s)
- the impulse response is $h(t) = s'(t)$, so now you can predict output for *any* input signal
- widely used

Fading memory

we say the convolution system has *fading memory* if $h(\tau) \to 0$ as $\tau \to \infty$

- means current output $y(t)$ depends less and less on $u(t-\tau)$ as τ gets large ($i.e.$, the remote past input)
- if $h(\tau) = 0$ for $\tau > T$, then system has *finite memory*: $y(t)$ depends only on $u(\tau)$ for $t - T \leq \tau \leq t$

if H is rational, fading memory means poles of H are in left halfplane

(poles in right halfplane or on the imaginary axis give terms in h that don't decay to zero)

DC gain

the *DC* (direct current) or *static gain* of a convolution system is

$$H(0) = \int_0^\infty h(\tau)\, d\tau$$

(if finite, $i.e.$, if $s = 0$ is in ROC of H)

in terms of step response:

$$H(0) = \lim_{t\to\infty} s(t)$$

interpretation: if u is constant, then for large t,

$$y(t) = u \int_0^t h(\tau)\, d\tau \approx H(0)u$$

so $H(0)$ gives the gain for static (constant) signals

Vehicle suspension example

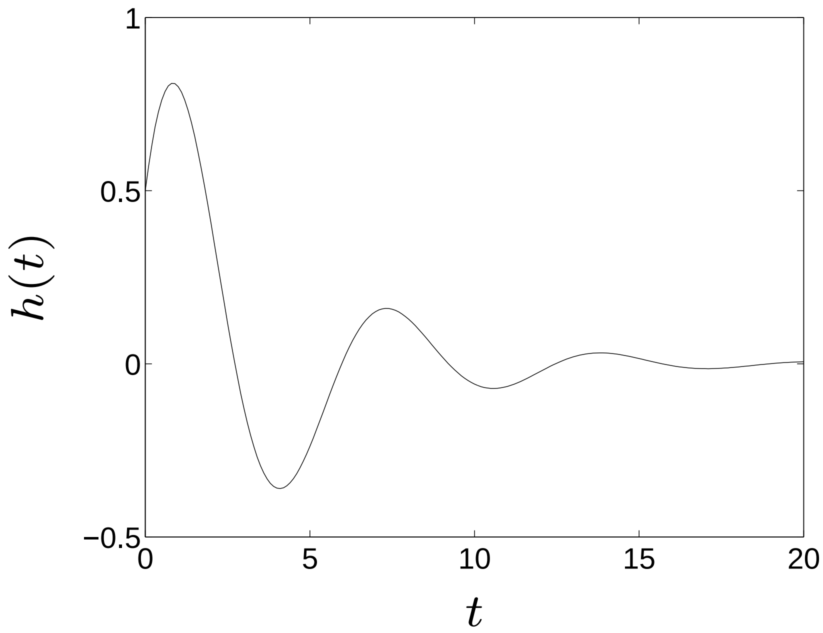
transfer function from road to vehicle height (page 7-7):

