Volterra Series: Engineering Fundamentals

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ABSTRACT

In the last century engineers have achieved great success in the analysis, control, and design of circuits and systems which are *linear* and *time-invariant*. For such systems we have the *convolution formula* for the output $y(t)$ in terms of the input $u(t)$ to the system:

$$y(t) = \int h(\tau) u(t-\tau) d\tau \tag{1}$$

A *Volterra series expansion* is a representation for *nonlinear systems* analogous to (1):

$$y(t) = h_0 + \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1, \ldots, \tau_n) u(t-\tau_1) \ldots u(t-\tau_n) d\tau_1 \ldots d\tau_n \tag{2}$$

The purpose of this thesis is to address some fundamental engineering issues surrounding the Volterra series (2). These issues are:

(I) When does (2) make sense and what exactly does it mean?

We show that (2) can be interpreted as a *Taylor series*, and so it is not surprising that (2) makes sense for inputs u smaller than a positive number which has the interpretation of the *radius of convergence* of the Volterra series (2).

(II) For what nonlinear systems is the expansion (2) appropriate?

Unlike (1), which is valid for essentially *all* linear time-invariant operators arising in engineering, the Volterra series expansion (2) is only appropriate for *some* nonlinear operators. We show that it is appropriate precisely for those operators with *fading memory*.

(III) How can the kernels h_n be measured in the laboratory?

Measuring the kernels by classical methods is extremely slow. We develop a new quick method for measuring the kernels and apply it to various real systems.

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Table of Contents

List of Symbols

Notation	Meaning
$\mathbf{R}_+$, $\mathbf{R}_-$	$\{t \in \mathbf{R} \mid t \geq 0\}$, $\{t \in \mathbf{R} \mid t \leq 0\}$
$\mathbb{C}_+$	$\{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$
$\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_-$, $\mathbb{N}$	$\{\cdots -1,0,1,\cdots\}$, $\{0,1,2,\cdots\}$, $\{\cdots -2,-1,0\}$, $\{1,2,3\cdots\}$
B_ρ	Zero-centered open ball of radius ρ, $\{x \mid \|x\| < \rho\}$.
$\mathbf{B}(\mathbf{R}_+^n)$	Bounded (signed) measures on $\mathbf{R}_+^n$, with norm $\|\mu\| \triangleq \int_{\mathbf{R}_+^n} d\lvert\mu\rvert$.
$\mathbf{L}^\infty$	Bounded functions on $\mathbf{R}$,† with norm $\|u\| \triangleq \sup\{\lvert u(t)\rvert \mid t \in \mathbf{R}\}$.
$\mathbf{L}^1(\mathbf{R}_+^n)$, $\mathbf{L}^2(\mathbf{R}_+^n)$	Integrable (square-integrable) functions on $\mathbf{R}_+^n$, with norms $\|u\|_1 \triangleq \int_{\mathbf{R}_+^n} \lvert u\rvert$ and $\|u\|_2 \triangleq (\int_{\mathbf{R}_+^n} u^2)^{1/2}$.
$\mathbf{l}^\infty$, $\mathbf{l}^\infty(\mathbb{Z}_+)$, $\mathbf{l}^\infty(\mathbb{Z}_-)$	Bounded sequences on $\mathbb{Z}$ ($\mathbb{Z}_+$, $\mathbb{Z}_-$), with norm $\|u\| \triangleq \sup_k \lvert u(k)\rvert$.
$\mathbf{l}^1$, $\mathbf{l}^1(\mathbb{Z}_+)$, $\mathbf{l}^1(\mathbb{Z}_-)$	Summable sequences on $\mathbb{Z}$ ($\mathbb{Z}_+$, $\mathbb{Z}_-$), with norm $\|u\|_1 \triangleq \sum_k \lvert u(k)\rvert$.
$\mathbf{l}^2$, $\mathbf{l}^2(\mathbb{Z}_+)$, $\mathbf{l}^2(\mathbb{Z}_-)$	Square-summable sequences on $\mathbb{Z}$ ($\mathbb{Z}_+$, $\mathbb{Z}_-$), with norm $\|u\|_2 \triangleq (\sum_k u(k)^2)^{1/2}$.
$C(\mathbf{R})$, $C[a,b]$	Subspace of $\mathbf{L}^\infty$ consisting of continuous functions (supported on $[a,b]$).

† Technically, equivalence classes of essentially bounded functions agreeing off a set of measure zero. We will not be so precise in the sequel.

Chapter 1

Introduction and Overview

1. Overview

In the last century engineers have achieved great success in the analysis, control, and design of circuits and systems which are *linear* and *time-invariant*. For such systems we have the *convolution formula* for the output $y(t)$ in terms of the input $u(t)$ to the system:

$$y(t) = \int h(\tau)u(t-\tau)d\tau \tag{1}$$

If the input u to such a system is $2\pi\omega^{-1}$-periodic, we have the *frequency-domain formula*

$$\hat{y}(m) = H(j\omega m)\hat{u}(m) \tag{2}$$

expressing the Fourier coefficients of the output in terms of the Fourier coefficients of the input.

In this thesis we will study *Volterra series,* a representation for *nonlinear systems* analogous to (1):

$$y(t) = h_0 + \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1, \ldots, \tau_n)u(t-\tau_1)\ldots u(t-\tau_n)d\tau_1\ldots d\tau_n \tag{3}$$

Corresponding to (2) we have the formula

$$\hat{y}(m) = \sum_{n=1}^{\infty} \left\{ \sum_{k_1+ \cdots +k_n=m} \right\} \hat{u}(k_1)\ldots\hat{u}(k_n)H_n(j\omega k_1,\ldots,j\omega k_n) \tag{4}$$

which expresses the Fourier coefficients of the output in terms of the Fourier coefficients of a $2\pi\omega^{-1}$-periodic input.

The purpose of this thesis is to address some fundamental issues surrounding the Volterra series (3).

Our first topic in chapter 2, *Analytical Foundations*, is, *what exactly do we mean by equation (3)?* Just as equation (1) can be interpreted in several different ways, so can (3). But the precise interpretation of (3) is more complicated than that of (1)- we will see, for example, that the Volterra series (3) is in general meaningful only for signals bounded by a certain number which we call the *radius of convergence* of the Volterra series (3).

We then carefully derive formulas for the kernels h_n of various system interconnections. Here our observation that Volterra series can be interpreted as a *Taylor series* helps us make sense of these (admittedly awful) formulas. We close chapter 2 by applying these arguments to show that a simple dynamical system has a Volterra series description.

In chapter 3, *Frequency Domain Topics,* we study the formula (4), which undoubtedly appears complicated and strange to the reader. In fact it *is* complicated; it is not always true, and indeed it does not always make sense. Our two main objectives in chapter 3 are, first, to give an intuitive interpretation to the formula (4), and, second, to give some simple conditions under which it holds. At the end of chapter 3 we present two applications of the material, the first related to a question involving modeling nonlinear devices; the second concerns linearizing a nonlinear operator with a nonlinear compensator.

Chapter 4, *Approximating Nonlinear Operators with Volterra Series*, is motivated by the following fundamental question:

> *When can a system be represented by the Volterra series (3), that is, for what systems is the Volterra series representation appropriate?*

The answer to this question depends on what we mean by *represent*. Suppose our system is described by a circuit with ideal circuit elements, a block diagram of simple operators, or as a finite-dimensional dynamical system, with the property that it determines an input/output operator N from one function space to another.

We might ask, is N given *exactly* by a Volterra series operator, that is, are there kernels h_n such that for the inputs u of interest we have

$$Nu(t) = h_0 + \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1, \ldots, \tau_n) u(t-\tau_1) \ldots u(t-\tau_n) d\tau_1 \ldots d\tau_n$$

The answer to this question is quite often yes. The interconnection formulas of chapter 2 can sometimes establish this; for the general case we refer the reader to the work of Sandberg.[1,2,3] Very roughly speaking, the requirement is that the nonlinearities which describe N are *analytic*, that is, have a power series which converges at least for small arguments. For a circuit we would require the constitutive relations of the various elements to be analytic; for a dynamical system we would require the vector field to be analytic.†

Thus our question is answered, but the answer is not really in terms of direct engineering significance. We would not ask a technician to check whether the V–I curve of some device is analytic, as opposed to merely infinitely differentiable, since all modeling of *real systems* applies only within a certain *precision*, for a certain set of input signals. Thus we are led to a different question: when can an operator N be approximated within some precision ϵ over some useful set K of signals by a Volterra series operator $\hat{N}$, that is

$$||Nu - \hat{N}u|| \leq \epsilon$$

for all $u \in K$? This is the topic of chapter 4.

The answer is simply that N must have *fading memory*, which makes good engineering sense. Thus we may answer our (modified) question:

The Volterra series representation (S) is appropriate for systems with fading memory.

One important comment: while the class of systems with fading memory is a very wide class of systems, it is by no means all the nonlinear systems important in engineering. To give one example, dynamical systems with several stable equilibria do not have fading memory. Thus the scope of Volterra series is a strict subset (it could be argued, a *very strict subset*) of the nonlinear systems arising in engineering. This is in marked contrast to (1), which suffices to describe all the linear time-invariant systems of engineering.

In chapter 5, *Measuring Volterra Kernels*, we show that the Volterra kernels can actually be measured in the laboratory, though of course the measurements are not so simple as those of a linear system (1).

2. A Comment

Perhaps the most obvious difference between (1) and (2) and their analogs (3) and (4) is that (3) and (4) are much more complicated. This greater complication persists throughout our study

† Analyticity is not the *only* requirement; for example the dynamical system $\dot{y}(t) = -y(t)^3 + u(t)$ can be shown *not* to have an exact Volterra series representation.

of Volterra series. Where equations (1) and (2) are simple to interpret, equations (3) and (4) are quite tricky; where the representation (2) is easy to manipulate, we will see in the next chapter formulas for Volterra kernels which fill two lines. Where (1) and (2) are appropriate for *all* linear time-invariant systems in engineering, the Volterra series representation (3) is only appropriate for *some* nonlinear systems. This greater complication is not due to the representation (3), but rather to the vastly more complicated behavior of *nonlinear systems* over *linear systems*.

3. Contribution of This Thesis

The material of chapter 2 is classic. Informal expositions of this material can be found in the early MIT work on Volterra series[4,5] or Rugh's book,[6] and the results could be rigorously derived from the work of Sandberg.[1,2]

The material of chapter 3 is mostly new, though it overlaps the work of Sandberg, particularly his work on almost periodic forcing functions.[7] The observation that the frequency domain formula (4) above does not always make sense, and the conditions under which it holds, are new, as are the applications presented in the last two sections of chapter 3. Chapters 2 and 3 are based on the paper:

> S. Boyd, L. O. Chua, and C. A. Desoer, Analytical Foundations of Volterra Series, to appear, *IMA Journal of Mathematical Control and Information,* Oxford University Press, (also UCB/ERL memo M84/14).

§6 of chapter 3 comes from the papers:

> S. Boyd and L. O. Chua, Uniqueness of a Basic Nonlinear Structure, *IEEE Trans. Circuits and Systems,* CAS-30 #9, Sept 1983, p648-651 (also UCB/ERL memo M83/8).
>
> S. Boyd and L. O. Chua, Uniqueness of Circuits and Systems Containing One Nonlinearity, *Math. Theory of Networks and Systems Conf. Proc.* Lecture Notes in Control and Information Sciences vol. 58, p101-119, June 1983. To appear in *IEEE Trans. on Automatic Control,* TAC-30 #7, July 1985 (also UCB/ERL memo M83/30).

All of the results of chapter 4 are new, though the topic of approximating nonlinear operators with Volterra series operators is an old one. Chapters 4 is based on the paper:

> S. Boyd and L. O. Chua, Fading Memory and the Problem of Approximating Nonlinear Operators with Volterra Series, submitted to *IEEE Trans. Circuits and Systems,* (also UCB/ERL memo M84/96).

Most of the techniques presented in chapter 5 are new, especially the quick method of measuring the second kernel and the idea of choosing the phases in the probing signal so as to minimize its crest factor. It is based on the paper:

> S. Boyd, Y. S. Tang, and L. O. Chua, Measuring Volterra Kernels, *IEEE Trans. Circuits and Systems,* CAS-30 #8, August 1983, p571-577 (also UCB/ERL memo M83/7).

Chapter 2

Analytical Foundations

In this chapter we carefully study the *analysis* involved with the formal Volterra series (0.1):

$$Nu(t) = h_0 + \sum_{n=1}^{\infty} \int \ldots \int h_n(\tau_1, \ldots, \tau_n) u(t-\tau_1) \ldots u(t-\tau_n) d\tau_1 \ldots d\tau_n \tag{0.1}$$

Our first topic is the interpretation of (0.1). For example, what are the kernels h_n, ordinary functions or distributions? When do the integrals and sum in (0.1) make sense? What restrictions are there on the input signal u? There are many ways to answer these questions, depending on what systems we will model with the Volterra series (0.1) and what we propose to use the model for.

If we intend to design controllers for nonlinear dynamical systems, for example, (0.1) should be able to model *unstable* systems which have some *smoothing* (roughly speaking, are strictly proper). In this case the kernels might be taken to be continuous functions supported on the positive orthant $\mathbf{R}_+^n$. The inputs could then be restricted to be continuous or piecewise-continuous functions supported on $\mathbf{R}_+$ and such that the integrals and sum in (0.1) converge absolutely. This is a common interpretation of (0.1).

On the other hand if we intend to model systems which, roughly speaking, are *stable,* but need not have any smoothing effect (example: memoryless nonlinearity), then another interpretation of (0.1) is more appropriate. This is the formulation we will develop. In fact, it is not difficult (just cumbersome) to extend or modify the following to apply to unstable or multi-input multi-output systems.

After stating in detail what we mean by (0.1), we examine some elementary properties of Volterra series operators, such as continuity and differentiability. We will show that they can be interpreted as *Taylor series*, and so are analogous to ordinary power series. This analogy guides the rest of the chapter. For example, we prove the Uniqueness theorem, which asserts that two operators of the form (0.1) (with symmetric kernels) are equal if and only the corresponding kernels are equal (cf. ordinary power series).

We then prove some well-known formulas for the kernels of various "system interconnections". We give an elementary and complete proof of the Inversion theorem for Volterra series, and work through an illustrative example.

1. What are the Kernels?

In most treatments the kernels $h_n(\tau_1,...,\tau_n)$ in equation (0.1) are interpreted as *functions* from $\mathbf{R}^n$ to $\mathbf{R}$. Unfortunately this interpretation rules out some operators common in engineering. We start with two examples:

Example 1:

$$\dot{x} = -x + u$$
$$y = x^2$$

and $x(0)=0$. Then for $t \geq 0$

$$y(t) = \left\{\int_0^t e^{-\tau} u(t-\tau)d\tau\right\}^2$$
$$= \int\int 1(\tau_1)1(\tau_2)e^{-(\tau_1+\tau_2)}u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2$$

where

$$1(t) \triangleq \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Hence this operator has a Volterra series description with just one nonzero kernel,

$$h_2(\tau_1,\tau_2) = 1(\tau_1)1(\tau_2)e^{-(\tau_1+\tau_2)}$$

This kernel h_2 is an ordinary function $:\mathbf{R}^2 \rightarrow \mathbf{R}$.

