Multiobjective H_2/H_∞-Optimal Control via Finite Dimensional Q-Parametrization and Linear Matrix Inequalities

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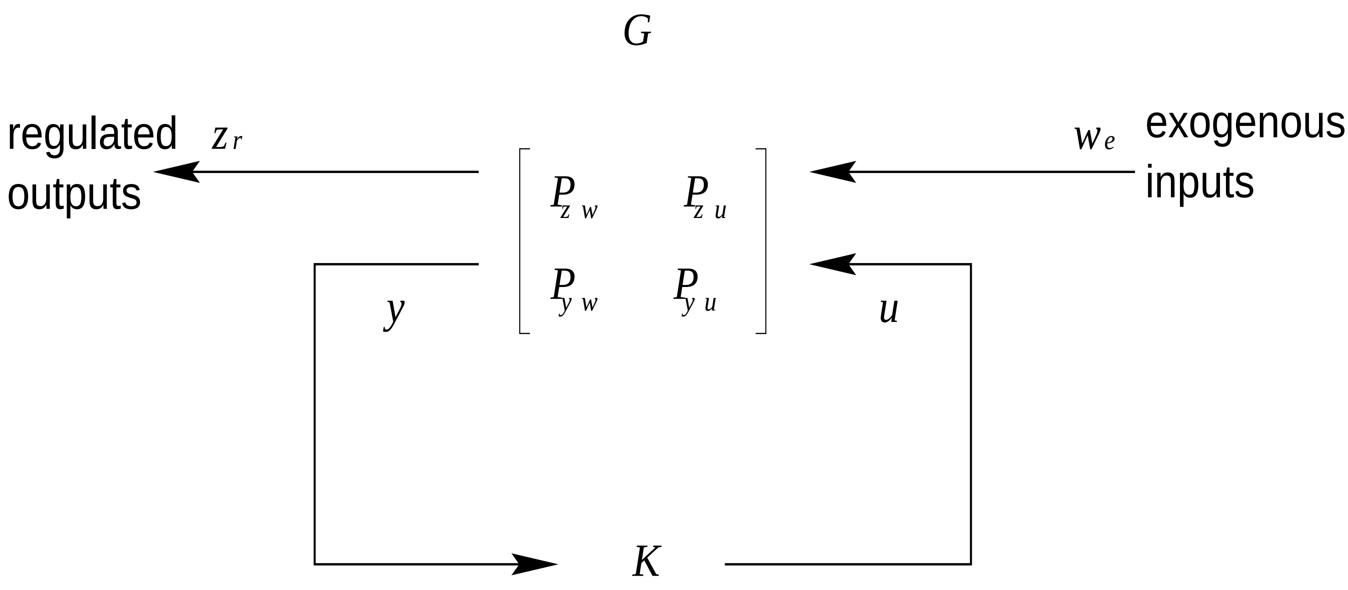
In this talk, we focus on H_∞. Same ideas carry through for H_2/H_∞ as well - see paper.

Q-Parametrization



- Set of all achievable **stable closed loop maps** is:

$$\{G = P_{z_r w_e} + P_{z_r u} K (I - P_{yu} K)^{-1} P_{y w_e} \mid K \text{ stabilizing}\}$$