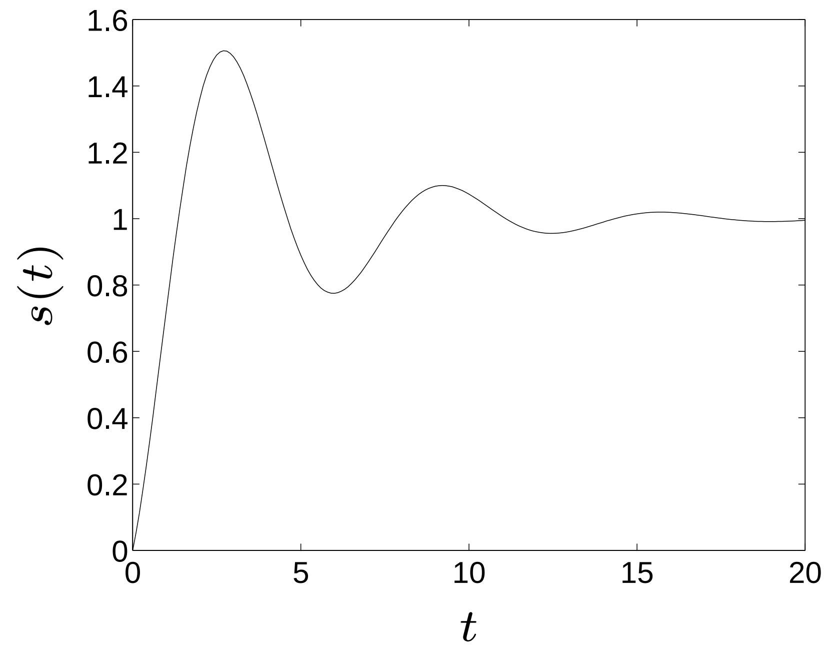
$$H(s) = \frac{bs + k}{ms^2 + bs + k}$$

- for $m > 0$, $b > 0$, $k > 0$ poles are in LHP, hence system has fading memory
- DC gain: $H(0) = 1$ (obvious!)

step response gives vehicle height after going over unit high curb at $t = 0$

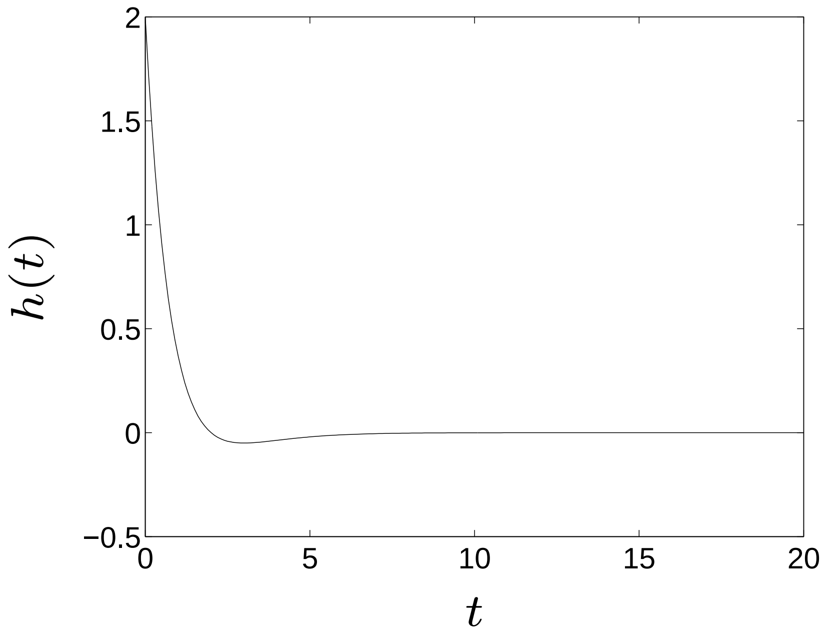
impulse response and step response for $k = 1$, $b = 0.5$, $m = 1$

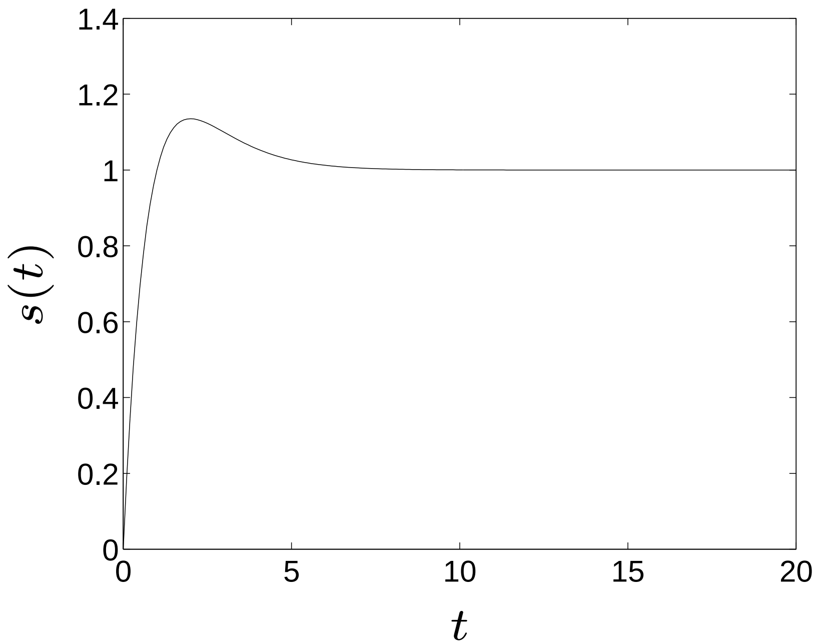




- poles are $-0.25 \pm j\,0.97$ (underdamped)
- step response 'overshoots' about 50%; settles at one in about 20sec