Example 2:

$$\dot{x} = -x + u^2$$
$$y = x$$

and $x(0)=0$. Here

$$y(t) = \int_0^t e^{-\tau} u(t-\tau)^2 d\tau =$$
$$= \int\int 1(\tau_1)1(\tau_2)e^{-\tau_1}\delta(\tau_1-\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2$$

if you will condone the notation. So here the kernel h_2 is not a *function* as it was in example 1 but a *measure* supported on the line $\tau_1=\tau_2$, informally given by

$$h_2(\tau_1,\tau_2) = 1(\tau_1)1(\tau_2)\delta(\tau_1-\tau_2)e^{-\tau_1}$$

These examples are typical- in general the Volterra series of dynamical systems with the vector field *affine* in the input u (e.g. *bilinear systems*) have kernels which are ordinary functions whereas in other cases more general measures may be necessary.[8,9,10,3] In the latter case Sandberg has called the series "Volterra-like".[3] §A1 contains an in-depth discussion of Volterra-like series.

A less exotic but widely occurring nonlinear operator whose description requires kernels which are measures is the *memoryless operator*

$$y(t) = f(u(t))$$

where $f:\mathbf{R}\rightarrow\mathbf{R}$ is analytic near 0.

We will allow our kernels to be measures. We will see that the analysis is no harder, and the resulting theory then includes *all* the examples above.

2. When the Series Converges

Recall that the ordinary power series $g(x)=\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<\rho$, where the radius of convergence is given by $\rho=(\limsup_{n\rightarrow\infty}|a_n|^{1/n})^{-1}$. Similarly a radius of convergence ρ can be associated with a formal Volterra series

$$Nu(t) = y(t) = h_0 + \sum_{n=1}^{\infty}\int\cdots\int h_n(\tau_1,\ldots,\tau_n)u(t-\tau_1)\ldots u(t-\tau_n)d\tau_1\ldots d\tau_n \qquad (2.1)$$

such that the series will converge for input signals with $|u(t)| < \rho$.

More precisely, let $\mathbf{B}(\mathbb{R}_+^n)$ be the bounded measures on $\mathbf{R}_+^n$ ($\mathbf{R}_+ \triangleq \{\tau|\tau\geq 0\}$),† with $||\mu|| = \int d|\mu|$. For convenience we will write elements of $\mathbf{B}(\mathbf{R}_+^n)$ as if they were absolutely continuous ("Physicists' style"), e.g. $h_2(d\tau_1,d\tau_2)=\delta(\tau_1-\tau_2)e^{-\tau_1}d\tau_1 d\tau_2$. For signals $||\ ||$ will denote the ∞-norm, i.e. $||u|| = ||u||_\infty \triangleq \sup_t |u(t)|$.††

† We thus consider only *causal* operators, but in fact all of the following holds for kernels which are bounded measures on $\mathbb{R}^n$.

†† An excellent reference on bounded measures and these norms (and analysis in general) is Rudin's book

Definition: By a *Volterra series operator* we will henceforth mean an operator given by equation (0.1) above and satisfying assumptions

(A1) $h_0 \in \mathbf{R}$, $h_n \in \mathbf{B}(\mathbf{R}_+^n)$, and

(A2) $\limsup_{n\to\infty} \|h_n\|^{1/n} < \infty$, that is, $\{\|h_n\|^{1/n}\}$ is bounded.

Our first task is to determine for which u's equation (2.1) makes sense.

Definition: If N is a Volterra series operator with kernels h_n, we define the *gain bound function* of N to be, for $x \geq 0$, $f(x) \triangleq |h_0| + \sum_{n=1}^{\infty} \|h_n\| x^n$ (with extended values, that is, $f(x)$ may be ∞). The *radius of convergence* of N is defined by $\rho = \mathrm{Rad} N \triangleq (\limsup_{n\to\infty} \|h_n\|^{1/n})^{-1}$.

Assumption (A2) implies that $\rho > 0$ and that the gain bound function f is analytic at 0, with normal radius of convergence ρ. Since all the terms in the series for f are positive ρ is also given by $\rho = \inf\{x \mid f(x) = \infty\}$, a formula which will be useful in §3. We can now say when (2.1) makes sense:

Theorem 2.1 (Gain bound theorem):

Suppose N is a Volterra series operator with kernels h_n, gain bound function f, and radius of convergence ρ. Then

(I) the integrals and sum in equation (2.1) above converge absolutely for inputs with $\|u\| < \rho$, that is, in B_ρ, the ball of radius ρ in $\mathbf{L}^\infty$.

(II) N satisfies $\|Nu\| \leq f(\|u\|)$ and consequently N maps B_ρ into $\mathbf{L}^\infty$.

(II) is partial justification for naming f the gain bound function, we'll soon see more. Theorem 2.1 is well known (in various forms).[6, 12, 8, 13, 14, 10, 15, 1]

Proof:

$$\int \cdots \int | h_n(\tau_1, \ldots, \tau_n) u(t-\tau_1) \ldots u(t-\tau_n) | \, d\tau_1 \ldots d\tau_n \leq \|h_n\| \, \|u\|^n$$

In particular, the integrals make sense. If $\|u\| < \rho$, then

[11].

$$|h_0| + \sum_{n=1}^{\infty} | \int \cdots \int h_n(\tau_1,...,\tau_n)u(t-\tau_1)...u(t-\tau_n)d\tau_1...d\tau_n |$$

$$\leq |h_0| + \sum_{n=1}^{\infty} \int \cdots \int | h_n(\tau_1,...,\tau_n)u(t-\tau_1)...u(t-\tau_n) | d\tau_1...d\tau_n$$

$$\leq |h_0| + \sum_{n=1}^{\infty} ||h_n|| \, ||u||^n = f(||u||) < \infty$$

which establishes absolute convergence of the series and the gain bound in (II). □

For convenience we adopt the notational convention that throughout this chapter N will denote a Volterra series operator with kernels h_n, gain bound function f, and radius of convergence ρ.

The Gain Bound theorem has many simple applications. For example, the tail of the gain bound function gives a bound on the truncation error for a Volterra series.

Corollary 2.2 (Error Bound for Truncated Volterra Series):

The truncated Volterra series operator defined by

$$N^{(k)}(t) \triangleq h_0 + \sum_{n=1}^{k} \int \cdots \int h_n(\tau_1,...,\tau_n)u(t-\tau_1)...u(t-\tau_n)d\tau_1...d\tau_n$$

satisfies

$$||Nu - N^{(k)}u|| \leq \sum_{n=k+1}^{\infty} ||h_n|| \, ||u||^n$$

which is $o(||u||^k)$. Thus if $\sum_{n=k+1}^{\infty} ||h_n|| M^n$ is small, the operator N is well modeled by the truncated Volterra series operator $N^{(k)}$ for inputs whose peaks do not exceed M.

3. Elementary Properties: Continuity

We will now show that N is continuous on B_ρ and Lipschitz continuous on any B_r, $r < \rho$.

Lemma 3.1: Suppose $||u|| + ||v|| < \rho$. Then

$$||N(u+v) - N(u)|| \leq f(||u|| + ||v||) - f(||u||) \leq f'(||u|| + ||v||)||v||$$

Proof: Assume $||u|| + ||v|| < \rho$. Then $||u+v|| < \rho$ so $N(u+v)$ makes sense and

$$|N(u+v)(t) - N(u)(t)| \leq \tag{3.1a}$$

$$\leq \sum_{n=1}^{\infty} \int \cdots \int \left| h_n(\tau_1,...,\tau_n) \left\{ \prod_{i=1}^{n} (u+v)(t-\tau_i) - \prod_{i=1}^{n} u(t-\tau_i) \right\} \right| d\tau_1...d\tau_n \tag{3.1b}$$

$$\leq \sum_{n=1}^{\infty} \|h_n\| \sum_{j=0}^{n-1} \binom{n}{j} \|u\|^j \|v\|^{n-j} \tag{3.1c}$$

$$= \sum_{n=1}^{\infty} \|h_n\| \left\{ (\|u\| + \|v\|)^n - \|u\|^n \right\} \tag{3.1d}$$

$$= f(\|u\| + \|v\|) - f(\|u\|) \tag{3.1e}$$

This technique will recur so careful explanation is worthwhile. In (3.1b) the first product, when expanded, has 2^n terms; the second product is precisely the first term in the expansion. Replacing the remaining 2^n-1 terms by their norms and integrating yields (3.1c).

The final inequality in lemma 3.1 follows from the mean value theorem, since

$$f(\|u\| + \|v\|) - f(\|u\|) = f'(\varsigma)\|v\|$$

where $\|u\| \leq \varsigma \leq \|u\| + \|v\|$ and f' is increasing. Thus f' can be interpreted as an *incremental gain bound function* for N. □

Theorem 3.2 (Incremental gain theorem): Let B_r be the ball of radius r in $\mathbf{L}^\infty$, and suppose $r < \rho$. Then

(I) $N: B_r \rightarrow B_{f(r)}$ is *Lipschitz continuous*,

(II) $N: B_\rho \rightarrow \mathbf{L}^\infty$ is *continuous*.

Proof: Suppose u and v are in B_r. From the Gain Bound theorem

$$\|Nu - Nv\| \leq f(\|u\|) + f(\|v\|) \tag{3.2}$$

We claim that

$$\|Nu - Nv\| \leq f(\|u-v\| + \|v\|) - f(\|v\|) \tag{3.3}$$

For $\|u-v\| + \|v\| < \rho$ (3.3) is simply lemma 3.1; for $\|u-v\| + \|v\| \geq \rho$ (3.3) is true since its right-hand side is ∞. From (3.2) and (3.3) we deduce

$$\|Nu-Nv\| \leq \min\{ f(\|u-v\| + \|v\|) - f(\|v\|),\ f(\|u\|) + f(\|v\|) \}$$
$$\leq \|u-v\| \min\left\{ \frac{f(r+\|u-v\|)-f(r)}{\|u-v\|},\ \frac{2f(r)}{\|u-v\|} \right\} \leq K\|u-v\|$$

where K is the max of the expression $\min\{\cdots\}$ for $0 \leq \|u-v\| \leq 2r$ and is finite. (K is in fact $2f(r)/(f^{-1}(3f(r)) - r)$: see figure 1). This establishes (I); since (I) is true for any $r < \rho$ (II) follows.

□

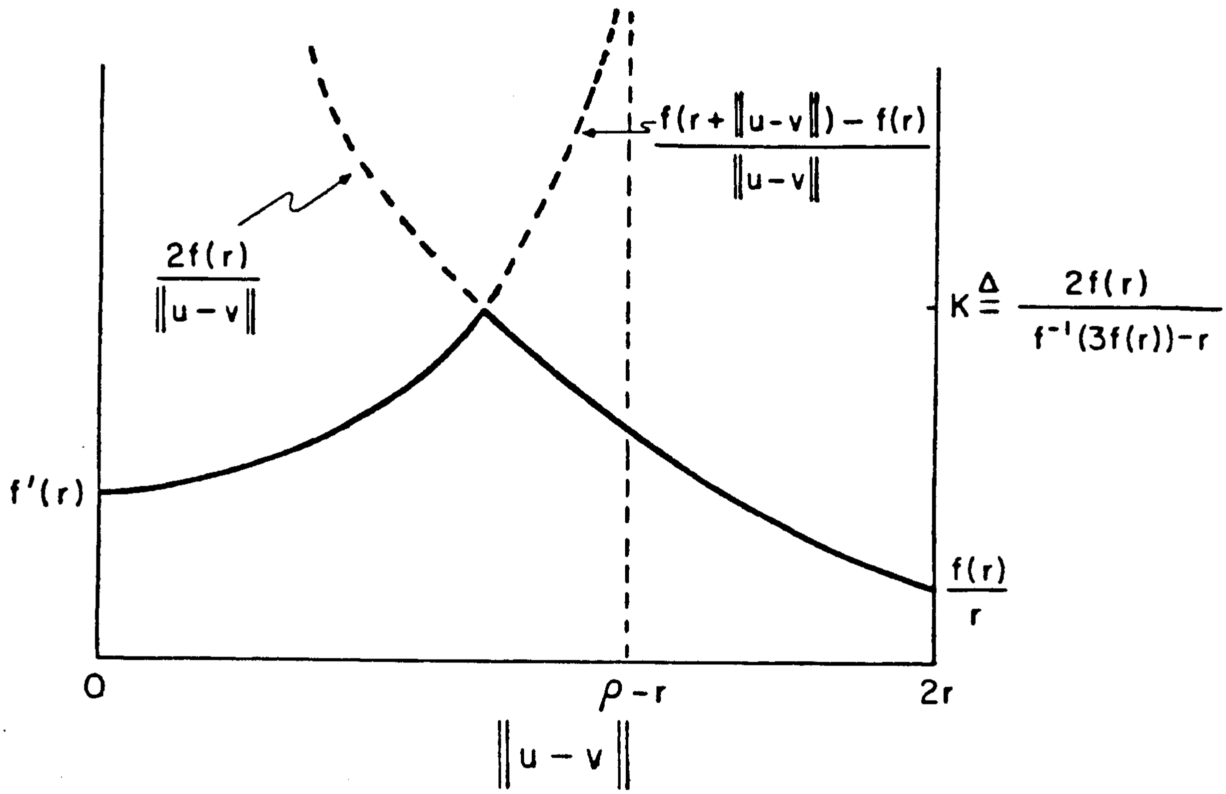

figure 1

We will soon see that N is much more than merely continuous; for example, N has Frechet derivatives of all orders on B_ρ. But before moving on, we present an extension of the last theorem which will be important in the next chapter.

Recall that for linear systems $y = h_1 * u$ we have the result $\|y\|_p \leq \|h_1\| \|u\|_p$, for $1 \leq p \leq \infty$.[16] It turns out that when *properly reformulated* the Gain Bound theorem and the Incremental Gain theorem are also true with general p-norms. First some warnings for $p < \infty$: a Volterra series operator need not be defined on any open subset of $\mathbf{L}^p$ (e.g. $Nu(t) = u(t)/(1-u(t))$), and even when it is, it need not map $\mathbf{L}^p$ back into $\mathbf{L}^p$ (e.g. $Nu(t) = u(t)^2$). For more details and discussion we refer the reader to §A2.

Theorem 3.3 (Gain Bound theorem for $\mathbf{L}^p$): Suppose $h_0 = 0$. Then for $1 \leq p \leq \infty$

$$\|Nu\|_p \leq \|u\|_p \frac{f(\|u\|)}{\|u\|}$$

(unmarked norms are ∞-norms).

Even though our next theorem is stronger, we give the proof here to demonstrate the basic argument.

