a2dr: Anderson Accelerated Douglas-Rachford Splitting

Open-sourced Python Solver for Prox-Affine Distributed Convex Optimization

https://github.com/cvxgrp/a2dr

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Joint work with Anqi Fu and Stephen P. Boyd

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Overview

1. Motivation and Problem Statement
2. Douglas-Rachford Splitting
3. Anderson Acceleration & A2DR
4. Numerical experiments
5. Conclusion

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Prox-affine form of generic convex optimization

We consider the following **prox-affine** representation/formulation of a **generic** convex optimization problem:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^N f_i(x_i) \\ \text{subject to} & \sum_{i=1}^N A_i x_i = b. \end{array}$$

with variable $x = (x_1, \ldots, x_N) \in \mathbf{R}^{n_1+\cdots+n_N}$, $A_i \in \mathbf{R}^{m \times n_i}$, $b \in \mathbf{R}^m$.

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- $f_i : \mathbf{R}^{n_i} \to \mathbf{R} \cup \{+\infty\}$ is closed, convex and proper (CCP).
- Each f_i can **only** be accessed through its proximal operator:

$$\mathbf{prox}_{tf_i}(v_i) = \mathrm{argmin}_{x_i} \ \left(f_i(x_i) + \frac{1}{2t}\|x_i - v_i\|_2^2 \right).$$