impulse response and step response for $k = 1$, $b = 2$, $m = 1$

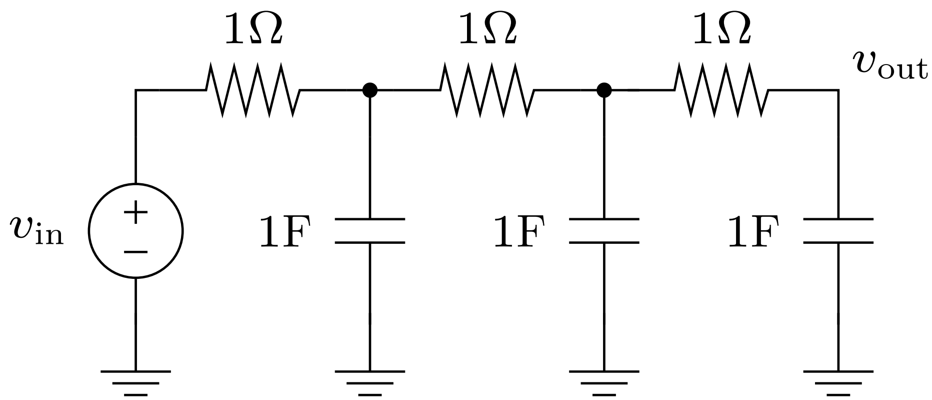




- repeated pole at -1 (critical damping)
- about 15% overshoot; step response settles in about 5sec

Example

wire modeled as 3 RC segments:



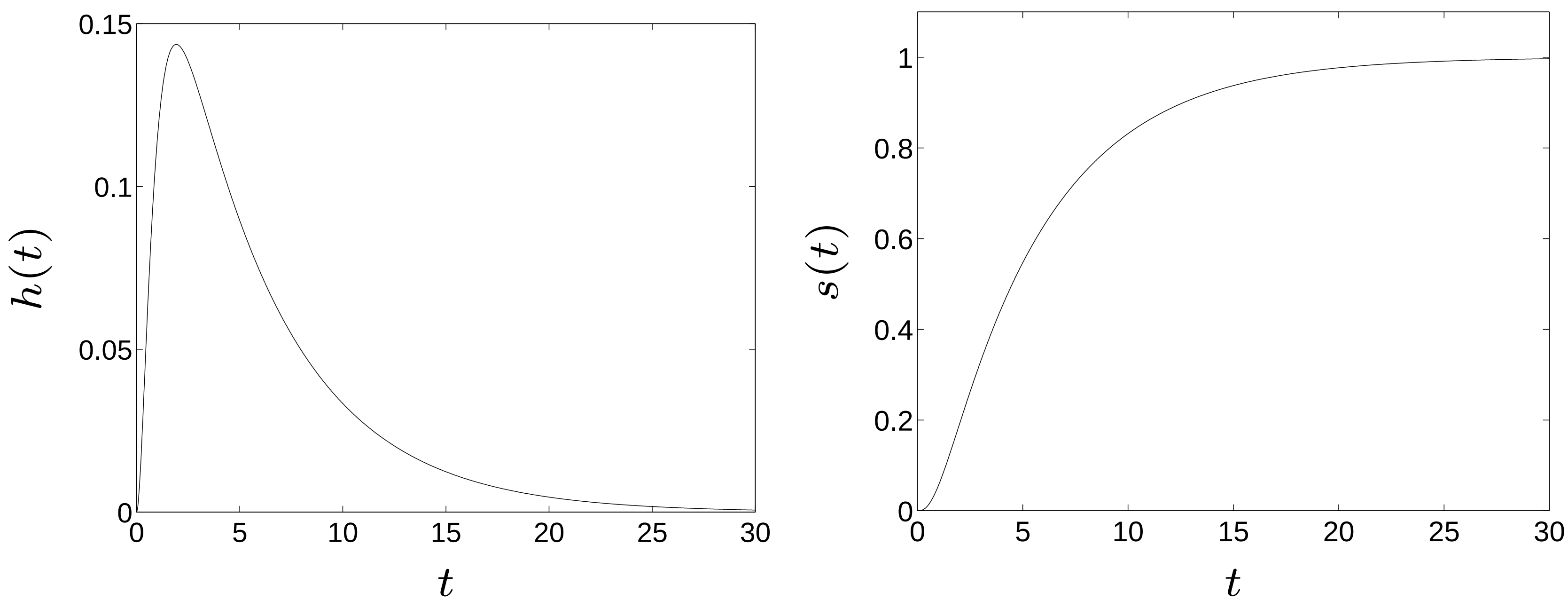
(except for values, could model interconnect wire in IC)