Proof:

$$|y_n(t)| \leq \int \cdots \int |h_n(\tau_1,\ldots,\tau_n)u(t-\tau_1)\ldots u(t-\tau_n)| \, d\tau_1\ldots d\tau_n$$
$$\leq ||u||^{n-1} \int \left\{ \int \cdots \int |h_n(\tau_1,\ldots,\tau_n)| \, d\tau_2 \ldots d\tau_n \right\} |u(t-\tau_1)| \, d\tau_1 \tag{3.4}$$

Now the bracketed expression in (3.4) is a *measure* in τ_1 with norm $||h_n||$, hence using the result for *linear systems* cited above we have[16]

$$||y_n||_p \leq ||u||^{n-1} ||h_n|| \, ||u||_p$$

Thus

$$||Nu||_p \leq \sum_{n=1}^{\infty} ||y_n||_p \leq ||u||_p \sum_{n=1}^{\infty} ||h_n|| ||u||^{n-1} = ||u||_p \frac{f(||u||)}{||u||}$$

which establishes theorem 3.3. □

Lemma 3.4:

$$||(N(u+v)-Nu)||_p \leq ||v||_p \frac{f(||u||+||v||)-f(||u||)}{||v||} \leq ||v||_p f'(||u||+||v||)$$

The proof combines the proof above with the proof of the Incremental Gain theorem and is in §A2.

Theorem 3.5 (Incremental Gain theorem for $\mathbf{L}^p$): Let B_r be the ball of radius r in $\mathbf{L}^\infty$, with $r < \rho$. Then there is a K such that

$$||Nu-Nv||_p \leq K||u-v||_p$$

Note that here the Volterra series N may have a nonzero 0th order kernel h_0. The proof of theorem 3.3 is identical to that of theorem 3.2 and so is omitted.

4. Multilinear and Polynomial Mappings

This section contains mostly background material for the next section and may be skipped by those familiar with the topic. There are many good references on this material, both in mathematics[17, 18] and engineering.[19, 20]

Note that the nth term y_n in a Volterra series is homogeneous of degree n in the input u. Indeed much more is true; it is a *polynomial* mapping in u.

Definition: Let V and W be vector spaces over $\mathbf{R}$. Then $M: V^n \to W$ is said to be *multilinear* or n-linear if it is linear in each argument separately, i.e. if

$$M(v_1, \ldots v_j + \alpha w, \ldots v_n) = M(v_1, \ldots v_j, \ldots v_n) + \alpha M(v_1, \ldots w, \ldots v_n)$$

Example 1: $V = \mathbf{R}^n$, $M(v_1, v_2) = v_1^T A v_2$, $A \in \mathbf{R}^{n \times n}$

Example 2: Let $V = W = \mathbf{L}^\infty$, $h \in \mathbf{B}(\mathbf{R}_+^2)$ and

$$M(u_1, u_2) = \int\int h(\tau_1, \tau_2) u_1(t-\tau_1) u_2(t-\tau_2) d\tau_1 d\tau_2$$

Definition: Let $M: V^n \to W$ be n-linear. Then a map $P: V \to W$ of the form

$$P(v) = M(v, \ldots v)$$

is said to be an n-order *polynomial mapping.*

Example 3: Let $V = W = \mathbf{L}^\infty$, $h \in \mathbf{B}(\mathbf{R}_+^2)$ and

$$P(u) = \int\int h(\tau_1, \tau_2) u(t-\tau_1) u(t-\tau_2) d\tau_1 d\tau_2$$

And in general the nth term of a Volterra series operator is an n-order polynomial mapping in the input u.

Theorem 4.1: An n-order polynomial mapping is homogeneous of degree n, but the converse is not true.

Proof: $P(\alpha v) = M(\alpha v, \ldots \alpha v) = \alpha^n M(v, \ldots v) = \alpha^n P(v)$.

To see that the converse is not in general true, let $V = \mathbf{R}^2$, $W = \mathbf{R}$, and consider

$$F(x_1, x_2) = (|x_1| + |x_2|)^2 = x_1^2 + x_2^2 + 2|x_1 x_2|$$

F is homogeneous of degree two but is not a polynomial mapping, since a second order polynomial mapping satisfies $P(x_1 + x_2) + P(x_1 - x_2) = 2P(x_1) + 2P(x_2)$; F does not.

This distinction between a homogeneous mapping and a polynomial mapping is like the difference between a general norm and a norm which comes from an inner product. To bring the discussion home to engineering consider the nonlinear TI operator N_{homog} given by

$$N_{homog} u(t) = F(u(t), u(t-1))$$

N_{homog} is homogeneous of degree two. We will see later that the response of a second order Volterra series operator to the input $u(t)=\cos t$ has, at most, two components: one at D.C. and one at 2 rad/sec. $N_{homog}(\cos t)$, however, has infinitely many harmonics. □

We need just a few more definitions:

Definition: An n-linear map M is said to be *symmetric* if for any permutation $\sigma \in S^n$

$$M(v_{\sigma 1},...,v_{\sigma n}) = M(v_1,...,v_n)$$

Thus the bilinear map of example 1 is symmetric iff $A=A^T$, and the bilinear map of example 2 is symmetric iff $h(\tau_1,\tau_2)=h(\tau_2,\tau_1)$.

Definition: **SYMM**M is the multilinear mapping defined by

$$\mathbf{SYMM}M(v_1,...,v_n) \triangleq \frac{1}{n!}\sum_{\sigma \in S^n} M(v_{\sigma 1},...,v_{\sigma n})$$

and similarly if h_n is a function or measure, we define

$$\mathbf{SYMM}h_n(\tau_1,...,\tau_n) \triangleq \frac{1}{n!}\sum_{\sigma \in S^n} h_n(\tau_{\sigma 1},...,\tau_{\sigma n})$$

SYMMM derives its importance from:

Theorem 4.2: Suppose the polynomial maps P_1 and P_2 are induced by multilinear maps M_1 and M_2, respectively. Then $P_1=P_2$ iff $\mathbf{SYMM}M_1=\mathbf{SYMM}M_2$.

Thus two bilinear maps of the form of example 1 induce the same polynomial map if and only if $A_1+A_1^T = A_2+A_2^T$.

Proof: First note that **SYMM**M and M always induce the same polynomial map, since

$$\mathbf{SYMM}M(v,...,v) = \frac{1}{n!}\sum_{\sigma \in S^n} M(v,...,v) = M(v,...,v)$$

The "if" part follows. In the next section we will prove more than the "only if" part, so here we will give just an informal sketch of how the "only if" proof goes. The key is the formula

$$\frac{1}{n!}\frac{\partial^n}{\partial\alpha_1...\partial\alpha_n}\bigg|_{\alpha=0} P(\sum_{i=1}^{n}\alpha_i v_i) = \mathbf{SYMM}M(v_1,...,v_n) \tag{4.1}$$

so that $P_1=P_2$ implies $\mathbf{SYMM}M_1=\mathbf{SYMM}M_2$. To "establish" the formula, note that

$$P(\sum_{i=1}^{n}\alpha_i v_i) = \sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}\alpha_{i_1}...\alpha_{i_n}M(v_{i_1},...,v_{i_n})$$

The only terms which contribute to

$$\frac{1}{n!}\frac{\partial^n}{\partial\alpha_1 \ldots \partial\alpha_n}\Big|_{\alpha=0}$$

are the $n!$ terms where the (i_j) are a permutation of $(1,2,\ldots,n)$, and the resulting sum is $\mathbf{SYM}M(v_1,\ldots,v_n)$. Of course we don't know yet that these derivatives exist, but we will see later that if the multilinear operators are bounded, then these derivatives are just Frechet derivatives.

This process of determining $\mathbf{SYM}M$ from the polynomial map P induced by M is known as *polarization*. In fact, we could replace the formula (4.1) above involving partial derivatives with a purely algebraic one; for example for $n=2$ we have the polarization formula

$$\mathbf{SYM}M(u_1,u_2) = P(\frac{u_1+u_2}{2}) - P(\frac{u_1-u_2}{2})$$

We gave the formula (4.1) because it generalizes to whole Volterra Series; the algebraic identities do not.

Let us now assume that V and W are Banach spaces. Then an n-linear map $M: V^n \to W$ is *bounded* if

$$\sup_{\|v_i\| \le 1} \|M(v_1,\ldots,v_n)\| < \infty \tag{4.2}$$

in which case we call the lefthand side of (4.2) the norm of M as a multilinear operator and denote it $\|M\|_{ML}$. The bilinear operator of example 1 is bounded, with $\|M\|_{ML} = \bar{\sigma}(A)$.† The bilinear operator in example 2 is bounded with norm at most $\|h_2\|$.††

We now quickly review derivatives in Banach space.[18,17] Recall that $L(V,W)$ denotes the Banach space of bounded linear maps from V into W, with the operator norm $\|A\| \triangleq \sup\{\, \|Av\| \mid \|v\| \le 1\}$. A map $N: G \to W$, where G is an open subset of V, is said to have a *Frechet* or *strong* derivative $DN(u_0) \in L(V,W)$ at $u_0 \in G$ if

$$\|N(u_0+u) - N(u_0) - DN(u_0)u\| = o(\|u\|)$$

If the map $u_0 \to DN(u_0)$ has a Frechet derivative, we say N has a second Frechet derivative $D^{(2)}N(u_0)$ and it is an element of $L(V, L(V, W))$. Fortunately this space can be identified with

† $\bar{\sigma}(A)$ means the largest singular value of A; here we assume the Euclidean norm on $\mathbb{R}^n$.
†† The actual norm, rather than this upper bound, is hard to compute; see §A3.

$L_2(V,W)$, the space of bounded bilinear maps $:V^2 \to W$, with the norm $\| \ \|_{ML}$ defined above. Similarly the nth Frechet derivative, if it exists, can be thought of as a bounded n-linear map $:V^n \to W$. It can be shown that $D^{(n)}N(u_0)$ is symmetric, e.g. if $D^{(n+1)}N(u_0)$ exists.

We now have all the background material necessary for

5. Relation to Taylor Series; Uniqueness of Volterra Series

We will now see that Volterra series operators *are* Taylor series of operators from some open ball in $\mathbf{L}^\infty$ into $\mathbf{L}^\infty$.

Theorem 5.1 (Frechet Derivatives of Volterra Series Operators):

On B_ρ N has Frechet derivatives of all orders with

$$D^{(k)}N(u_0)(u_1,...,u_k)(t) = \tag{5.1a}$$

$$= \mathbf{SYM} \sum_{n=k}^{\infty} n(n-1)...(n-k+1) \int ... \int h_n(\tau_1,...,\tau_n) \prod_{i=1}^{k} u_i(t-\tau_i)d\tau_i \prod_{i=k+1}^{n} u_0(t-\tau_i)d\tau_i \tag{5.1b}$$

Thus $\|D^{(k)}N(u_0)\| \leq f^{(k)}(\|u_0\|)$ and $(n!)^{-1}D^{(n)}N(0)$ is the n-linear mapping associated with the nth term of the Volterra series and given by:

$$\frac{1}{n!} D^{(n)}N(0)(u_1,...,u_n)(t) = \int ... \int h_n(\tau_1,...,\tau_n) u_1(t-\tau_1)...u_n(t-\tau_n)d\tau_1...d\tau_n \tag{5.1c}$$

Remark: (5.1c) of theorem 5.1 tells us that the Volterra series we have considered so far are in fact Taylor series of operators $:\mathbf{L}^\infty \to \mathbf{L}^\infty$. The reader may wonder whether the Volterra series constitute *all* of the Taylor series of TI nonlinear maps $:\mathbf{L}^\infty \to \mathbf{L}^\infty$. In §A3 we show that this is *not* true, but that the Taylor series left out are not important in engineering.

Proof: Let M_k denote the multilinear map given in (5.1b) above. We will show that

$$N(u_0 + u) - \sum_{k=0}^{n} \frac{1}{k!} M_k(u,...,u) = o(\|u\|^{n+1})$$

which will prove $M_k = D^{(k)}N(u_0)$ as claimed. First note that

$$\|M_k\| \leq \sum_{n=k}^{\infty} n(n-1)...(n-k+1) \|h_n\| \|u_0\|^{n-k} = f^{(k)}(\|u_0\|)$$

Now using the symmetry of $\mathbf{SYM}h_n$

$$N(u_0 + u) = \sum_{n=1}^{\infty} \int \cdots \int \mathbf{SYM} h_n(\tau_1,...,\tau_n) \sum_{k=0}^{n} \binom{n}{k} \prod_{i=1}^{k} u(t-\tau_i)d\tau_i \prod_{i=k+1}^{n} u_0(t-\tau_i)d\tau_i \tag{5.2}$$

For $||u||$ small enough ($||u||+||u_0||<\rho$ will do) the entire righthand side of equation (5.2) is absolutely convergent so we may rewrite it as:

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\substack{n=k\\ n\geq k}} n(n-1)...(n-k+1)\int\cdots\int \mathbf{SYM}h_n(\tau_1,...,\tau_n)\prod_{i=1}^{k}u_i(t-\tau_i)d\tau_i\prod_{i=k+1}^{n}u_0(t-\tau_i)d\tau_i$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}M_k(u,...,u)$$

Thus we have

$$||N(u_0+u)-\sum_{k=0}^{n}\frac{1}{k!}M_k(u,...,u)|| \leq \sum_{k=n+1}^{\infty}\frac{1}{k!}||M_k||\,||u||^k$$

$$\leq \sum_{k=n+1}^{\infty}\frac{1}{k!}f^{(k)}(||u_0||)||u||^k = f(||u_0||+||u||)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(||u_0||)||u||^k$$

which is indeed $o(||u||^{n+1})$. □

Theorem 5.2 (Uniqueness theorem for Volterra series): Suppose N and M are Volterra series operators with kernels h_n and g_n, respectively.

Then $N=M$ iff $\mathbf{SYM}h_n=\mathbf{SYM}g_n$ for all n.

Note that $N=M$ asserts equality of maps from some ball in $\mathbf{L}^\infty$ into $\mathbf{L}^\infty$, whereas the conclusion asserts equality of a sequence of measures.

Proof: The "if" part is clear, (see theorem 4.2). To show the "only if" part we will show that the measures $\mathbf{SYM}h_n$ are determined by the operator N. A measure $\mu\in\mathbf{B}(\mathbf{R}_+^n)$ is determined by its integral over all n-rectangles in $\mathbf{R}^n$, i. e. by the integrals

$$\int\cdots\int \mu(\tau_1,...,\tau_n)u_1(-\tau_1)...u_n(-\tau_n)d\tau_1...d\tau_n \tag{5.3}$$

where each u_i is the characteristic function of an interval. Now by theorem 5.1 we have

$$\int\cdots\int \mathbf{SYM}h_n(\tau_1,...,\tau_n)u_1(-\tau_1)...u_n(-\tau_n)d\tau_1...d\tau_n$$

$$= \frac{1}{n!}D^{(n)}N(u_1,...,u_n)(0)$$

so that N determines the integrals in (5.3) and hence the measure $\mathbf{SYM}h_n$. A more explicit formula for these integrals is:

$$\int\cdots\int \mathbf{SYM}h_n(\tau_1,...,\tau_n)u_1(-\tau_1)...u_n(-\tau_n)d\tau_1...d\tau_n$$

$$= \frac{1}{n!}\frac{\partial^n}{\partial\alpha_1...\partial\alpha_n}\Big|_{\alpha=0}N(\sum_{i=1}^{n}\alpha_i u_i)(0)$$

which is the formula mentioned in the previous section. □

The Uniqueness theorem tells us that we may as well choose our kernels h_n to be symmetric, and from now on we will assume that all kernels are symmetric. Of course other canonical forms are possible and in some cases more convenient. For example the *triangular* kernels satisfy

$$h_{tri\,n}(\tau_1,...,\tau_n) = 0 \qquad \text{unless} \quad 0 \leq \tau_1 \leq \ldots \leq \tau_n$$

and the Volterra series is then

$$Nu(t) = h_0 + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} h_{tri\,n}(\tau_1,...,\tau_n) u(t-\tau_1)...u(t-\tau_n) d\tau_1...d\tau_n$$

These kernels are often convenient in the study of dynamical systems.