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Interface of a2dr:

```
x_vals, primal, dual, num_iters, solve_time = a2dr(p_list, A_list, b)
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Try it out! Simply provide a list of proximal functions $\mathbf{prox}_{tf_i}(v_i)$ (p_list), list of A_i's (A_list), and b (b), and you are done!

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Finally: CVXPY + a2dr – Expression tree complier exists: Epsilon (Wytock et al., 2015).

Previous Work

Most common approaches for prox-affine formulation (sometimes goes by the name "distributed optimization"):

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These are typically slow to converge – acceleration techniques:

- Adaptive penalty parameters.
- Momentum methods.
- Quasi-Newton or Newton-type method with line search.

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- Why **DRS**?
  - Allows for a natural NEFP representation (ADMM not), and amenable to proximal evaluation (ALM not).

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Theory: First **globally** convergent type-II AA variant in **non-smooth**, potentially **pathological** settings.
Practice: An open-source Python solver `a2dr` based on **A2DR**:

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DRS Algorithm

- Rewrite problem as ($\mathcal{I}_S$ is the indicator of set S)

$$\text{minimize} \quad \overbrace{\sum_{i=1}^{N} f_i(x_i)}^{f(x)} + \overbrace{\mathcal{I}_{Ax=b}(x)}^{g(x)}.$$

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- DRS iterates for $k = 1, 2, \ldots,$

$$\begin{aligned} x_i^{k+1/2} &= \mathbf{prox}_{tf_i}(v^k), \quad i = 1, \ldots, N \\ v^{k+1/2} &= 2x^{k+1/2} - v^k \\ x^{k+1} &= \Pi_{Av=b}(v^{k+1/2}) \\ v^{k+1} &= v^k + x^{k+1} - x^{k+1/2} \end{aligned}$$

$\Pi_S(v)$ is Euclidean projection of v onto S.

Convergence of DRS

- DRS iterations can be conceived as a fixed point (FP) mapping

$$v^{k+1} = F(v^k)$$

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In practice, this convergence is often rather slow.

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  - Better stability for **general purpose** solvers and distributed settings.
    - **prox** operators have much larger diversity than solvable cones in SCS.

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Extrapolation perspective of type-II AA:

- *Extrapolates* next iterate using $M+1$ most recent iterates

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- Minimizing the FP residual of extrapolated point $\sum_{j=0}^{M} \alpha_j^k v^{k-M+j}$ when F is affine.

Regularization

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- (Scieur et al., 2016) showed that adding **constant quadratic regularization** to the objective leads to local convergence improvement.
- **Insufficient** for global convergence both in theory and practice.

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- Adaptive quadratic regularization: (adaptive LS)

