Proceedings of the
American Control Conference
Seattle, Washington • June 1995

CRCD Program: Convex Optimization for Engineering Analysis and Design

Stephen Boyd
Information Systems Laboratory
Electrical Engineering Department
Stanford University
boyd@isl.stanford.edu

Lieven Vandenberghe
Katholieke Universiteit Leuven
ESAT, Departement Elektrotechniek
K. Mercierlaan 94, 3001 Leuven, Belgium
vandenbe@isl.stanford.edu

1. Overview

The focus of this NSF-supported combined research and curriculum development (CRCD) project is *convex optimization applied to engineering analysis and design*. The basic idea is that many analysis and design problems arising in engineering can be cast, or recast, in the form of a convex optimization problem, i.e., minimizing a convex objective of some decision variable subject to some convex constraints on the variable. Although such problems can appear difficult—they may have hundreds of variables and a nonlinear, nondifferentiable objective function—they can in fact be solved (numerically) very efficiently by recently developed interior-point methods that exploit convexity and the particular problem structure. Thus, the original engineering problem is effectively solved.

Of course, not all problems arising in engineering are convex and hence amenable to rapid numerical solution. But the number of problems that are convex or can be transformed to a convex problem is vastly larger than the number of problems that have an analytical solution. And in cases where the engineering problem is not convex, convex approximations can yield suboptimal solutions that are very useful in practice (see, e.g., [2]). Quite generally, then, it is a useful skill to recognize convexity in an engineering problem and to know how to exploit it.

The basic program therefore involves two parts: expressing an engineering analysis or design problem as a convex optimization problem, and developing an interior-point algorithm that solves the resulting problem very efficiently. When successful, this program results in an efficient solution to an engineering problem that might, at first glance, appear quite difficult, and in particular is very unlikely to have an analytical solution.

Development of a convex programming-based solution has important practical ramifications. An implementation of the algorithm can be used as the core "computational engine" in a customized computer-aided design or analysis tool for the engineering problem. If the user-interface for such a tool is designed properly, the user can specify the problem entirely in its original, engineering setting. The results are also displayed in the form most natural for the original problem. The translation from engineering problem to convex optimization problem, solution of the resulting problem via an interior-point method, and translation of the results back into the engineering problem setting are transparent to the user of such a tool. For more discussion of this topic see [3].

This program has been carried out successfully over the last eight years, by many researchers, for some problems arising in control engineering. As a very partial list of work in this area, see the references cited in the books [4, 5]. At least one commercial computer-aided design tool [6] as well as several academic/research tools [7, 8] have already been developed.

There are many other engineering application areas in which the basic program sketched above has been carried out, at least to some extent. In other areas, convexity of some important problems has been noted but, in our opinion, not yet fully exploited, To give an idea of the breadth, we mention some applications areas with a sample citation for each. The formulation of FIR filter design as an LP goes back to the sixties [9]; a related application is antenna array weight design [10]. VLSI transistor sizing/interconnect optimization is considered in [11]. Image processing and reconstruction via convex optimization is considered in, e.g., [12]. Convex optimization plays an important role in statistics in general (see e.g., [13]) and optimal design of experiments [14] and design centering and yield maximization [15, 16] in particular. For optimal design of computer networks and routing in such networks see [17]. Convexity of several problems in communications has been noted (see, e.g., [18]). In mechanical engineering the use of convex programming, especially linear programming and monotone variational inequalities, is well developed and has a long history; see, e.g., [19, 20].

2. Convex Analysis and Optimization

In the area of convex optimization our program rests on some very recent research. The mathematical foundation, convex analysis, has been developed over the past fifty years or so by researchers such as Fenchel and Rockefellar [21]; a recent and complete reference is [22]. Convex analysis has been successfully used in several fields including economics, operations research, and statistics, mostly for theoretical purposes such as concluding existence of a saddle point, deriving optimality conditions or duality theories.

Nevertheless rapid development in the area of algorithms for *convex optimization* is much more recent. The most obvious landmarks are:

- the ellipsoid algorithm developed in the 1970s and later used to prove that linear programs can be solved in polynomial time [23]
- Karamarkar's 1984 linear programming algorithm, which sparked intense research on interior-point methods for linear and quadratic programming [24]
- Nesterov and Nemirovsky's 1994 extension of interior-point methods to virtually all convex optimization problems arising in engineering [25]

3. Research Overview

The main questions addressed in the research component are:

- What engineering problems can be cast as convex optimization problems? How can one recognize such problems? What problems cannot be cast as convex problems, and what partial or approximate solutions can be obtained via convex programming in this case?
- Which interior-point algorithms are best suited to the problems arising from engineering? What is the best way to exploit problem structure to gain efficiency? Are some algorithms better for use as a computational engine in a CAD tool?

Let us describe one specific area that merits further research. As mentioned above, interior-point methods have recently been developed for the (often nonlinear) convex problems that arise in engineering [25, 8]. For these algorithms the number of iterations is small (typical numbers lie between 10 and 50) and grows very slowly with problem size. The main computational effort of each iteration is the solution of a large, dense, but often highly structured least-squares problem. Exploiting this structure, via conjugate-gradients and related techniques, seems to be the key to an efficient implementation. We have carried out this plan for a few specialized problems arising in control engineering with success [26, 27]. By exploiting the underlying structure of the engineering problem we were able to develop algorithms that solve certain nonlinear nondifferentiable convex optimization problems with more than 1000 variables and tens of thousands of constraints in times on the order of ten minutes on a small workstation [27]. We envision this method being widely applied to convex problems that arise in engineering.