One point worth mentioning: the triangle inequality implies

$$||\mathbf{SYM} h_n|| = ||h_{tri\,n}|| \leq ||h_n||$$

Thus using the symmetric (or triangular) kernels can only *decrease* the gain bound function f and hence increase the radius of convergence ρ. In the sequel we will refer to the gain bound function and radius of convergence computed from the symmetric kernels as *the* gain bound function and radius of convergence of N.

6. Final Comments on the Formulation

The formulation we have given is by no means the only possible, even for stable systems. For example, we could interpret the norms on input signals and kernels as $\mathbf{L}^2$ norms, leaving the norm on output signals (i.e. $y = Nu$) an $\mathbf{L}^\infty$ norm. Input signals and kernels would thus be $\mathbf{L}^2$ functions with $||h_n|| \triangleq ||h_n||_2 = (\int ... \int h_n(\tau_1,...,\tau_n)^2 d\tau_1...d\tau_n)^{1/2}$. Then with the exception of the $\mathbf{L}^p$ material of §3 *all the preceding results hold.* This is essentially the *Fock* space framework proposed by deFigueiredo et al.†[21]

7. Applications to Systems Theory: Sum and Product Operator

In the next few sections we apply the ideas of the previous sections to give simple rigorous proofs of some well-known theorems. We show that the sum, pointwise product, and composition

† Our Volterra series with radii exceeding r would be almost all of the Fock space with weights $n!r^n$.

of two Volterra series operators have Volterra series and we bound their gain functions. We proceed to find the condition under which a Volterra series operator has an inverse and compute its kernels. This is applied to show that the I/O operator of a simple dynamical system is given by a Volterra series.

This program of working out the Volterra series of various "system interconnections" was first carried out at MIT in the late 1950's,[12,4] but none of this work is rigorous. This constructive approach is not really a fully modern approach, where one powerful general theorem would prove all these theorems (and more).[1] Unfortunately this one powerful theorem may be so general and abstract that the underlying simplicity of the formulas may be lost. In this section we want to demonstrate two things: First, that supplying the analytical details in the MIT work is relatively straightforward; and second, that the resulting formulas, though complicated, are just simple extensions of the same formulas for ordinary power series. This of course should be expected in view of theorem 5.1.

The notation for the next three sections (§7-§9) is as follows: A and B will denote Volterra series operators with kernels a_n and b_n, gain bound functions f_A and f_B, and radii of convergence ρ_A and ρ_B, respectively. To simplify some of the formulas, we will assume the constant terms (0th order kernels) a_0 and b_0 are zero.

The pointwise product of A and B is defined by

$$[A \cdot B]u(t) = [Au](t)[Bu](t)$$

Definition: if $a \in \mathbf{B}(\mathbf{R}_+^n)$, $b \in \mathbf{B}(\mathbf{R}_+^k)$ then the *symmetric tensor product* $a \vee b \in \mathbf{B}(\mathbf{R}_+^{n+k})$ is defined by:

$$a \vee b(\tau_1,\ldots,\tau_{n+k}) \triangleq \mathbf{SYM}\, a(\tau_1,\ldots,\tau_n) b(\tau_{n+1},\ldots,\tau_{n+k})$$

By the product we mean the normal product measure. (Thus $h(\tau)g(\tau)$ doesn't necessarily make sense, but $h(\tau_1)g(\tau_2)$ does.) Note that

$$\|a \vee b\| = \int\ldots\int |\mathbf{SYM}\, a(\tau_1,\ldots\tau_n) b(\tau_{n+1},\ldots\tau_{n+k})| d\tau_1\ldots d\tau_{n+k}$$
$$\leq \frac{1}{(n+k)!} \sum_{\sigma \in S^{n+k}} \int\ldots\int |a(\tau_{\sigma 1},\ldots,\tau_{\sigma n})||b(\tau_{\sigma n+1},\ldots,\tau_{\sigma n+k})| d\tau_1\ldots d\tau_{n+k}$$

$$= ||a|| \, ||b||$$

Theorem 7.1 (Product Operator): $A \cdot B$ is a Volterra series operator with kernels

$$h_n = \sum_{k=1}^{n-1} a_k \vee b_{n-k}$$

and characteristic gain function $f_{A\,B} \leq f_A f_B$. In particular, $\rho_{A\,B} \geq \min\{\rho_A, \rho_B\}$.

Remark: If we write a Volterra series as a formal sum

$$a_1 + \cdots + a_n + \cdots$$

then we can write the formal symmetric tensor product of $a_1 + \cdots$ and $b_1 + \cdots$ as

$$(a_1 + \cdots) \vee (b_1 + \cdots) = (a_1 \vee b_1) + (a_1 \vee b_2 + a_2 \vee b_1) + \cdots$$

so the Volterra series of $A \cdot B$ is the formal symmetric tensor product of the Volterra series of A and B. Note the similarity with the formula for the coefficients of the product of two power series.

Proof: Let $||u|| < \min\{\rho_A, \rho_B\}$. Then Au and Bu make sense and

$$A \cdot Bu(t) = \left\{ \sum_{m=1}^{\infty} \int \ldots \int a_m(\tau_1, \ldots, \tau_m) \prod_{i=1}^{m} u(t-\tau_i) d\tau_i \right\} \left\{ \sum_{n=1}^{\infty} \int \ldots \int b_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \right\} \quad (7.1a)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int \ldots \int a_m(\tau_1, \ldots \tau_m) b_n(\tau_{m+1}, \ldots \tau_{m+n}) \prod_{i=1}^{m+n} u(t-\tau_i) d\tau_i \quad (7.1b)$$

$$= \sum_{n=1}^{\infty} \int \ldots \int \left\{ \sum_{k=1}^{n-1} a_k \vee b_{n-k} \right\} u(t-\tau_1) \ldots u(t-\tau_n) d\tau_1 \ldots d\tau_n \quad (7.1c)$$

All of the changes in the order of summation and integration in equation (7.1) are justified by the Fubini theorem, since

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int \ldots \int |a_m(\tau_1, \ldots \tau_m) b_n(\tau_{m+1}, \ldots \tau_{m+n}) \prod_{i=1}^{m+n} u(t-\tau_i)| d\tau_i$$

$$\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} ||a_n|| \, ||b_m|| \, ||u||^{m+n} = f_A(||u||) f_B(||u||) < \infty$$

Since equation (7.1) holds for *any* u with $||u|| < \min\{\rho_A, \rho_B\}$, the Uniqueness theorem tells us that $\sum_{k=1}^{n-1} a_k \vee b_{n-k}$ are *the* kernels of $A \cdot B$. Now

$$f_{A\,B}(x) = \sum_{n=1}^{\infty} ||h_n|| x^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} ||a_k \vee b_{n-k}|| x^n$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} ||a_k|| \, ||b_{n-k}|| x^{n-k}$$

$$= \left\{ \sum_{n=1}^{\infty} ||a_n|| x^n \right\} \left\{ \sum_{n=1}^{\infty} ||b_n|| x^n \right\} = f_A(x) f_B(x)$$

The final conclusion $\rho_{A\,B} \geq \min\{\rho_A, \rho_B\}$ follows from $f_{A\,B} \leq f_A f_B$ and $\rho_{A\,B} = \inf\{x | f_{A\,B} = \infty\}$.

□

Theorem 7.2 (Sum Operator): $A+B$ is a Volterra series operator with kernels

$$h_n(\tau_1,...,\tau_n) = a_n(\tau_1,...,\tau_n) + b_n(\tau_1,...,\tau_n)$$

and gain bound function $f_{A+B} \leq f_A + f_B$. Thus $\rho_{A+B} \geq \min\{\rho_A, \rho_B\}$.

The proof is left to the reader.

8. Composition Operator

The composition of A and B, which we denote by the juxtaposition AB, is defined by

$$[AB]u(t) \triangleq A(Bu)(t)$$

To motivate the formula for the kernels of AB, recall that the nth coefficient of the composition of the ordinary power series $\sum_{l=1} a_l x^l$ and $\sum_{l=1} b_l x^l$ is given by

$$\sum_{k=1}^{n} \left\{ \sum_{\substack{i_1,...,i_k \geq 1 \\ i_1+...+i_k=n}} \right\} a_k b_{i_1} \cdots b_{i_k} \tag{8.1}$$

Theorem 8.1 (Composition theorem): AB is a Volterra series operator with kernels

$$h_n(t_1,...,t_n) = \textbf{SYM} \sum_{k=1}^{n} \left\{ \sum_{\substack{i_1,...,i_k \geq 1 \\ i_1+...+i_k=n}} \right\} \int \cdots \int a_k(\tau_1,...,\tau_k) \cdot \tag{8.2a}$$

$$\cdot b_{i_1}(t_1-\tau_1,...t_{i_1}-\tau_1) \cdots b_{i_k}(t_{n-i_k+1}-\tau_k,...t_n-\tau_k) d\tau_1...d\tau_k \tag{8.2b}$$

Moreover $f_{AB}(x) \leq f_A(f_B(x))$. Thus $\rho_{AB} \geq \min\{\rho_B, f_B^{-1}(\rho_A)\}$.

Proof: Let h_n be defined by the formula (8.2) above. First note that

$$||h_n|| \leq \sum_{k=1}^{n} \left\{ \sum_{\substack{i_1,...,i_k \geq 1 \\ i_1+...+i_k=n}} \right\} ||a_k|| ||b_{i_1}|| \cdots ||b_{i_k}|| \tag{8.3}$$

and the righthand side of (8.3) is the nth coefficient of $f_A(f_B(\cdot))$, so $f_H(x) \leq f_A(f_B(x))$. This computation justifies the changes of order of integration and summation in the following.

Suppose $f_A(f_B(||u||)) < \infty$. Then Bu makes sense and $||Bu|| \le f_B(||u||)$ so ABu makes sense and:

$$ABu(t) = \sum_{k=1}^{\infty} \int \cdots \int a_k(\tau_1,\ldots,\tau_k) Bu(t-\tau_1) \cdots Bu(t-\tau_k) d\tau_1 \ldots d\tau_k$$

$$= \sum_{k=1}^{\infty} \int \ldots \int a_k(\tau_1,\ldots,\tau_k) \prod_{i=1}^{k} \left\{ \sum_{m=1}^{\infty} \int \ldots \int b_m(t_1,\ldots t_m) u(t-\tau_i-t_1) \ldots u(t-\tau_i-t_m) dt_1 \ldots dt_m \right\} d\tau_i$$

$$= \sum_{k=1}^{\infty} \int \ldots \int a_k(\tau_1,\ldots,\tau_k) \left\{ \sum_{i_1 \ge 1,\ldots,i_k \ge 1} \right\} b_{i_1}(t_1,\ldots,t_{i_1}-\tau_1) \ldots b_{i_k}(t_{i_1+\ldots+i_{k-1}+1},\ldots,t_{i_1+\ldots+i_k}) \cdot$$

$$\cdot u(t-\tau_1-t_1) \ldots u(t-\tau_1-t_{i_1}) \ldots u(t-\tau_k-t_{i_1+\ldots+i_{k-1}+1}) \ldots u(t-\tau_k-t_{i_1+\ldots+i_k}) dt_1 \ldots dt_{i_1+\ldots+i_k} d\tau_1 \ldots d\tau_k$$

We now collect terms by degree in u to get:

$$= \sum_{n=1}^{\infty} \int \cdots \int \left\{ \sum_{k=1}^{n} \left\{ \sum_{\substack{i_1,\ldots,i_k \ge 1 \\ i_1+\ldots+i_k=n}} \right\} a_k(\tau_1,\ldots,\tau_k) \cdot \right.$$

$$\left. \cdot b_{i_1}(t_1,\ldots,t_{i_1}) \ldots b_{i_k}(t_{n-i_k+1},\ldots,t_n) u(t-\tau_1-t_1) \ldots u(t-\tau_1-t_{i_1}) \ldots u(t-\tau_k-t_{n-i_k+1}) \ldots u(t-\tau_k-t_n) \right\} dt_1 \ldots dt_n \, d\tau_1 \ldots d\tau_k$$

Finally, we change the t_i variables:

$$= \sum_{n=1}^{\infty} \int \cdots \int \left\{ \sum_{k=1}^{n} \left\{ \sum_{\substack{i_1,\ldots,i_k \ge 1 \\ i_1+\ldots+i_k=n}} \right\} \int \ldots \int a_k(\tau_1,\ldots,\tau_k) \cdot \right.$$

$$\left. \cdot b_{i_1}(t_1-\tau_1,\ldots,t_{i_1}-\tau_1) \cdots b_{i_k}(t_{n-i_k+1}-\tau_k,\ldots,t_n-\tau_k) d\tau_1 \ldots d\tau_k \right\} u(t-\tau_1) \ldots u(t-\tau_n) dt_1 \ldots dt_n$$

$$= \sum_{n=1}^{\infty} h_n(\tau_1,\ldots,\tau_n) u(t-\tau_1) \ldots u(t-\tau_n) d\tau_1 \ldots d\tau_n$$

and the uniqueness theorem tells us h_n are the kernels of AB. Equation (8.3) establishes the bound $f_{AB} \le f_A f_B$, and the lower bound on the radius of convergence of AB follows. □

9. Inverses of Volterra Series Operators

We now ask the question: when does a Volterra series operator have a local inverse near 0 given by a Volterra series operator and what are its kernels? Whole papers have been written on this important topic.[19,22] Just like ordinary power series, the condition is just that the first term be invertible.

Theorem 9.1 (Inversion theorem for Volterra Series):

A has a local inverse at 0 if and only if its first kernel a_1 is invertible in $\mathbf{B}(\mathbf{R}_+)$, i.e. there exists a measure $b_1 \in \mathbf{B}(\mathbf{R}_+)$ with $a_1 * b_1 = \delta$.†

Remark: Since the Frechet derivative of A at 0 is given by convolution with a_1 (theorem 5.1), the Inversion theorem can be thought of as a generalized *Inverse function theorem*. We will not pursue this idea further: instead we take a constructive approach.

Proof: To see the "only if" part, suppose A has a local inverse B, that is

$$AB = BA = I \tag{9.1}$$

where I is the identity operator ($I_1 = \delta$, $I_n = 0$ for $n > 0$). Using the composition theorem to compute the first kernel of the operators in equation (9.1) yields

$$a_1 * b_1 = b_1 * a_1 = \delta$$

Thus a_1 is invertible in $\mathbf{B}(\mathbf{R}_+)$.