$$\text{minimize} \quad \|g^k - Y_k\gamma^k\|_2^2 + \eta\left(\|S_k\|_F^2 + \|Y_k\|_F^2\right)\|\gamma^k\|_2^2,$$

where $\eta \geq 0$ is a regularization parameter and

$$g^k = G(v^k), \quad y^k = g^{k+1} - g^k, \quad Y_k = [y^{k-M} \ \ \ldots \ \ y^{k-1}]$$
$$s^k = v^{k+1} - v^k, \quad S_k = [s^{k-M} \ \ \ldots \ \ s^{k-1}]$$

A2DR

- A2DR iterates for $k = 1, 2, \ldots,$ ($\epsilon > 0$, M positive integer)

  1. Compute $v_{\text{DRS}}^{k+1} = F(v^k)$, $\quad g^k = v^k - v_{\text{DRS}}^{k+1}$.
  2. Update Y_k and S_k to include the new information

     & Compute α^k by solving the **adaptive LS** w.r.t. γ^k.
  3. Compute $v_{\text{AA}}^{k+1} = \sum_{j=0}^{M} \alpha_j^k v_{\text{DRS}}^{k-M+j+1}$.
  4. If the residual $\|G(v^k)\|_2 = O(1/n_{\text{AA}}^{1+\epsilon})$: (**safeguard**)

     Adopt $v^{k+i} = v_{\text{AA}}^{k+i}$ for $i = 1, \ldots, M$.

     (n_{AA}: # of adopted AA candidates)
  5. Otherwise, take $v^{k+1} = v_{\text{DRS}}^{k+1}$.

Stopping Criterion of A2DR

- Stop and output $x^{k+1/2}$ when $\|r^k\|_2 \leq \epsilon_{\mathsf{tol}} = \epsilon_{\mathrm{abs}} + \epsilon_{\mathrm{rel}}\|r^0\|_2$:

$$\begin{aligned} r^k_{\mathsf{prim}} &= Ax^{k+1/2} - b, \\ r^k_{\mathsf{dual}} &= \tfrac{1}{t}(v^k - x^{k+1/2}) + A^T\lambda^k, \\ r^k &= (r^k_{\mathrm{prim}}, r^k_{\mathrm{dual}}). \end{aligned}$$

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- Remark:
  - Just KKT conditions. Notice that $(v^k - x^{k+1/2})/t \in \partial f(x^{k+1/2})$.
  - $\mathbf{prox}_f$ is enough, and no need for access to f or its sub-gradient.
- Dual variable is solution to least-squares problem

$$\lambda^k = \text{argmin}_\lambda \ \|r^k_{\text{dual}}\|_2$$

Key lemmas to the proof

Lemma (Bounded approximate inverse Jacobian)

We have $v_{AA}^{k+1} = v^k - H_k g^k$, *where* $g^k = G(v^k)$ *is the FP residual at* v^k, *and* $\|H_k\|_2 \leq 1 + 2/\eta$, *where* $\eta > 0$ *is the regularization parameter in the adaptive LS subproblem.*

Lemma (Connecting FP residuals with OPT residuals)

Suppose that $\liminf_{j\to\infty} \|G(v^j)\|_2 \leq \epsilon$ *for some* $\epsilon > 0$, *then*

$$\liminf_{j\to\infty} \|r_{\mathrm{prim}}^j\|_2 \leq \|A\|_2\epsilon, \quad \liminf_{j\to\infty} \|r_{\mathrm{dual}}^j\|_2 \leq \frac{1}{t}\epsilon.$$

Convergence of A2DR

Theorem (Solvable Case)

If the problem is solvable (e.g., feasible and bounded), then

$$\liminf_{k\to\infty} \|r^k\|_2 = 0$$

and the AA candidates are adopted infinitely often. Furthermore, if F has a fixed point, then

$$\lim_{k\to\infty} v^k = v^\star \textit{ and } \lim_{k\to\infty} x^{k+1/2} = x^\star,$$

where $v^\star$ is a fixed-point of F and $x^\star$ is a solution to our problem.

Remark. when the proximal operators and projections are evaluated with *errors* bounded by ϵ, then $\liminf_{k\to\infty} \|r^k\|_2 = O(\sqrt{\epsilon})$.

Convergence of A2DR

Theorem (Pathological Case)