4. Curriculum Development Overview

The curriculum development component involves the creation and development of a complete set of teaching materials for a course entitled *Convex Optimization with Engineering Applications*, which will be given at Stanford starting Spring quarter 1995. The course is aimed at senior undergraduates and first year graduate students from all fields of engineering. We hope to expose students to the basics of convex analysis and optimization, but more importantly to concentrate on developing the background, experience, and skills required to recognize and exploit convexity in engineering applications. A secondary goal is to expose students to applications of convex optimization in many areas of engineering.

We believe that such a course would be of enormous value. While we know of other courses that cover related material, e.g., convex analysis and theory of optimization, monotone variational inequalities in theoretical mechanics, we believe that none has the same emphasis and goals. Preliminary and informal discussions with colleagues at other universities indicate a great interest in adopting such a course once appropriate curriculum and teaching materials are available.

We hope that courses similar to the one we develop are eventually adopted at most universities with an advanced engineering program. Our ultimate goal is that convex optimization and the recently developed interior-point techniques for solving convex problems should make their way into the mainstream of numerical mathematics, and—our main point—into engineering practice.

References

[1] L. Vandenberghe and S. Boyd. Semidefinite programming. Submitted to *SIAM Review*, July 1994.

[2] F. Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM Journal on Optimization*, 5(1):13–51, February 1995.

[3] S. Boyd. Robust control tools: Graphical user-interfaces and LMI algorithms. *Systems, Control and Information*, 38(3):111–117, March 1994. Special issue on Numerical Approaches in Control Theory.

[4] S. Boyd and C. Barratt. *Linear Controller Design: Limits of Performance.* Prentice-Hall, 1991.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics.* SIAM, Philadelphia, PA, June 1994.

[6] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali. The LMI control toolbox. In *Proc. IEEE Conf. on Decision and Control*, pages 2038–2041, 1994.

[7] S. Boyd, N. Khraishi, C. Barratt, S. Norman, V. Balakrishnan, and D. Meyer. *QDES Version 1.5.* Information Systems Laboratory, Electrical Engineering Dept., Stanford University, Stanford, CA 94305, 1990.

[8] L. Vandenberghe and S. Boyd. *SP. Software for Semidefinite Programming. User's Guide. Beta Version.* K.U. Leuven and Stanford University, October 1994.

[9] A. W. Potchinkov and R. M. Reemtsen. Fir filter design in the complex domain by a semi-infinite programming technique. ii. examples. *AEU, Archiv fuer Elektronik und Uebertragungstechnik: Electronics and Communication*, 48(4):200–209, July 1994.

[10] I. Thng, A. Cantoni, and Y. H. Leung. Derivative constrained optimum broad-band antenna arrays. *IEEE Transactions on Signal Processing*, 41(7):2376–2388, July 1993.

[11] P. Sancheti and S. Sapatnekar. Interconnect design using convex optimization. *IEEE Custom Integrated Circuits Conference Proceedings*, pages 549–552, 1994.

[12] C. Byrne. Iterative image reconstruction algorithms based on cross-entropy minimization. *IEEE Transactions on Image Processing*, 2(1):96–103, January 1993.

[13] J. E. Pečarić, F. Proschan, and Y. L. Tong. *Convex Functions, Partial Orderings, and Statistical Applications*, volume 187 of *Mathematics in Science and Engineering.* Academic Press, 1992.

[14] V. V. Fedorov. *Theory of Optimal Experiments.* Academic Press, 1971.

[15] S. S. Sapatnekar. *A convex programming approach to problems in VLSI design.* PhD thesis, Coordinated Science Laboratory, University of Illinois at Urbana-Campaign, August 1992.

[16] R. K. Brayton, S. W. Director, and G. D. Hachtel. Yield maximization and worst-case design with arbitrary statistical distributions. *IEEE Transactions on Circuits and Systems*, 27:756–764, 1980.

[17] T. M. J. Ng and D. B. Hoang. Joint optimization of capacity and flow assignement in a packet-switched communications network. *IEEE Transactions on Communications*, 35(2):202–209, February 1994.

[18] R. E. Blahut. Computation of channel capacity and rate-distortion functions. *IEEE Transactions on Information Theory*, 18(4):460–473, 1972.

[19] P. D. Panagiotopoulos. *Inequality Problems in Mechanics and Applications: Convex and Nonconvex Energy Functions.* Birkhauser, 1985.

[20] U. Kirsch. *Structural Optimization.* Springer-Verlag, 1993.

[21] R. T. Rockafellar. *Convex Analysis.* Princeton Univ. Press, Princeton, second edition, 1970.

[22] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms I*, volume 305 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, New York, 1993.

[23] R. G. Bland, D. Goldfarb, and M. J. Todd. The ellipsoid method: A survey. *Operations Research*, 29(6):1039–1091, 1981.

[24] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4(4):373–395, 1984.

[25] Yu. Nesterov and A. Nemirovsky. *Interior-point polynomial methods in convex programming*, volume 13 of *Studies in Applied Mathematics.* SIAM, Philadelphia, PA, 1994.

[26] L. Vandenberghe and S. Boyd. Primal-dual potential reduction method for problems involving matrix inequalities. To be published in *Math. Programming*, 1995.

[27] S. Boyd, L. Vandenberghe, and M. Grant. Efficient convex optimization for engineering design. In *Proceedings IFAC Symposium on Robust Control Design*, pages 14–23, September 1994.