The proof of the "if" part will proceed as follows: we first construct a right inverse for A under the assumption that the first kernel is just δ. Using this we show that A has a right inverse in the general case, a_1 invertible in $\mathbf{B}(\mathbf{R}_+)$. We finish the proof by showing that the right inverse constructed is in fact also a left inverse for A.

Special Case: Assume for now that $a_1 = \delta$. To motivate what follows, consider an ordinary power series $a(x) \triangleq \sum_{n=1}^{\infty} a_n x^n$ with $a_1 = 1$. Since $a'(0) = 1$, $a(\cdot)$ has an analytic inverse $b(x) \triangleq \sum_{n=1}^{\infty} b_n x^n$ near 0. Using formula (3.3) for the coefficients of the composition $a(b(x)) = x$ yields $b_1 = 1$ and the following recursive formula for b_n:

$$b_n = -\sum_{k=2}^{n} \left\{ \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} \right\} a_k b_{i_1} \cdots b_{i_k} \tag{9.2}$$

Note that since the index k starts at two, the righthand side of (9.2) refers only to $b_1,\ldots,b_{n-1}$. Incidently this process of recursively computing the coefficients of the inverse of an analytic function is known as *reversion* of a power series.[23]

† Since the convolution of measures in $\mathbf{B}(\mathbf{R}_+)$ is commutative, $a_1 * b_1 = \delta$ implies $b_1 * a_1 = \delta$.

We now use the same construction for Volterra series. Let $b_1 = \delta$ and for $n > 1$ define measures $b_n \in \mathbf{B}(\mathbb{R}_+^n)$ recursively by

$$b_n(t_1, \ldots, t_n) = -\mathbf{SYM} \sum_{k=2}^{n} \left\{ \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n}} \right\} \int \cdots \int a_k(\tau_1,\ldots,\tau_k) \cdot \tag{9.3a}$$

$$\cdot b_{i_1}(t_1-\tau_1,\ldots t_{i_1}-\tau_1) \cdots b_{i_k}(t_{n-i_k+1}-\tau_k,\ldots t_n-\tau_k) d\tau_1 \ldots d\tau_k \tag{9.3b}$$

As in (9.2) above this comes directly from the composition formula and $(AB)_n = 0$, $n > 1$. We now have to show that b_n, as defined in (9.3) above, are actually the kernels of a Volterra series operator: we must verify that assumptions (A1) and (A2) hold.

We establish (A1) by induction. First note that $b_1 = \delta \in \mathbf{B}(\mathbb{R}_+)$. Assuming that $b_j \in \mathbf{B}(\mathbb{R}_+^j)$ for $j=1,\ldots,n-1$ (9.3) implies that $b_n \in \mathbf{B}(\mathbb{R}_+^n)$, with

$$||b_n|| \leq \sum_{k=2}^{n} \left\{ \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n}} \right\} ||a_k|| \, ||b_{i_1}|| \cdots ||b_{i_k}|| \tag{9.4}$$

We now establish (A2). Let $g(x) \triangleq 2x - f_A(x)$. Since $g'(0)=1$ (recall that $a_1 = \delta$) g has an analytic inverse $h(x) \triangleq \sum_{n=1}^{\infty} \alpha_n x^n$ near 0. We claim that $f_B(x) \leq h(x)$ and thus $\rho_B \geq \mathrm{Rad}\, h(\cdot)$. The coefficients α_n are given by formula (9.2): $\alpha_1 = 1$ and for $n > 1$

$$\alpha_n = \sum_{k=2}^{n} \left\{ \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k=n}} \right\} ||a_k|| \alpha_{i_1} \cdots \alpha_{i_k} \tag{9.5}$$

By induction we now show

$$||b_n|| \leq \alpha_n \tag{9.6}$$

for all n. (9.6) is true for $n=1$, suppose (9.6) has been established for $n < m$. Then (9.4), (9.5), and the inductive hypothesis establish (9.6) for $n = m$ and hence for all n. Consequently

$$f_B(x) = \sum_{n=1}^{\infty} ||b_n|| x^n \leq \sum_{n=1}^{\infty} \alpha_n x^n = h(x)$$

which proves our claim above that the measures b_n do satisfy assumption (A2) and hence are the kernels of a Volterra series operator which we naturally enough call B. From the formula (9.3) for b_n we conclude

$$AB = I$$

B is thus a right inverse for A. This concludes the proof for the *special case.*

General Case: Suppose now that a_1 is invertible in $\mathbf{B}(\mathbf{R}_+)$. We will use the proof of the *special case* presented above to prove the *general case.* Let $b_1 \in \mathbf{B}(\mathbf{R}_+)$ satisfy $a_1 * b_1 = \delta$. Let A_{lin} be the Volterra series operator with first kernel a_1 and other kernels zero. A_{lin} is invertible, with inverse A_{lin}^{-1} (which has first kernel b_1 and other kernels zero). Consider the operator $A_{lin}^{-1}A$ whose kernels we could easily compute with the composition theorem. Its first kernel is δ, so using the construction above find a local right inverse C to $A_{lin}^{-1}A$. Then $B = CA_{lin}^{-1}$ is the local right inverse of A, since

$$AB = A_{lin}A_{lin}^{-1}ACA_{lin}^{-1} = A_{lin}A_{lin}^{-1} = I \tag{9.7}$$

Our final task is to show that the right inverse B is also a *left* inverse for A. Since the first kernel of B is invertible (indeed it has inverse a_1) we can find a right inverse D *for* B. Then we have

$$A = AI = A(BD) = (AB)D = ID = D \tag{9.8}$$

(9.8) and $BD = I$ shows

$$BA = I$$

which with (9.7) proves that B really is the local inverse of A at 0 and completes the proof of theorem 9.1. □

Remark 1: If $a_1 \in \mathcal{A}$, the subalgebra of $\mathbf{B}(\mathbf{R}_+)$ of those measures lacking singular continuous part, then we have the criterion[16]

$$A \text{ is invertible iff } \inf_{\mathrm{Re}\, s \geq 0} |\hat{a}_1(s)| > 0$$

where $\hat{a}_1(s)$ denotes the Laplace transform of a_1.

Remark 2: It is interesting to note that the Special Case considered above has the interpretation of *unity feedback around a strictly nonlinear operator*, which is an important system-theoretic topic in its own right.

10. Dynamical System Example

To illustrate the theorems of this section we now work an example.

Example: Consider the one-dimensional dynamical system:

$$\dot{x} = f(x) + g(u) \tag{10.1a}$$

$$x(0) = 0 \tag{10.1b}$$

$$y = q(x) \tag{10.1c}$$

Suppose f, g and q are analytic near 0, $f(0)=g(0)=q(0)=0$, and $f'(0)<0$. Then the system is exponentially stable at 0, and for $||u||$ small there is a unique state trajectory x satisfying (10.1).

We will now show that the I/O map $:u \rightarrow y$ is a Volterra series operator.

Proof: We first use a *loop transformation* to reexpress equations (10.1a) and (10.1b) in terms of Volterra series operators. (10.1a) and (10.1b) are equivalent to

$$x = e^{f'(0)t} * (f_{snl}(x) + g(u))$$

where $f_{snl}(x) \triangleq f(x) - f'(0)x$ (the *strictly nonlinear* part of f). (See figure (2)). Let H_{lin} be the Volterra series operator with first kernel $1(\tau)e^{f'(0)\tau}$ and other kernels 0. Let F_{snl}, G, and Q be the memoryless Volterra series operators associated with the functions f_{snl}, g, and q, respectively, e.g. $Q_n(\tau_1,...,\tau_n) = n!^{-1}q^{(n)}(0)\delta(\tau_1)...\delta(\tau_n)$. Then the system equations (10.1) are equivalent to

$$x = H_{lin}(F_{snl}(x) + G(u)) \tag{10.2a}$$

$$y = Q(x) \tag{10.2b}$$

Since H_{lin} is linear

$$(I - H_{lin}F_{snl})x = H_{lin}Gu \tag{10.3}$$

By the sum and composition theorems $(I - H_{lin}F_{snl})$ is a Volterra series operator with first kernel δ. By the inversion theorem $(I - H_{lin}F_{snl})$ has a Volterra series local inverse $(I - H_{lin}F_{snl})^{-1}$ near 0. Since as mentioned above (10.3) has only one solution x when $||u||$ is small, it must be

$$x = (I - F_{snl}H_{lin})^{-1}Gu$$

Thus for $||u||$ small, the output y is given by a Volterra series operator in u:

$$y = Q\,(I - F_{snl}H_{lin})^{-1}Gu$$

A few comments are in order. (10.1) may have *multiple equilibria* when $u = 0$ (for example if $f(x) = -\sin x$), or even a finite escape time for some u's (for example if $f(x) = -x + x^2$). We've

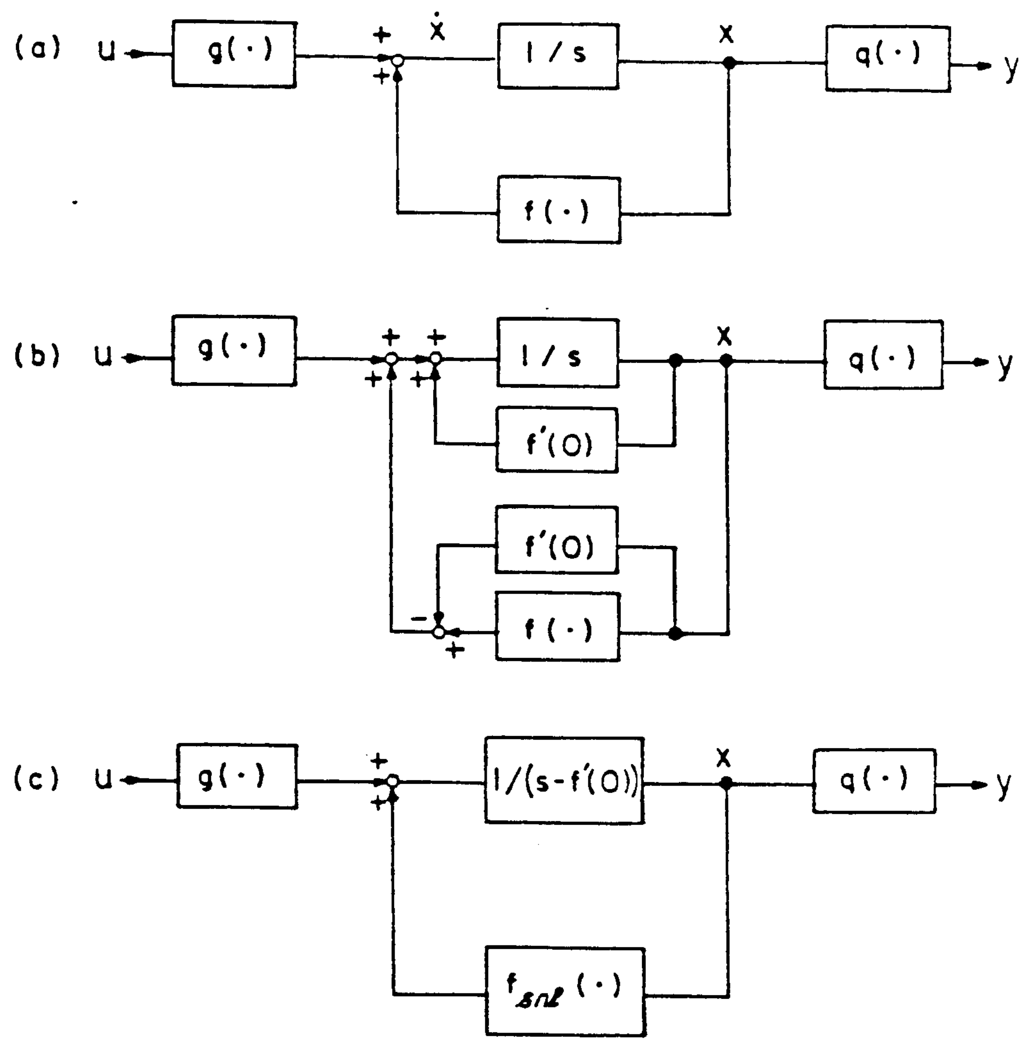

figure 2

shown that as long as $||u||$ is small enough, say less than K, then the state x and the output y are given by a Volterra series in u. In particular $||u|| < K$ must keep the state x from leaving the domain of attraction of 0, for otherwise the Steady State theorem (see §1 chapter 3) or the Gain Bound theorem would be violated.

Chapter 3

Frequency Domain Topics

In this chapter we consider frequency domain topics, concentrating on the simplest case: periodic inputs. As with linear systems, the frequency domain Volterra kernels can be given a simple interpretation in terms of the steady state response to a periodic input. To put this notion of steady-state response on sound footing, we start by proving the steady-state theorem for Volterra series operators. We then establish the validity of a general formula for the spectrum of the output in terms of the spectrum of a periodic or almost periodic input. We present two applications of the material in this and the previous chapter.

1. The Steady State Theorem

Theorem 1.1 (Steady state theorem): Let u and u_s be any signals with $||u||$, $||u_s|| < \rho = \mathrm{Rad}N$, and suppose that $u(t) \rightarrow u_s(t)$ as $t \rightarrow \infty$.

Then $Nu(t) \rightarrow Nu_s(t)$ as $t \rightarrow \infty$.

Remark: This is a very different concept from N's being continuous as a map from $\mathbf{L}^\infty \rightarrow \mathbf{L}^\infty$, which tells us that if $u_n \rightarrow u$ *uniformly* as $n \rightarrow \infty$, then $Nu_n \rightarrow Nu$ (uniformly). In engineering terms, continuity means that if the peak difference of two input signals is small enough, then the peak difference of the corresponding outputs is small, whereas the steady-state theorem asserts that if two input signals approach each other as time evolves, then the corresponding outputs approach each other as time evolves. Indeed we will see in chapter 4 that there are simple TI continuous operators $:\mathbf{L}^\infty \rightarrow \mathbf{L}^\infty$ for which the steady state theorem does not hold.

In fact the steady state theorem is implied by a stronger property of Volterra series operators, *fading memory*, which we will study in chapter 4. But the steady state theorem will suffice to put the notion "periodic steady state" on a sound footing.