If the problem is pathological (strongly primal infeasible or strongly dual infeasible), then

$$\lim_{k\to\infty} \left(v^k - v^{k+1}\right) = \delta v \neq 0.$$

Furthermore, if $\lim_{k\to\infty} Ax^{k+1/2} = b$*, then the problem is unbounded and* $\|\delta v\|_2 = t\,\mathbf{dist}(\mathbf{dom}\, f^*, \mathbf{range}(A^T))$*.*
Otherwise, it is infeasible and $\|\delta v\|_2 \geq \mathbf{dist}(\mathbf{dom}\, f, \{x : Ax = b\})$ *with equality when the dual problem is feasible.*

Implementation

Pre-conditioning (convergence greatly improved by rescaling problem):

- Replace original A, b, f_i with

$$\hat{A} = DAE, \quad \hat{b} = Db, \quad \hat{f}_i(\hat{x}_i) = f_i(e_i \hat{x}_i)$$

- D and E are diagonal positive, $e_i > 0$ corresponds to ith block diagonal entry of E, and chosen by equilibrating A
- Proximal operator of $\hat{f}_i$ can be evaluated using proximal operator of f_i

$$\mathbf{prox}_{t\hat{f}_i}(\hat{v}_i) = \tfrac{1}{e_i}\mathbf{prox}_{(e_i^2 t)f_i}(e_i \hat{v}_i)$$

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Parallelization: `multiprocessing` package in Python.

1. Motivation and Problem Statement

2. Douglas-Rachford Splitting

3. Anderson Acceleration & A2DR

4. Numerical experiments

5. Conclusion

Nonnegative Least Squares (NNLS)

$$\begin{array}{ll} \text{minimize} & \|Fz - g\|_2^2 \\ \text{subject to} & z \geq 0 \end{array}$$

with respect to $z \in \mathbf{R}^q$

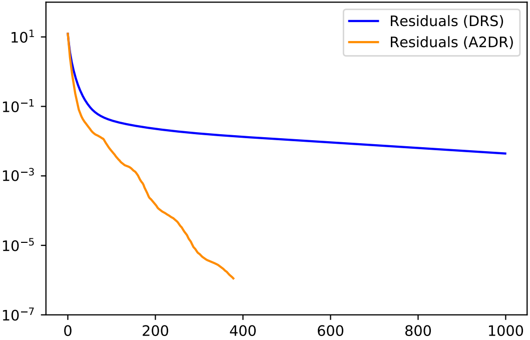
- Problem data: $F \in \mathbf{R}^{p \times q}$ and $g \in \mathbf{R}^p$
- Can be written in standard form with

$$f_1(x_1) = \|Fx_1 - g\|_2^2, \quad f_2(x_2) = \mathcal{I}_{\mathbf{R}_+^n}(x_2)$$
$$A_1 = I, \quad A_2 = -I, \quad b = 0$$

- We evaluate proximal operator of f_1 using LSQR

NNLS: Convergence of $\|r^k\|_2$

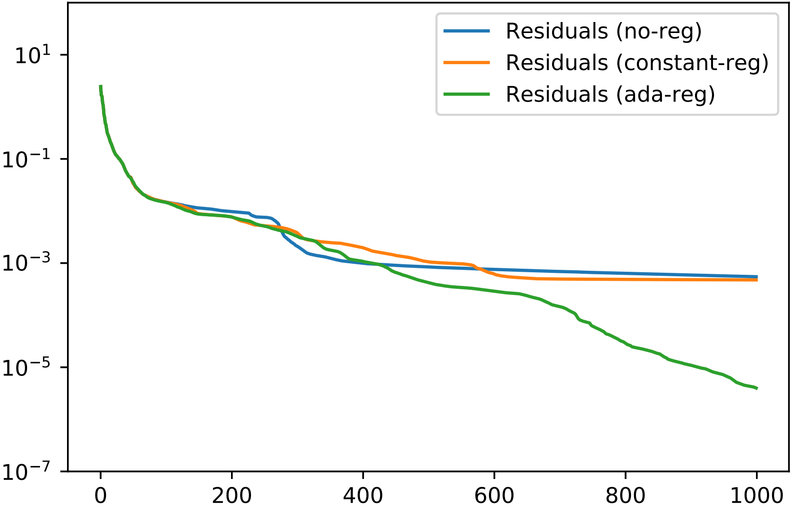
$p = 10^4$, $q = 8000$, F has 0.1% nonzeros



OSQP and SCS took respectively 349 and 327 seconds, while A2DR only took 55 seconds.

NNLS: Effect of regularization

$p = 300,\ q = 500,\ F$ has 0.1% nonzeros



Sparse Inverse Covariance Estimation

- Samples $z_1, \dots, z_p$ IID from $\mathcal{N}(0, \Sigma)$
- Know covariance $\Sigma \in \mathbf{S}_+^q$ has **sparse** inverse $S = \Sigma^{-1}$
- One way to estimate S is by solving the penalized log-likelihood problem

$$\text{minimize} \quad -\log\det(S) + \mathrm{tr}(SQ) + \alpha\|S\|_1,$$

where Q is the sample covariance, $\alpha \geq 0$ is a parameter

- Note $\log\det(S) = -\infty$ when $S \nsucc 0$

Sparse Inverse Covariance Estimation

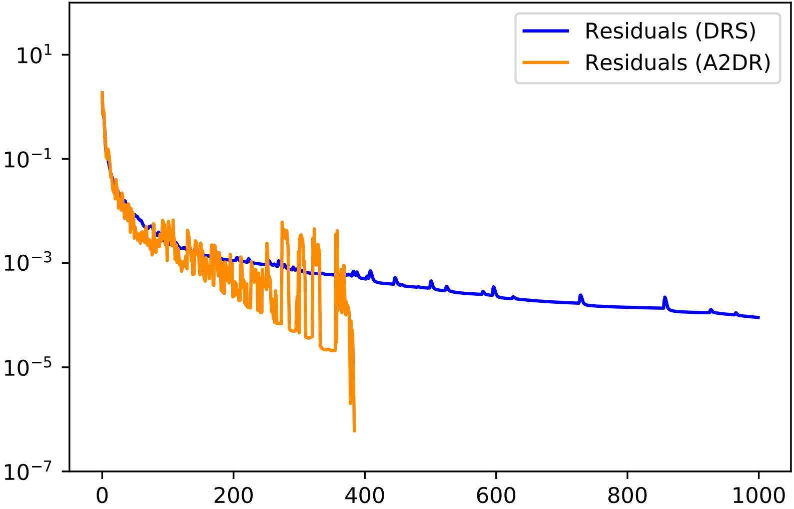
- Problem can be written in standard form with

$$f_1(S_1) = -\log\det(S_1) + \mathrm{tr}(S_1 Q), \quad f_2(S_2) = \alpha \|S_2\|_1,$$
$$A_1 = I, \quad A_2 = -I, \quad b = 0.$$

- Both proximal operators have closed-form solutions.

Covariance Estimation: Convergence of $\|r^k\|_2$

$p = 1000,\ q = 100,\ S$ has 10% nonzeros



Covariance Estimation: larger examples