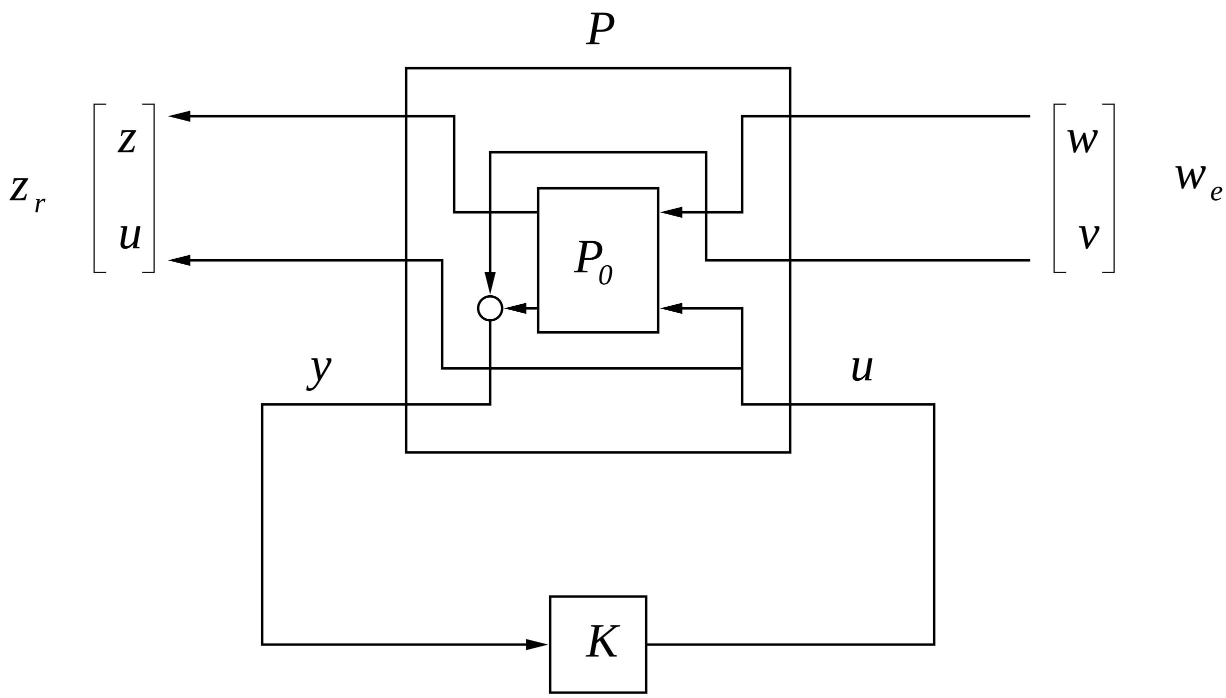
  – set of stabilizing K's **not obvious**
  – parametrization is **linear fractional**
  – P's and K can be **unstable**

- Can transform into **equivalent parametrization**:

$$\{G = H - U\,Q\,V \mid Q \text{ stable}\}$$

  – now H, U, V and Q **stable**
  – **affine** in Q $\rightarrow$ Good for optimization

General Regulator Problem



$$z_r = \begin{bmatrix} z \\ u \end{bmatrix} = G\, w_e = \begin{bmatrix} G_{zw_e} \\ G_{uw_e} \end{bmatrix} w_e$$

- Typically want:
  1. Small $\|G_{zw_e}\|_\infty$ for "good regulation"
  2. Small $\|G_{uw_e}\|_\infty$ for "efficient control"
  3. "Reject" disturbances $w_e = \begin{bmatrix} w \\ v \end{bmatrix}$
- Usually **conflicting** requirements:
  "good" regulation requires "large" control

Multiobjective Design Paradigm

- Define:

$$J_\lambda^M(Q) = (1-\lambda)\,\|G_{zw_e}(Q)\|_\infty^2 + \lambda\,\|G_{uw_e}(Q)\|_\infty^2$$

- Compute **tradeoff curve**:

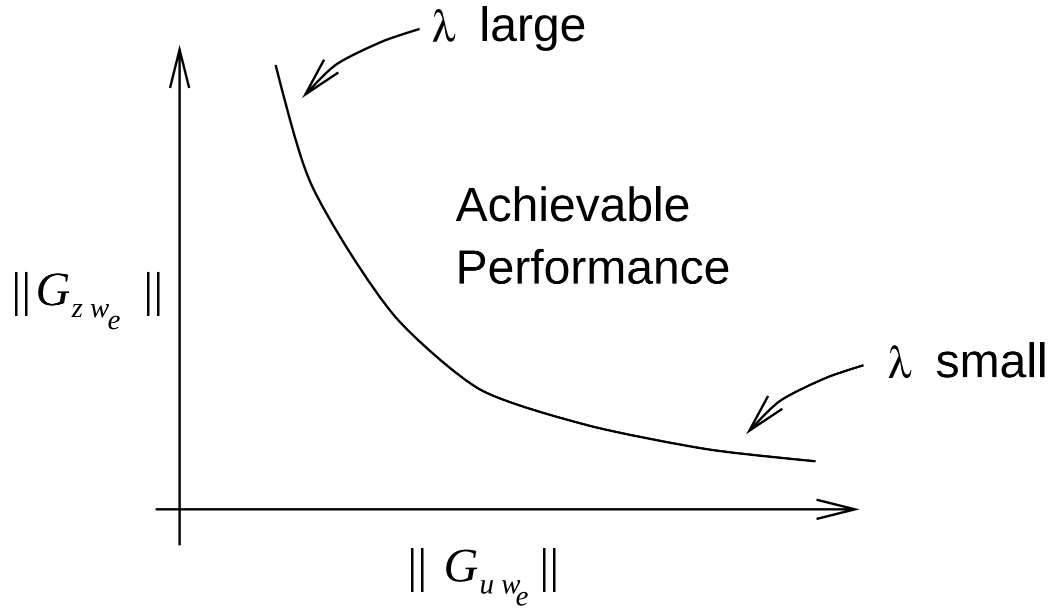
  for $\lambda = 0$ to 1

  solve for Q_λ: $\inf_{Q\in H_\infty} J_\lambda^M(Q)$

  plot $\|G_{zw_e}(Q_\lambda)\|_\infty$ versus $\|G_{uw_e}(Q_\lambda)\|_\infty$

  end

- Tradeoff curve gives **limits of performance** - very useful in practice!



Standard vs Multiobjective H^∞

Standard H^∞ Problem: $z_r = \begin{bmatrix} (1-\lambda)^{\frac{1}{2}}\, z \\ \lambda^{\frac{1}{2}}\, u \end{bmatrix}$ minimize

$$J_\lambda^S(Q) = \left\| \begin{bmatrix} (1-\lambda)^{\frac{1}{2}}\, G_{zw_e}(Q) \\ \lambda^{\frac{1}{2}}\, G_{uw_e}(Q) \end{bmatrix} \right\|_\infty^2$$

$$= \sup_{w_e \neq 0} \frac{((1-\lambda)\|z\|_2^2 + \lambda\|u\|_2^2)}{\|w_e\|_2^2}$$

Multiobjective H^∞ Problem: minimize

$$J_\lambda^M(Q) = (1-\lambda)\|G_{zw_e}(Q)\|_\infty^2 + \lambda\, \|G_{uw_e}(Q)\|_\infty^2$$

$$= (1-\lambda) \sup_{w_e \neq 0} \frac{\|z\|_2}{\|w_e\|_2} + \lambda\, \sup_{w_e \neq 0} \frac{\|u\|_2}{\|w_e\|_2}$$

Comments

- In multiobjective design maximization of z and u over w_e is done **independently**
- In standard design maximization of z and u over w_r is done **simultaneously** - artificially **couples** z and u
- Why would we care about the gain from w_e to the **sum** of z and u? They might peak at different frequencies.

More Remarks

- Note that since

$$J_\lambda^S = \text{“sup of sum”}$$

$$J_\lambda^M = \text{“sum of sups”}$$

we have

$$J_\lambda^S(Q) \le J_\lambda^M(Q) \ \forall\, Q \in H_\infty$$

$$\inf_{H_\infty} J_\lambda^S \le \inf_{H_\infty} J_\lambda^M$$

- Also, since $G_{zw_e}(Q)$ and $G_{uw_e}(Q)$ are both **affine** in Q $\Longrightarrow$ both problems **convex**
- Finally, note that the problems are **infinite dimensional**
- In Standard problem, **state space** structure provides means for **minimizing exactly** via **bisection** applied to **Riccati equations** or **LMI**s .
- In Multiobjective problem **cannot solve exactly** in general. Can only minimize **conservative upper bound**. But **no analysis** for degree of conservativeness.
- So why not use finite dimensional Q-based approach which fell out of favor because **no analysis** was available for degree of approximation?

Previous Research

- State space, upper bound on H_∞/H_2
  '89: Bernstein & Haddad
  '91: Khargonekar & Rotea

- Finite dimensional Q, convex optimization
  '88: Boyd, Barratt, Balakrishnan, Kabamba & Meyer
  '94: Sznaier, Rotstien & Sideris

- Finite dimensional Q and LMIs
  '95: Chen & Wen
  '95: Scherer - our method was first proposed

- Lyapunov Shaping, LMIs
  '95: Scherer, Gahinet & Chilali
  '95: El-Ghaoui & Folcher