(after *alot* of algebra) we find

$$H(s) = \frac{1}{s^3 + 5s^2 + 6s + 1}$$

- poles are -3.247, -1.555, -0.198
- DC gain is $H(0) = 1$ (again, obvious)

step response gives v_{out} when v_{in} is unit step (as in $0 \rightarrow 1$ logic transition)



wire delays transition about 20sec or so

(Peak) gain

$$y(t) = \int_0^t h(\tau)u(t-\tau)\, d\tau$$

the peak values of the input & output signals as

$$\textsf{peak}(y) = \max_{t\geq 0} |y(t)|, \quad \textsf{peak}(u) = \max_{t\geq 0} |u(t)|$$

question: how large can $\dfrac{\textsf{peak}(y)}{\textsf{peak}(u)}$ be?

answer is given by the *peak gain* of the system, defined as

$$\alpha = \max_{u\neq 0} \frac{\textsf{peak}(y)}{\textsf{peak}(u)} = \int_0^\infty |h(\tau)|\, d\tau$$

$i.e.$, for any signal u we have $\textsf{peak}(y) \leq \alpha\, \textsf{peak}(u)$ and there are signals where equality holds for any t we have

$$\begin{aligned}
|y(t)| &= \left| \int_0^t h(\tau) u(t-\tau)\, d\tau \right| \\
&\leq \int_0^t |h(\tau)|\, |u(t-\tau)|\, d\tau \\
&\leq \mathsf{peak}(u) \int_0^t |h(\tau)|\, d\tau \\
&\leq \mathsf{peak}(u) \int_0^\infty |h(\tau)|\, d\tau
\end{aligned}$$

which shows that $\mathsf{peak}(y) \leq \alpha\, \mathsf{peak}(u)$

conversely, we can find an input signal with

$$\frac{\mathsf{peak}(y)}{\mathsf{peak}(u)} \approx \int_0^\infty |h(\tau)|\, d\tau$$

choose T large and define

$$u(t) = \begin{cases} \mathsf{sign}(h(T-t)) & t \le T \\ 0 & t > T \end{cases}$$

then $\mathsf{peak}(u) = 1$ and

$$y(T) = \int_0^T h(\tau)\mathsf{sign}(h(\tau))\, d\tau = \int_0^T |h(\tau)|\, d\tau,$$

for large T this signal satisfies

$$\frac{\mathsf{peak}(y)}{\mathsf{peak}(u)} \approx \int_0^\infty |h(\tau)|\, d\tau$$

example: $H(s) = 1/(s+1)$, so $h(t) = e^{-t}$

- DC gain is one, $i.e.$, constant signals are amplified by one
- peak gain is $\displaystyle\int_0^\infty |e^{-t}|\, d\tau = 1$ which is the same as the DC gain

so for this system, peak of the output is no more than the peak of the input

more generally,

- peak gain always at least as big as DC gain since

$$\int_0^\infty |h(\tau)|\, d\tau \geq \left| \int_0^\infty h(\tau)\, d\tau \right| = |H(0)|$$

- they are equal only when impulse response is always nonnegative (or nonpositive), $i.e.$, step response is monotonic

Stability

a system is *stable* if its peak gain is finite

interpretation: bounded inputs give bounded outputs

$$\mathsf{peak}(y) \leq \alpha\, \mathsf{peak}(u)$$

also called *bounded-input bounded-output stability* (to distinguish from other definitions of stability)

if H is rational, stability means poles of H are in left halfplane