Proof: Suppose $||u||$, $||u_s|| < \rho$ and $u(t) \rightarrow u_s(t)$ as $t \rightarrow \infty$. Let $v = u_s - u$ so $v(t) \rightarrow 0$ as $t \rightarrow \infty$. The proof is a modification of the proof of the incremental gain theorem, we simply break the estimate into two parts, one due to the recent past only. For $T \geq 0$

$$(Nu_s - Nu)(t) = (N(u+v) - Nu)(t) = I_1(t,T) + I_2(t,T)$$

where

$$I_1(t,T) \triangleq \sum_{n=1}^{\infty} \int_{[0,T]^n} \cdots \int h_n(\tau_1,\ldots,\tau_n)\left\{\prod_{i=1}^{n} u_s(t-\tau_i) - \prod_{i=1}^{n} u(t-\tau_i)\right\} d\tau_1 \ldots d\tau_n$$

$$I_2(t,T) \triangleq \sum_{n=1}^{\infty} \int_{\mathbb{R}^n - [0,T]^n} \cdots \int h_n(\tau_1,\ldots,\tau_n)\left\{\prod_{i=1}^{n} u_s(t-\tau_i) - \prod_{i=1}^{n} u(t-\tau_i)\right\} d\tau_1 \ldots d\tau_n$$

We now estimate I_1 and I_2 separately.

$$I_1(t,T) = \sum_{n=1}^{\infty} \int_{[0,T]^n} \cdots \int h_n(\tau_1,\ldots,\tau_n)\left\{\prod_{i=1}^{n} (u+v)(t-\tau_i) - \prod_{i=1}^{n} u(t-\tau_i)\right\} d\tau_1 \ldots d\tau_n$$

$$= \sum_{n=1}^{\infty} \int_{[0,T]^n} \cdots \int h_n(\tau_1,\ldots,\tau_n) \sum_{k=1}^{n} \binom{n}{k} \prod_{i=1}^{k} v(t-\tau_i)\, d\tau_i \prod_{i=k+1}^{n} u(t-\tau_i)\, d\tau_i$$

using the symmetry of h_n. Thus

$$|I_1(t,T)| \le \sum_{n=1}^{\infty} ||h_n|| \sum_{k=1}^{n} \binom{n}{k} ||v||_{[t-T,t]}^{k} ||u||^{n-k} \tag{1.1a}$$

$$= f(||u|| + ||v||_{[t-T,t]}) - f(||u||) \tag{1.1b}$$

where $||v||_{[t-T,t]}$ means $\sup\{\,|v(\tau)| \mid t-T \le \tau \le t\,\}$ and f is the gain bound function of N. Note that (1.1b) may be ∞ for some t,T. But as $t-T\to\infty$, $||v||_{[t-T,t]}\to 0$ so (1.1b) eventually becomes and stays finite and in fact converges to zero. Thus we conclude:

$$I_1(t,T) \to 0 \qquad \text{as} \qquad t-T\to\infty \tag{1.2}$$

Now we estimate I_2:

$$|I_2(t,T)| \le \sum_{n=1}^{\infty} ||h_n||_{\mathbb{R}^n - [0,T]^n} (||u_s||^n + ||u||^n) \tag{1.3}$$

where

$$||h_n||_{\mathbb{R}^n - [0,T]^n} = \int_{\mathbb{R}^n - [0,T]^n} \cdots \int |h_n(\tau_1,\ldots,\tau_n)|\, d\tau_1 \ldots d\tau_n \tag{1.4}$$

For each n (1.4) decreases to zero as T increases to ∞, since each h_n is a bounded measure. Hence each term in the sum in (1.3) decreases to zero as T increases to ∞. The righthand side of (1.3) is always less than $f(||u_s||) + f(||u||)$, so the dominated convergence theorem tells us that the righthand side of (1.3), and hence $I_2(t,T)$, converges to zero as $T\to\infty$.

If we now set $T=t/2$ then as $t\to\infty$ both T and $t-T$ increase to ∞. Hence as $t\to\infty$ $Nu_s(t) - Nu(t) = I_1(t,t/2) + I_2(t,t/2) \to 0$. □

Remark: unlike linear systems, the rate of convergence can depend on the *amplitude* of the input. For example, consider N given by

$$Nu = \sum_{k=1}^{\infty} u(t-k)^k$$

N has radius of convergence one. Now consider step inputs of amplitude α, $0<\alpha<1$. As α increases to one, the time to convergence to within, say, 1% of the steady state grows like $(1-\alpha)^{-1}$. For *linear systems* the time to convergence is independent of the amplitude of the input.

Although in the steady state theorem u_s can be any signal with $||u_s|| < \rho$, usually u_s has the interpretation of a *steady state* input, for example in

Theorem 1.2 (Periodic steady state theorem): If the input u is periodic with period T for $t \geq 0$ then the output Nu approaches a steady state, also periodic with period T. (Soon we'll compute the Fourier Series coefficients of Nu).

Proof: Let u_s be u extended periodically to $t=-\infty$. Clearly $u(t) \rightarrow u_s(t)$ as $t \rightarrow \infty$ (indeed $u(t)=u_s(t)$ for $t \geq 0$) so by the steady state theorem $Nu(t) \rightarrow Nu_s(t)$ as $t \rightarrow \infty$. Nu_s is periodic with period T since

$$(Nu_s(\cdot))(t+T) = N(u_s(\cdot + T))(t) = Nu_s(t)$$

where the first equality is due to the time-invariance of N and the second equality is due to the T-periodicity of u_s. □

Note in particular that Volterra series operators *cannot* generate *subharmonics*. A related application of the steady state theorem is:

Theorem 1.3 (Almost periodic steady state theorem): If the input u is *almost periodic* for $t \geq 0$ then the output approaches an almost periodic steady state. (we'll compute the frequencies and spectral amplitudes of the output in §A5).

Proof: The hypothesis simply means that *there is* some u_s which is almost periodic and agrees with u for $t \geq 0$. By the steady state theorem we know $y(t) \rightarrow y_s(t) \triangleq Nu_s(t)$ so we need only show that Volterra series operators take almost periodic inputs into almost periodic outputs. The proof of this, as well as the formula for the spectral amplitudes of the output, are in §A5. This last topic has been studied by Sandberg.[7]

2. Frequency Domain Volterra Kernels

As with linear systems, it is often convenient to use the Laplace transforms of the kernels, defined by

$$H_n(s_1,\ldots,s_n) = \int \cdots \int h_n(\tau_1,\ldots,\tau_n) e^{-(s_1\tau_1+\ldots+s_n\tau_n)} d\tau_1 \ldots d\tau_n$$

We call H_n the nth *frequency domain kernel* or just kernel of the operator N. Since $h_n \in \mathbf{B}(\mathbf{R}_+^n)$, H_n is defined *at least* in $\overline{\mathbf{C}_+^n}$ ($\mathbf{C}_+^n$ means $\{ \mathbf{s} \mid \mathrm{Re}\, s_k > 0 \}$). H_n is symmetric, bounded and uniformly continuous there; it is analytic in $\mathbf{C}_+^n$. We should mention that the *unicity theorem* for Laplace transforms tells us that two measures in $\mathbf{B}(\mathbf{R}_+^n)$ are equal $(h_n = g_n)$ if and only if their Laplace transforms are equal $(H_n = G_n)$.

The formulas of §7 and §8 of chapter 2 are somewhat simpler in the frequency domain. Using the notational convention that C_n denotes the nth frequency domain kernel of a Volterra series operator C, we have:

Theorem 2.1: Suppose A and B are Volterra series operators. Then the frequency domain kernels of $A+B$, $A \cdot B$ (pointwise product), and AB (composition) are given by:

$$(A+B)_n(s_1,\ldots,s_n) = A_n(s_1,\ldots,s_n) + B_n(s_1,\ldots,s_n)$$

$$(A \cdot B)_n = \sum_{k=1}^{n-1} A_k \vee B_{n-k} \triangleq \mathbf{SYM} \sum_{k=1}^{n} A_k(s_1,\ldots,s_k) B_{n-k}(s_{k+1},\ldots,s_n)$$

$$(AB)_n(s_1,\ldots,s_n) = \mathbf{SYM} \sum_{k=1}^{n} \left\{ \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\ldots+i_k = n}} \right\} A_k(s_1+\ldots+s_{i_1},\ldots,s_{n-i_k+1}+\ldots+s_n) \cdot$$

$$\cdot B_{i_1}(s_1,\ldots,s_{i_1}) \cdots B_{i_k}(s_{n-i_k+1},\ldots,s_n)$$

These well-known formulas follow easily from the formulas of §7 and §8 of chapter 2.

3. Multitone Inputs; the Fundamental Frequency Domain Formula

We start with a simple calculation. Suppose that $u(t)$ is a (real) *trigonometric polynomial,* that is

$$u(t) = \sum_{k=-M}^{M} \alpha_k e^{j\omega k t}$$

where $\alpha_{-k} = \bar{\alpha}_k$. Suppose also that $||u|| < \rho = \mathrm{Rad} N$. Then

$$y(t) = Nu(t) = \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1,\ldots,\tau_n) \prod_{i=1}^{n} \sum_{k=-M}^{M} \alpha_k e^{j\omega(t-\tau_i)} d\tau_i \tag{3.1a}$$

$$= \sum_{n=1}^{\infty} \left\{ \sum_{-M \le k_1,\ldots,k_n \le M} \right\} \alpha_{k_1} \ldots \alpha_{k_n} H_n(j\omega k_1, \ldots, j\omega k_n) e^{j(\omega k_1 + \cdots + \omega k_n)t} \tag{3.1b}$$

The term $\alpha_{k_1} \ldots \alpha_{k_n} H_n(j\omega k_1, \ldots, j\omega k_n) e^{j(\omega k_1 + \cdots + \omega k_n)t}$ is often called an nth order $(\omega k_1,\ldots,\omega k_n)$ intermodulation product. Since it is proportional to $H_n(j\omega k_1, \ldots, j\omega k_n)$ this suggests the interpretation of $H_n(j\omega k_1, \ldots, j\omega k_n)$ as a *measure* of the $(\omega k_1,\ldots,\omega k_n)$ intermodulation distortion of N.

Now we've already seen that the *first* sum in (3.1b) is an l^1 sum, i.e. absolutely convergent. In fact for each t,

$$\sum_{n=1}^{\infty} \left| \left\{ \sum_{-M \le k_1,\ldots,k_n \le M} \right\} \alpha_{k_1} \ldots \alpha_{k_n} H_n(j\omega k_1, \ldots, j\omega k_n) e^{j(\omega k_1 + \cdots + \omega k_n)t} \right| \le f(||u||)$$

where f is the gain bound function of N. Consequently we may evaluate the mth Fourier coefficient of y

$$\hat{y}(m) \triangleq \frac{\omega}{2\pi} \int_0^{2\pi\omega^{-1}} y(t) e^{-j\omega m t} dt$$

inside the *first* sum in (3.1b) as:

$$\hat{y}(m) = \sum_{n=1}^{\infty} \frac{\omega}{2\pi} \int_0^{2\pi\omega^{-1}} \left\{ \sum_{-M \le k_1,\ldots,k_n \le M} \right\} \alpha_{k_1} \ldots \alpha_{k_n} H_n(j\omega k_1, \ldots, j\omega k_n) e^{j(\omega k_1 + \cdots + \omega k_n)t} e^{-j\omega m t} dt$$

Each integral is easily evaluated (the integrands are *trigonometric polynomials*) yielding

$$\hat{y}(m) = \sum_{n=1}^{\infty} \left\{ \sum_{k_1 + \cdots + k_n = m} \right\} \hat{u}(k_1) \ldots \hat{u}(k_n) H_n(j\omega k_1, \ldots, j\omega k_n) \tag{3.2}$$

since $\hat{u}(k) = \alpha_k$ for $|k| \le M$ and 0 for $|k| > M$ (and thus the inner sum in (3.2) is *finite*). We call (3.2) the *fundamental frequency domain formula* since it expresses the output spectrum in terms of the input spectrum. Of course we've only established it for inputs which are trigonometric polynomials, but we will see that it is true for more general periodic inputs, and an analogous formula holds for *almost periodic* inputs as well (see § A5).

Remark: Suppose a trigonometric polynomial signal u is passed through a unit nth power law device so that $y(t) = u(t)^n$. Then

$$\hat{y}(m) = \hat{u}^{*n}(m) = \left\{ \sum_{k_1 + \cdots + k_n = m} \right\} \hat{u}(k_1) \ldots \hat{u}(k_n)$$

where $\hat{u}^{*n}$ means the n-fold convolution $\hat{u} * \hat{u} * \ldots * \hat{u}$ (the sum in the convolution is finite here!). The first equality makes sense: it is just the *dual* of the correspondence between *convolution in the time domain* and *multiplication in the frequency domain*. The second equality makes the fundamental formula (3.2) seem quite natural; the nth term of (3.2) can be thought of as an n-fold convolution power of $\hat{u}$, weighted by $H_n(j\omega k_1, \ldots, j\omega k_n)$.

Before establishing the fundamental formula for more general periodic inputs, we have to carefully examine the question of whether it even *makes sense* for more general periodic inputs. Despite its resemblance to the composition formula and the fact that every sum and integral encountered so far has converged absolutely, we have:

4. Fundamental Formula Doesn't Converge Absolutely

Remarkably the fundamental formula is *not* absolutely convergent even for u as simple as a two-tone input signal! That is

$$\sum_{n=1}^{\infty} \left\{ \sum_{k_1 + \cdots + k_n = m} \right\} | \hat{u}(k_1) \ldots \hat{u}(k_n) H_n(j\omega k_1, \ldots, j\omega k_n) |$$

can equal ∞ even in the case considered above, u a trigonometric polynomial (but our calculation was correct).

Remark: Practically, this means that we cannot arbitrarily rearrange the terms in the sum above. We must first perform the inner (bracketed) sum (which in this case is a finite sum), and then perform the outer sum over n.

Example: Let $u = \frac{5}{9}(\cos t + \sin 2t)$. Then $||u||$ can be shown to be

$$\frac{5}{9} \left[\frac{15 + \sqrt{33}}{32} \right]^{1/2} \left[\frac{3 + \sqrt{33}}{4} \right]$$

which is about $0.978 < 1$. Let N be the memoryless operator with $H_n = 1$ for all n, that is

$y(t)=u(t)/(1-u(t))$. Then $\rho=1$ so $y(t)$ makes sense and satisfies $||y|| \leq ||u||/(1-||u||)$ (which is about 45). According to the fundamental formula (3.2) of the last subsection, the D.C. term of y is given by:

$$\hat{y}(0) = \sum_{n=1}^{\infty}\left\{\sum_{k_1+\cdots+k_n=0}\right\}\hat{u}(k_1)\ldots\hat{u}(k_n) \tag{4.1}$$

Now we claim that (4.1) does not converge absolutely. To see this,

$$\sum_{n=1}^{\infty}\left\{\sum_{k_1+\cdots+k_n=0}\right\}|\hat{u}(k_1)\ldots\hat{u}(k_n)| \geq \sum_{n\ even}\left\{\sum_{k_1+\cdots+k_n=0}\right\}|\hat{u}(k_1)\ldots\hat{u}(k_n)|$$

$$= \sum_{n\ even}\left\{\sum_{k_1+\cdots+k_n=0}\right\}\hat{v}(k_1)\ldots\hat{v}(k_n)$$

where $v(t) \triangleq \frac{5}{9}(\cos t + \cos 2t)$ so that $|\hat{u}(k)| = \hat{v}(k)$ for all k.

$$= \sum_{n\ even}\frac{1}{2\pi}\int_0^{2\pi} v(t)^n\, dt \geq \sum_{n\ even}\frac{1}{4\pi} = \infty$$

since $v(t) \geq 1$ for $-.25 \leq t \leq .25$. Thus the fundamental formula isn't absolutely convergent in this simple case.

It is surprising that the trouble in (3.2) occurs when the input is a simple trigonometric polynomial signal; we might expect it to give us trouble only when, say, u does not have an absolutely convergent Fourier series.