- Solve nonconvex problem
  '98: Halder, Hassibi & Kailath

$\|G\|_\infty$ via Bounded Real Lemma

- To avoid truncation errors of QDES, we use **state space**
- Given closed loop system G with then

$$\|G\|_\infty \equiv \|D + C\,(zI - A)^{-1}B\|_\infty = \gamma^\star$$

if and only if $\gamma^\star$ is optimizer of

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \begin{bmatrix} A^T X\, A - X & A^T X\, B & C^T \\ B^T X\, A & B^T X\, B - \gamma\, I & D^T \\ C & D & -\gamma\, I \end{bmatrix} < 0 \\ & \qquad\qquad X > 0 \end{array}$$

- A, B, C, D are **closed loop** matrices - contain controller variables
- Due to **cross terms** between A, B, and X, have **nonlinear** matrix inequality
- In '93 '94, Gahinet & Apkarian and Iwasaki & Skelton showed that using **elimination lemma** can reduce to 3 LMI's

(Similar LMI's exist for H_2 norm)

Multiobjective $\mathcal{H}_\infty$ Problem

- We now want to minimize

$$(1-\lambda)\,\|G_z\|_\infty + \lambda\,\|G_u\|_\infty$$

- Apply **bounded real lemma** to G_z and G_u separately $\longrightarrow$ **SDP** in $\gamma_z, \gamma_u, X_z, X_u$, and closed loop matrices of G_z and G_u:

$$\min \quad (1-\lambda)\,\gamma_z \;+\; \lambda\,\gamma_u$$

$$\text{s.t.} \quad \begin{bmatrix} A_z^T X_z\, A_z - X_z & A_z^T X_z\, B_z & C_z^T \\ B_z^T X_z\, A_z & B_z^T X_z\, B_z - \gamma_z\, I & D_z^T \\ C_z & D_z & -\gamma_z\, I \end{bmatrix} < 0$$

$$X_z > 0$$

$$\begin{bmatrix} A_u^T X_u\, A_u - X_u & A_u^T X_u\, B_u & C_u^T \\ B_u^T X_u\, A_u & B_u^T X_u\, B_u - \gamma_u\, I & D_u^T \\ C_u & D_u & -\gamma_u\, I \end{bmatrix} < 0$$

$$X_u > 0$$

- Again **cross terms** between A's, X's and B's.
- But now **elimination lemma fails**
- Note C's and D's appear **linearly**
- If could put all controller variables in C's and D's, get LMI's - **done!**. This is our **goal**.

State Space SISO FIR

- Given FIR system Q with **pulse response**

$$\{q_0, q_1, q_2, q_3, 0, 0, \dots\}$$

- We have (control canonical form) **realization**

$$\left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] \equiv \left[\begin{array}{c|c} \begin{bmatrix} 0 & & \\ 1 & 0 & \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} & q_0 \end{array}\right]$$

- **All variables** q_i are in C_Q and D_Q matrices.
- Matrices A_Q and B_Q are **fixed**.
- Later on, we will **assume** that the component SISO systems Q_{ij} of Q in the Q-parametrization are all SISO FIRs.

Pulling Out Q

- Recall that in Q-parametrization: H, U, Q, and V are just **matrices** in $\mathcal{H}_\infty$.
- Want to write

$$G(Q) = H - U\,Q\,V$$

in terms of **SISO components** of Q **explicitly**.

- Decompose Q as sum of its **SISO components** Q_{rs} times elementary matrices $E_{rs} = e_r\,e_s^T$:

$$Q = \sum_{r,s} Q_{rs}\,e_r\,e_s^T$$

- Hence:

$$G(Q) = H - U\,(\sum_{r,s} Q_{rs}\,e_r\,e_s^T)\,V$$

$$= H - \sum_{r,s} Q_{rs}\,\big((U\,e_r)(e_s^T\,V)\big)$$

- Therefore:

$$G(Q) = H - \sum_{r,s} Q_{rs}\,T_{rs}$$

where $T_{rs} = (U\,e_r)(e_s^T\,V)$.