Ran A2DR on instances with $q = 1200$ and $q = 2000$ (vectorizations on the order of 10^6) and compared its performance to SCS:

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- In the former case, A2DR took 1 hour to converge to a tolerance of 10^{-3}, while SCS took 11 hours to achieve a tolerance of 10^{-1} and yielded a much worse objective value.
- In the latter case, A2DR converged in 2.6 hours to a tolerance of 10^{-3}, while SCS failed immediately with an out-of-memory error.

Multi-Task Logistic Regression

$$\text{minimize} \quad \phi(W\theta, Y) + \alpha \textstyle\sum_{l=1}^{L} \|\theta_l\|_2 + \beta\|\theta\|_*$$

with respect to $\theta = [\theta_1 \cdots \theta_L] \in \mathbf{R}^{s \times L}$

- Problem data: $W \in \mathbf{R}^{p \times s}$ and $Y = [y_1 \cdots y_L] \in \mathbf{R}^{p \times L}$
- Regularization parameters: $\alpha \geq 0, \beta \geq 0$
- Logistic loss function

$$\phi(Z, Y) = \sum_{l=1}^{L} \sum_{i=1}^{p} \log\left(1 + \exp(-Y_{il} Z_{il})\right)$$

Multi-Task Logistic Regression

- Rewrite problem in standard form with:

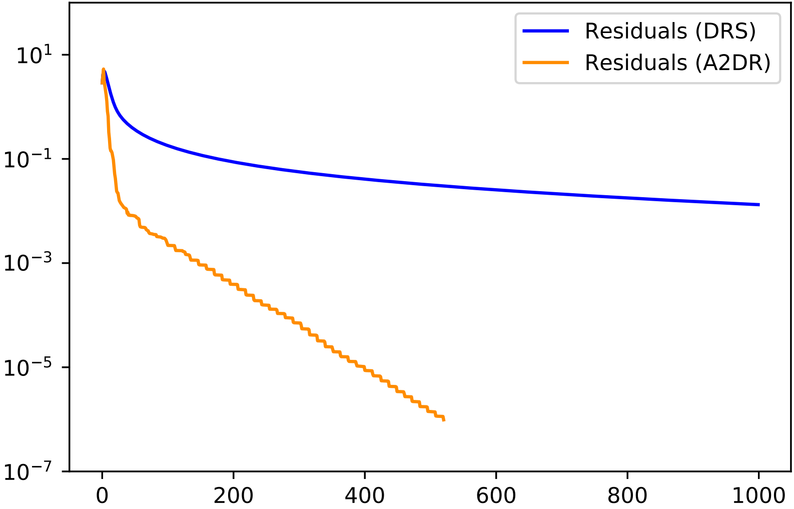
$$f_1(Z) = \phi(Z, Y), \quad f_2(\theta) = \alpha \sum_{l=1}^{L} \|\theta_l\|_2, \quad f_3(\tilde{\theta}) = \beta \|\tilde{\theta}\|_*,$$

$$A = \begin{bmatrix} I & -W & 0 \\ 0 & I & -I \end{bmatrix}, \quad x = \begin{bmatrix} Z \\ \theta \\ \tilde{\theta} \end{bmatrix}, \quad b = 0$$

- We evaluate proximal operator of f_1 using Newton-CG method, and the rest with closed-form formulae.

Multi-Task Logistic: Convergence of $\|r^k\|_2$

$p = 300,\ s = 500,\ L = 10,\ \alpha = \beta = 0.1$



Other examples

A (very) brief summary of other examples (see the paper for more details):

- l_1 trend filtering.
- Stratified models.
- Single commodity flow optimization (match the performance of OSQP, and largely outperform SCS).
- Optimal control (largely outperform both SCS and OSQP).
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Remark. The advantage compared to OSQP probably comes from the inclusion of AA, while the advantage compared to SCS (which includes type-I AA) is probably due to the more compact standard form representation.

Conclusion

- A2DR is a fast, robust algorithm for solving generic (non-smooth) convex optimization problems in the prox-affine form.

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- Python library:

  `https://github.com/cvxgrp/a2dr`

Future Work

- More work on feasibility detection.
- Expand library of proximal operators (non-convex proximal).
- User-friendly interface with CVXPY (with the help of `Epsilon`).
- GPU parallelization and cloud computing,

References

Fu, A.*, Zhang, J.* and Boyd, S. P. (2019). (*equal contribution) Anderson Accelerated Douglas-Rachford Splitting. *arXiv preprint arXiv:1908.11482.*

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    - Zhang, J., O'Donoghue, B. and Boyd, S. P. (2018).
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Thanks for listening!

Any questions?