There is one obvious but rare case in which (3.2) does converge absolutely. Suppose $\hat{u} \in \mathbf{l}^1$, i.e. u has an absolutely convergent Fourier series, and in addition $f(||\hat{u}||_1) < \infty$. Then $|\hat{u}| \in \mathbf{l}^1$ and $|\hat{u}|^{*n} \in \mathbf{l}^1$ with $|||\hat{u}|^{*n}||_1 \leq ||\hat{u}||_1^n$, thus we have the estimate

$$\sum_{n=1}^{\infty}\left\{\sum_{k_1+\cdots+k_n=m}\right\}|\hat{u}(k_1)\ldots\hat{u}(k_n)H_n(j\omega k_1, \ldots, j\omega k_n)| \leq \sum_{n=1}^{\infty}||\hat{u}||_1^n||h_n|| = f(||\hat{u}||_1)$$

In conclusion, then, we must proceed with extreme care in establishing the fundamental formula for more general periodic input signals.

5. Proof of Fundamental Formula for General Inputs

We start with some calculations. Suppose u is any periodic input with $||u|| < \rho$. Recall that the Mth *Cesaro* sum of the Fourier series of u is defined by

$$u_M(t) = \sum_{k=-M}^{M} (1 - \frac{|k|}{M}) \hat{u}(k) e^{j\omega kt} \triangleq \sum_k C_M(k) \hat{u}(k) e^{j\omega kt}$$

u_M is u convolved with an approximate identity and thus satisfies $||u_M|| \leq ||u||$ and $||u_M - u||_1 \to 0$ as $M \to \infty$.†[24] From the first fact we conclude that Nu_M makes sense since $||u_M|| \leq ||u|| < \rho = \mathrm{Rad}N$. Using the Incremental Gain theorem for $\mathbf{L}^1[0,2\pi\omega^{-1}]$ we conclude that $||Nu - Nu_M||_1 \to 0$ as $M \to \infty$. Hence $(Nu_M)\hat{} = \hat{y}_M$ converges uniformly to $\hat{y}$ as $M \to \infty$. u_M is a trigonometric polynomial so we know the fundamental formula holds for Nu_M; putting all this together we have shown

$$\hat{y}(m) = \lim_{M\to\infty} \sum_{n=1}^{\infty} \left\{ \sum_{k_1 + \cdots + k_n = m} \right\} \hat{u_M}(k_1) \ldots \hat{u_M}(k_n) H_n(j\omega k_1, \ldots, j\omega k_n) \tag{5.1}$$

$$= \sum_{n=1}^{\infty} \lim_{M\to\infty} \left\{ \sum_{k_1 + \cdots + k_n = m} \right\} \hat{u}(k_1) \ldots \hat{u}(k_n) \prod_{i=1}^{n} C_M(k_i) H_n(j\omega k_1, \ldots, j\omega k_n) \tag{5.2}$$

The dominated convergence theorem justifies the interchange of limit and sum in (5.1) since as we have mentioned before the first sum in (5.1) and (5.2) is always absolutely convergent and $|C_M| \leq 1$. Since $\lim_{M\to\infty} C_M(k) = 1$ for each k, if we knew that the inner sum also converged absolutely we could apply dominated convergence once again to conclude

$$= \sum_{n=1}^{\infty} \left\{ \sum_{k_1 + \cdots + k_n = m} \right\} \hat{u}(k_1) \ldots \hat{u}(k_n) H_n(j\omega k_1, \ldots, j\omega k_n) \tag{5.3}$$

which would establish the fundamental formula in the general case.

Unfortunately the inner sum

$$\left\{ \sum_{k_1 + \cdots + k_n = m} \right\} \hat{u}(k_1) \ldots \hat{u}(k_n) H_n(j\omega k_1, \ldots, j\omega k_n) \tag{5.4}$$

is *not* always absolutely convergent (and thus does not always make sense). In such a case formula (5.2) is as close as we can get to the fundamental formula. But in fact the inner sum *is* absolutely convergent in almost all situations arising in engineering. We now present two conditions which suffice:

---

† For a T-periodic u, $||u||_1$ means $T^{-1}\int_0^T |u(\tau)| d\tau$.

Lemma 5.1: Suppose u has bounded variation over one period. Then

$$\left\{ \sum_{k_1 + \cdots + k_n = m} \right\} |\hat{u}(k_1)...\hat{u}(k_n)| < \infty$$

In particular, the inner sum (5.4) in the fundamental formula converges absolutely. The proof is given in §A1.

Lemma 5.2: Suppose that $H_n(j\omega k_1, \ldots, j\omega k_n) = O(\frac{1}{k_1...k_n})$. Then the inner sum (5.4) is absolutely convergent.

Remark: This condition can be interpreted as: N is *strictly proper*. For example the kernels of the input/output operator of a dynamical system with vector field affine in the input have this property.

The proof is in §A4. We summarize the results of this section in

Theorem 5.3 (Fundamental frequency domain formula): Suppose $\|u\| < \rho$ and that either

(I) the input u has *bounded variation over one period*, or

(II) the operator N is *strictly proper*, that is, $H_n(j\omega k_1, \ldots, j\omega k_n) = O(\frac{1}{k_1...k_n})$.

Then the fundamental frequency domain formula is valid, that is:

$$\hat{y}(m) = \sum_{n=1}^{\infty} \left\{ \sum_{k_1 + \cdots + k_n = m} \right\} \hat{u}(k_1)...\hat{u}(k_n) H_n(j\omega k_1, \ldots, j\omega k_n)$$

Proof: Theorem 5.3 follows from the discussion at the beginning of this section and the lemmas above. □

6. Application: Uniqueness of a Basic Nonlinear Structure

In this section and the next we briefly present two applications of the preceding material. The first is motivated by a question concerning modeling of nonlinear circuit elements, and the second concerns the linearization of nonlinear systems.

Consider the problem of modeling a capacitor. If the capacitor is *linear* it can be described by $q = Cv + q_0$ or $i = C\dot{v}$, which are equivalent. But suppose now the capacitor is *not* linear.

We might model it as a voltage-controlled capacitor:

$$q = f(v) \tag{6.1}$$

where f is a function from $\mathbf{R}$ into $\mathbf{R}$.

Alternatively we might use the model:

$$i = g(\dot{v}) \tag{6.2}$$

where $g:\mathbf{R}\rightarrow\mathbf{R}$. We now make the following observation: (6.1) and (6.2) are *never* equivalent. To see this, differentiate (6.1) to get $f'(v)\dot{v} = g(\dot{v})$, and consider the driving voltage $v(t) = at$. Then $f'(at)a = g(a)$ for all t and a and hence $f'(x) = g(1)$ and $g(x) = g(1)x$ for all x, contradicting our assumption that the capacitor is not linear.

For the two capacitor models we have the block diagrams shown in figure 1.

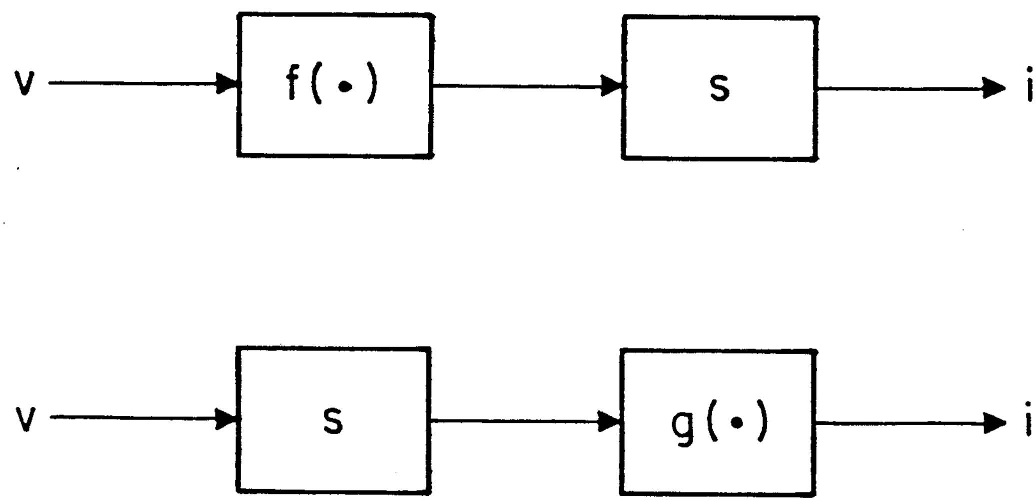

figure 1

Our argument above showed that unless g is linear (and f is affine) the two nonlinear operators shown in figure 1 are never equal. We might say that the two *structures* shown in figure 1 yield different operators.

Thus for a capacitor which is not linear, at least one of the models (6.1) or (6.2) is *incorrect*, which is in sharp contrast to the case of a linear capacitor, where both models are correct. This example is a little unrealistic, since device physics would normally tell us which model is correct (probably (6.1)), but it demonstrates the basic idea that the two structures shown in figure 1 are very different when the memoryless blocks are not linear. We will now establish a very strong generalization of this idea.

Motivated by the example above, we consider what is perhaps the simplest interconnection of LTI and memoryless operators, shown in figure 2, and ask the question: under what conditions could two such systems have the same I/O map?

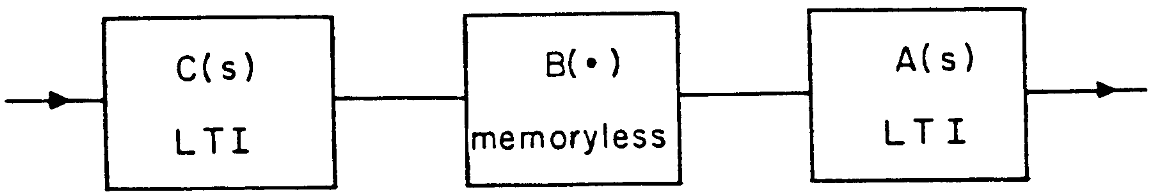

figure 2

Some conditions are easy to think of, for example we can rescale the operators or distribute any delay in A and C arbitrarily between them. Thus if $\tilde{A}=\alpha\exp(-sT)A$, $\tilde{C}=\gamma\exp(sT)C$, and $\tilde{B}(x)=\alpha^{-1}B(\gamma^{-1}x)$, then $ABC=\tilde{A}\tilde{B}\tilde{C}$ (see figure 3).

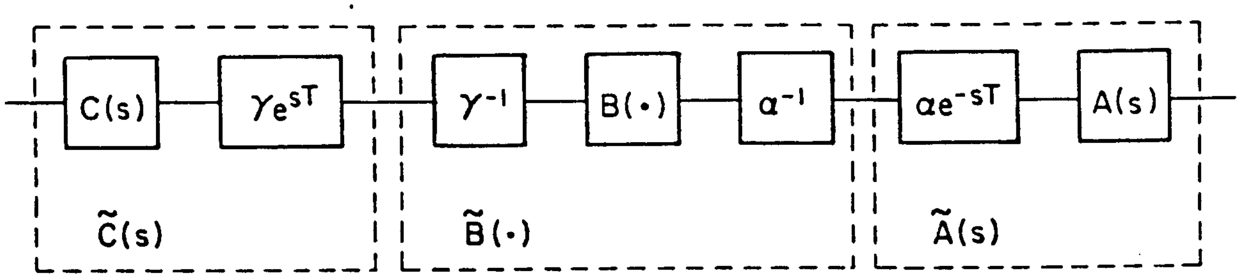

figure 3

We will now show that these are the *only* ways two such systems could have the same I/O operator.

Theorem 6.1 (Sandwich structure uniqueness theorem): Suppose A, $\tilde{A}$, C, $\tilde{C}$ are nonzero LTI operators, B and $\tilde{B}$ are memoryless operators, at least one of which is not linear. If $ABC=\tilde{A}\tilde{B}\tilde{C}$, then there are real constants α, γ, T such that:

$$\widetilde{A}(s)=\alpha\exp(-sT)A(s) \qquad \widetilde{C}(s)=\gamma\exp(sT)C(s)$$
$$\widetilde{B}(x)=\alpha^{-1}B(\gamma^{-1}x)$$

That is, systems of the form in figure 2 which are not linear have a unique representation of the form in figure 2, modulo scaling and delays.

Proof: Under the hypotheses of the theorem, the two systems have the same kernels

$$H_n(s_1,...,s_n) = A(s_1+\cdots+s_n)B_n C(s_1)C(s_2)\cdots C(s_n) \tag{6.3a}$$
$$= \widetilde{A}(s_1+\cdots+s_n)\widetilde{B}_n\widetilde{C}(s_1)\widetilde{C}(s_2)\cdots\widetilde{C}(s_n) \tag{6.3b}$$

Consider now any $n>1$ for which H_n is not identically zero (and there is at least one such n). Find an open ball D in $\mathbb{C}_+^n$ on which $H_n\neq 0$. On D define $Q=$

$$= \log\left[B_n(C/\widetilde{C})(s_1)\cdots(C/\widetilde{C})(s_n)\right] \tag{6.4a}$$
$$= \log\left[\widetilde{B}_n(\widetilde{A}/A)(s_1+\cdots+s_n)\right] \tag{6.4b}$$

Any branch of log will do. Then on D,

$$\frac{\partial^2 Q}{\partial s_1 \partial s_2} = 0 \tag{6.5}$$

when calculated from (6.4a) and

$$\frac{\partial^2 Q}{\partial s_1 \partial s_2} = [\log(\widetilde{A}/A)]''(s_1+\cdots+s_n) \tag{6.6}$$

when calculated from (6.4b). Note that $n>1$ is *crucial;* this is where the requirement that at least one of B or $\widetilde{B}$ be not linear enters. From (6.5) and (6.6) we conclude for some η and T,

$$\log(\widetilde{A}/A)(s_1+\cdots+s_n) = \eta - T(s_1+\cdots+s_n)$$

on D and hence *everywhere* in $\mathbb{C}_+^n$. Thus

$$\widetilde{A}(s) = \alpha\exp(-sT)A(s) \tag{6.7}$$

for $s\in\mathbb{C}_+$, where $\alpha=\exp\eta$. From $A(\overline{s})=\overline{A(s)}$ we conclude α and T are real. Substituting (6.7) into (6.3a) and (6.3b) yields

$$\widetilde{C}(s) = \gamma\exp(sT)C(s) \tag{6.8}$$

where $\gamma^n=B_n\widetilde{B}_n^{-1}\alpha^{-1}$ and as above γ real. Thus we have $\widetilde{B}_n=\alpha^{-1}B_n\gamma^{-n}$, which remains true for those n for which $B_n=\widetilde{B}_n=0$, hence

$$\widetilde{B}(x) = \alpha^{-1}B(\gamma^{-1}x) \tag{6.9}$$

and the theorem is proved. □

Actually the theorem is true under more general conditions than we have shown, for example if $A(s)=s$, which is not a LTI Volterra series operator in the sense of this thesis (see Boyd and Chua[25]). We have also established a more general result which applies to nonlinear systems and networks which contain one memoryless nonlinearity (as in figure 2), but possibly in a feedback loop.[26] We give a few interesting corollaries of the sandwich structure uniqueness theorem:

Corollary 6.2: Systems of the form HN are *completely disjoint* from systems of the form NH, where H is LTI *nonconstant* and N is memoryless and *not* linear.

Corollary 6.3: Given any operator N with at least two nonzero kernels, the only LTI operators which commute with N are delays (or delays and negation, if N is odd).