Kronecker Products

- So we have

$$G(Q) = H - \sum_{rs} Q_{rs} T_{rs}$$

- Now $Q_{rs} T_{rs}$ is just **scalar** (SISO) $\times$ **matrix** (MIMO) in $\mathcal{H}_\infty$. So

$$Q_{rs} T_{rs} \triangleq \begin{bmatrix} Q_{rs} T_{rs}^{(11)} & \cdots & Q_{rs} T_{rs}^{(1n)} \\ \vdots & \ddots & \vdots \\ Q_{rs} T_{rs}^{(m1)} & \cdots & Q_{rs} T_{rs}^{(mn)} \end{bmatrix}$$

$$= Q_{rs} \otimes T_{rs}$$

where $\otimes$ denotes **Kronecker multiplication**

$$A \otimes B \triangleq \begin{bmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & \ddots & \vdots \\ a_{m1} B & \cdots & a_{mn} B \end{bmatrix} \in \mathbf{R}^{mp \times nq}$$

- So to be **explicit** we write:

$$G(Q) = H - \sum_{rs} Q_{rs} \otimes T_{rs}$$

State Space Representation of $Q \otimes T$

- **Given:** $Q \in \mathcal{H}_\infty^{p\times q}$ and $T \in \mathcal{H}_\infty^{m\times n}$

$$Q \equiv \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] \quad \text{and} \quad T \equiv \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right]$$

Then: $Q \otimes T$ has state space

$$\left[\begin{array}{c|c} A_{Q\otimes T} & B_{Q\otimes T} \\ \hline C_{Q\otimes T} & D_{Q\otimes T} \end{array}\right] = \left[\begin{array}{cc|c} A_Q \otimes I_m & B_Q \otimes C_T & B_Q \otimes D_T \\ 0 & I_q \otimes A_T & I_q \otimes B_T \\ \hline C_Q \otimes I_m & D_q \otimes C_T & D_Q \otimes D_T \end{array}\right].$$

- If Q has **SISO FIR structure**, then **all coefficients** q_i of Q (contained in C_Q & D_Q) appear only in $C_{Q\otimes T}$ and $D_{Q\otimes T}$.

State Space for Closed Loop System G

- **Assume:** that Q is SISO, then there's just **one** Q and **one** T. Can then drop r and s indexes:

$$\sum_{r,s} Q_{rs} \otimes T_{rs} = Q \otimes T.$$

(**general case** same idea - see paper)

- Then closed loop transfer function

$$G(Q) = H - Q \otimes T$$

- This is just H **in parallel** with $-(Q \otimes T)$.
- Therefore it's easy to write down **state space** for G:

$$\left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] = \left[\begin{array}{cc|c} A_H & & B_H \\ & A_{Q\otimes T} & B_{Q\otimes T} \\ \hline C_H & -C_{Q\otimes T} & D_H - D_{Q\otimes T} \end{array}\right].$$

- Note that if Q is FIR, then **all coefficients** of Q are contained in C_G and D_G

State Space for Multiobjective Closed Loop

- Start with

$$G(Q) = H - \sum_{rs} Q_{rs} \otimes T_{rs}.$$

- **Partition** G, H, T according to $z_r = \begin{bmatrix} z \\ u \end{bmatrix}$:

$$\begin{bmatrix} G_z(Q) \\ G_u(Q) \end{bmatrix} = \begin{bmatrix} H_z \\ H_u \end{bmatrix} + \sum_{r,s} \begin{bmatrix} Q_{rs} \otimes T_{z,rs} \\ Q_{rs} \otimes T_{u,rs} \end{bmatrix}$$

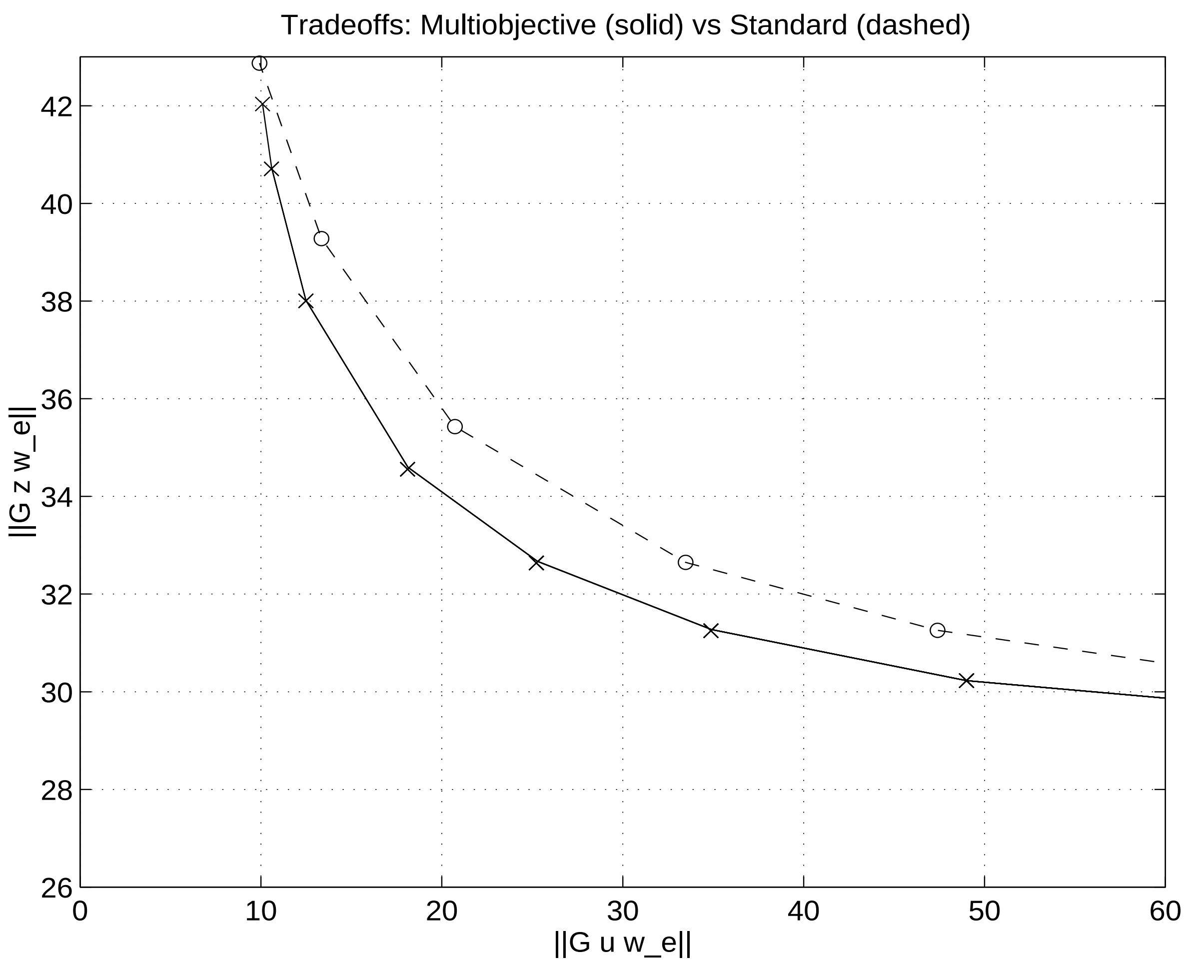
- Again **assume** just **one** Q and T.
- Now get **state space** of G_z and G_u:

$$\left[\begin{array}{c|c} A_z & B_z \\ \hline C_z & D_z \end{array}\right] = \left[\begin{array}{cc|c} A_{H_z} & & B_{H_z} \\ & A_{Q\otimes T_z} & B_{Q\otimes T_z} \\ \hline C_{H_z} & -C_{Q\otimes T_z} & D_{H_z} - D_{Q\otimes T_z} \end{array}\right]$$

$$\left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right] = \left[\begin{array}{cc|c} A_{H_u} & & B_{H_u} \\ & A_{Q\otimes T_u} & B_{Q\otimes T_u} \\ \hline C_{H_u} & -C_{Q\otimes T_u} & D_{H_u} - D_{Q\otimes T_u} \end{array}\right]$$

- Note that if Q is FIR, then **all coefficients** of Q are contained in C_z, D_z and C_u, D_u $\longrightarrow$ **done !**

Numerical Example



- System was:
  - unstable, second order, $f_0 = 1$, $\zeta = -0.5$.
  - discretized at $T_s \approx 1/6$
  - $0.9\,T_s$ delay in loop
- Stabilized with LQG to get H, U, and V
- Modified with **12-tap FIR** Q

Result: 25% reduction in control effort!

Conclusion

Proposed Method

- based on Q-parametrization & finite dimensional convex optimization

- conservative, but can outperform standard H_∞ and Lyapunov shaping

- extends to H_2/H_∞ (and other problems)

- involves more computation than standard methods, but structure can be exploited for speedup