Corollary 6.4: Chua has defined *algebraic* circuit elements as those with constitutive relations of the form $\Phi(v^{(\alpha)}, i^{(\beta)})=0$ (where $f^{(\alpha)}$ is the αth derivative, or integral if $\alpha<0$, of f).[27] Nonlinear resistors, capacitors, and inductors are examples. Under weak conditions our theorem shows that if such an element is *not linear* its order (α, β) and its characteristic curve $\Phi(x,y)=0$ are *unique*, that is, such elements have only one description as algebraic elements. This explains our capacitor modeling problem above.

Our theorem has other simple applications, for example

Application: Consider a communications system consisting of N cable-repeater sections, each with frequency response $R(s)$. Suppose the output stage of the kth repeater drifts off bias and starts distorting slightly. The faulted system I/O operator is then $R^{N-k}f(\cdot)R^k$, where $f(\cdot)$ represents the nonlinear output stage: see figure 4. The sandwich structure uniqueness theorem tells us that *from I/O measurements alone* (of the whole system) we can locate the faulty repeater.

This should be compared to a linear fault: suppose an element in the kth repeater amplifier drifts

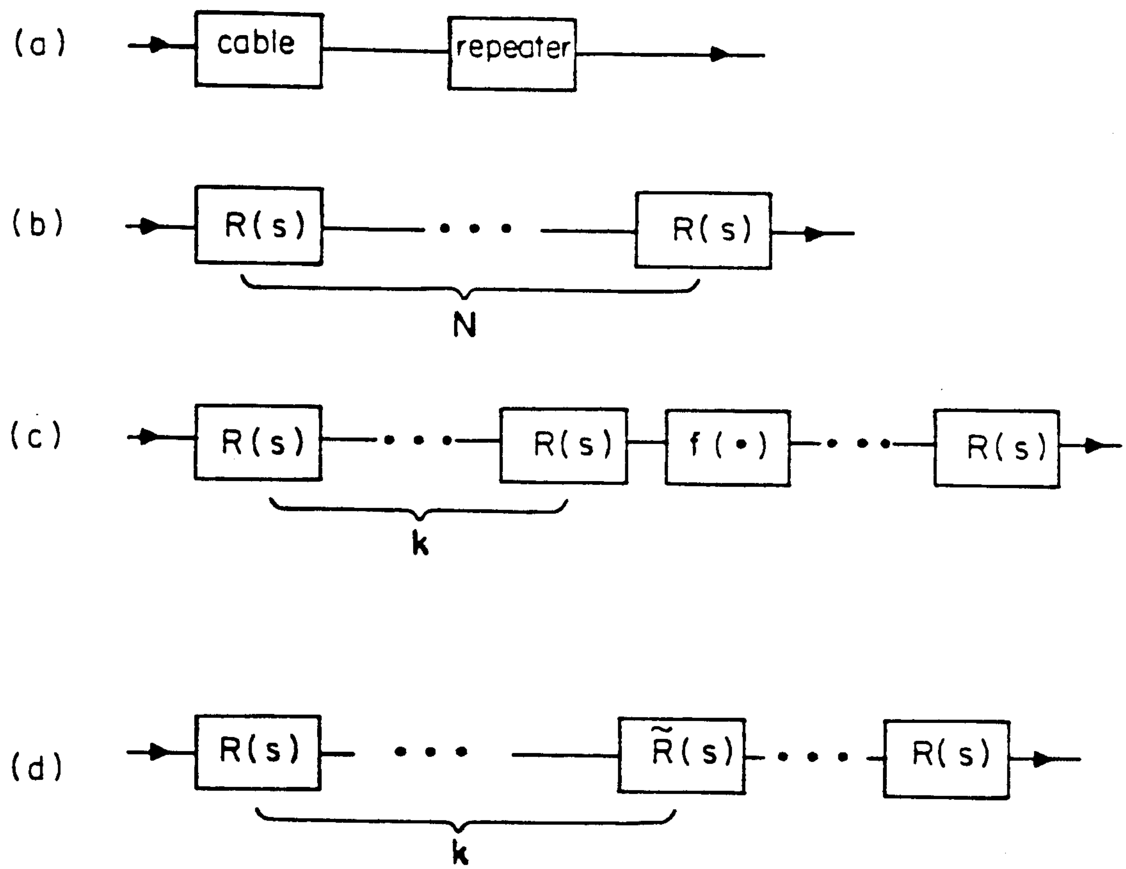

figure 4

in such a way as to, say, halve the bandwidth of the repeater. The kth repeater is still linear, but with frequency response $\tilde{R}(s)$. I/O measurements alone *cannot locate* this fault, since the system's linear (and only) frequency response is $R(s)^{N-1}\tilde{R}(s)$ no matter where the fault is.

7. Application: Linearization of Nonlinear Systems

Often we would like to make a nonlinear system linear, or at least more linear than it was. One very successful technique is to use strong negative linear feedback around the system; this is used for example in amplifiers. But in some cases it is not possible to use this technique, since the requirement of closed-loop stability limits the amount of feedback possible. This would be the case if the system had a delay d, where d^{-1} is on the order of (or smaller than) the bandwidth over which we'd like the closed-loop system to be nearly linear. In this case a *linearizing pre- (or post-) compensator* might be more appropriate. Let us define three different problems:

Pre-compensator Problem: Given a Volterra series operator P, find a Volterra series operator Q_{pre} such that PQ_{pre} is linear.

Post-compensator Problem: Given P, find Q_{post} such that $Q_{post}P$ is linear.

Pre- and Post-compensator Problem: Given P, find Q_{pre} and Q_{post} such that $Q_{post}PQ_{pre}$ is linear.

The pre-compensator problem would be appropriate if P were an electro-mechanical actuator (we envision Q realized electronically); similarly if P were a mechanico-electrical sensor the post-compensator problem is appropriate. If P is, say, a communications channel, then we might be able to use both a pre- and a post-compensator, and we have the third problem. It should be mentioned that the problem of finding a *feedback* which linearizes P can be shown to be equivalent to the pre-compensator problem.

To see that these three problems are in fact different, consider the three operators with block diagrams shown in figure 5. The memoryless nonlinearity is $f(x) \triangleq \exp x - 1$, which has inverse $g(x) = \log(x+1)$.

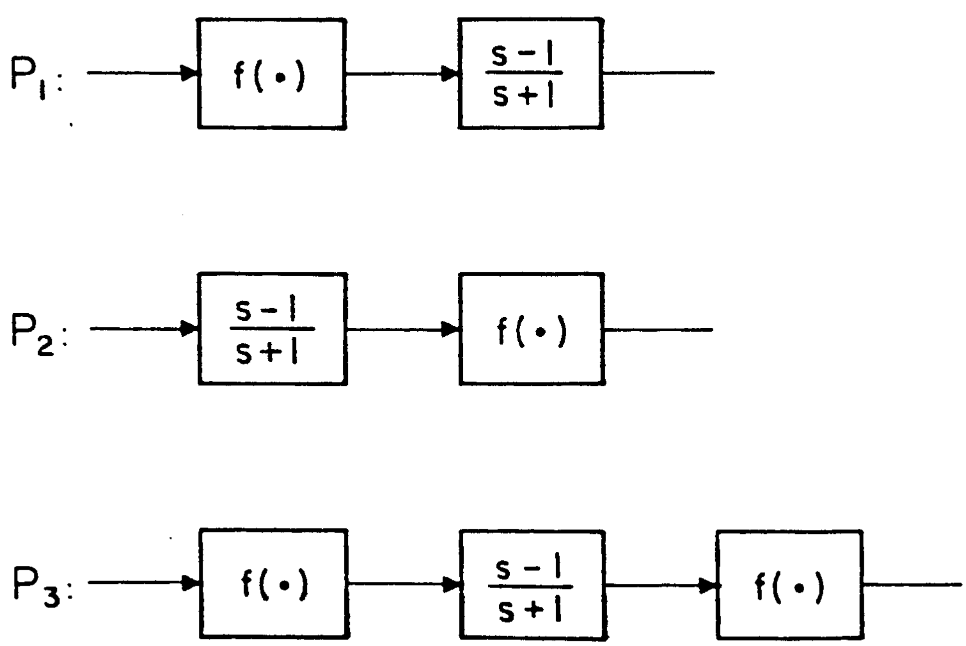

figure 5

It can be shown that for P_1, the pre-compensator problem is solvable (take $Q = g(\cdot)$) but the post-compensator problem is not. For P_2, the post-compensator problem is solvable, but the pre-compensator problem is not. For P_3, neither the pre- nor post-compensator problems is solvable, but the pre- and post-compensator problem is (take $Q_{pre} = Q_{post} = g(\cdot)$).

We do not know general conditions on P under which any of these problems is solvable. But we will show how the ideas of §9 of chapter 2 can be used to construct Q_{pre}, if only its first kernel $Q_{pre\,1}$ is known. Suppose that PQ is linear, that is

$$(PQ)_1(s) = P_1(s)Q_1(s), \quad (PQ)_n = 0, \; n > 1$$

Using the composition formula we have for $n > 1$:

$$0 = (PQ)_n(s_1,...,s_n) = \textbf{SYM} \sum_{k=1}^{n} \left\{ \sum_{\substack{i_1,...,i_k \geq 1 \\ i_1+...+i_k=n}} \right\} P_k(s_1+ ...+ s_{i_1},...,s_{n-i_k+1}+ ...+ s_n)\cdot$$
$$\cdot Q_{i_1}(s_1,...,s_{i_1}) \cdots Q_{i_k}(s_{n-i_k+1},...,s_n)$$

This is precisely the situation we had in the Inversion theorem, and the same recursive procedure determines Q_n for $n \geq 2$:

$$Q_n(s_1,...,s_n) = -P_1(s_1+ ...+ s_n)^{-1} \textbf{SYM} \sum_{k=2}^{n} \left\{ \sum_{\substack{i_1,...,i_k \geq 1 \\ i_1+...+i_k=n}} \right\} P_k(s_1+ ...+ s_{i_1},...,s_{n-i_k+1}+ ...+ s_n)\cdot$$
$$\cdot Q_{i_1}(s_1,...,s_{i_1}) \cdots Q_{i_k}(s_{n-i_k+1},...,s_n)$$

Note that the right-hand side of this equation refers only to Q_j for $j < n$, since $k > 1$.

This observation is by no means a solution to the pre-compensator problem, just a humble start.

Chapter 4

Approximating Nonlinear Operators with Volterra Series

The main purpose of this chapter is to make precise two folk theorems. The first is that *any time-invariant continuous nonlinear operator* can be approximated by a *Volterra series operator*, and the second is that the approximating operator can be realized as a *finite-dimensional linear dynamical system* with a *nonlinear readout map*.

The usefulness of Volterra series hinges on their ability to model a very wide class of nonlinear operators. Two general approaches can be taken to establish this. First, one can directly demonstrate that many explicitly described systems have I/O operators given by Volterra series. In chapter 2 we used the various interconnection theorems to show that a simple dynamical system had a Volterra series I/O operator; using the more general results of Sandberg[1] one can establish that a wide class of systems have I/O operators which are given by Volterra series, the requirement being, roughly speaking, that the nonlinearities are *analytic*. Thus an op-amp (with transistors modeled by the Ebers-Moll equations, which are analytic) has an I/O operator expressible, at least for small inputs, as a Volterra series.

But many common nonlinear systems are modeled with non-analytic nonlinearities. For example the I/O operator of a control system containing an ideal *saturator,* that is, a memoryless nonlinearity with characteristic

$$\mathrm{SAT}(a) \triangleq \begin{cases} \mathrm{sign}(a) & |a| > 1 \\ a & |a| \leq 1 \end{cases}$$

(which of course is not analytic) can easily be shown *not* to have a Volterra series representation valid for any inputs for which the saturator threshold is exceeded.† One could reasonably argue that even though the I/O operator of such a control system does not have an *exact* representation as a Volterra series operator, it could be *approximated* by one, for example by replacing the saturator with a polynomial approximation. But exactly what do we mean by *approximate* here, that is, over what set of signals and in what sense can the I/O operator be approximated by a

† The Uniqueness theorem (theorem 5.2 of chapter 2) implies that a Volterra series operator which is linear for small inputs is in fact linear for all inputs.

Volterra series operator? This is one of the questions addressed in this chapter.

The second approach to establishing the generality of Volterra series is *axiomatic* in style, and conceptually more satisfying. Here one demonstrates that under only a few physically reasonable assumptions about an operator N (such as causality, time-invariance, and some form of continuity) there is a Volterra series operator $\hat{N}$ which approximates, in some sense, N. No assumption whatever is made concerning the internal structure or realization of N.

The idea of such an approximation is not new, and in fact is discussed in the original work of Volterra,[28] who cites Frechet.[29] Even in this early work one can find the basic idea (clouded by archaic mathematics): there is an analogy between ordinary polynomials and finite Volterra series, and hence some analog of the Weierstrass approximation theorem should apply to approximating general nonlinear operators with finite Volterra series.

Wiener rekindled interest in this problem at MIT in the forties and fifties, [30,5,4] and since then various researchers have considered the problem.[31,32,33,34] A clear discussion of a typical approximation result can be found on pages 34-37 of Rugh's book.[6] The result presented there is:

Theorem: Let K be a compact subset of $\mathbf{L}^2[0,T]$ and suppose $N\colon K \to \mathbf{C}[0,T]$ is a TI causal continuous operator. Let $\epsilon > 0$.

Then there is a Volterra series operator $\hat{N}$ such that for all $u \in K$ and $0 \leq t \leq T$

$$|Nu(t) - \hat{N}u(t)| \leq \epsilon \tag{0.1}$$

Roughly speaking, all of this work has the following problems:

(1) The input signals are nonzero only on a bounded time interval $[0,T]$,

(2) The approximation is always on a *compact* subset of the input space,

(3) The approximation only holds over a bounded time interval $[0,T]$.

While demonstrating that Volterra series operators can, at least in a very weak sense, approximate a general TI causal continuous operator, these results are not really satisfying. (1), (2) and (3) are severe restrictions: we would really like an approximation which allows input signals defined on *unbounded time intervals* and which approximates the operator N over an *unbounded time interval.* (1)-(3) preclude, for example, periodic forcing signals which start at $t=0$. Rugh concludes his discussion with the following comments concerning (2): "...I should point out that the main drawback is in the restrictive input space K. The compactness requirement rules out many of the more natural choices for K."

The compactness requirement (2) and the finite time interval requirements (1) and (3) come from the use of the Stone-Weierstrass theorem, which underlies all of these approximation results, and so might seem unavoidable. Indeed we will see an example which demonstrates that without additional assumptions we *cannot* find an approximation for which (0.1) holds for all $t \in \mathbf{R}$. But we will demonstrate that all of these drawbacks can be overcome if the usual continuity assumption on N is strengthened slightly to ensure that N has *fading memory*. In particular, our approximation results (I) will hold over useful (noncompact) sets of signals, possibly nonzero for all $t \in \mathbf{R}$, and (II) will hold for all time, not just on an interval $[0,T]$.

The structure of this chapter is as follows: §1 contains the preliminaries, §2 introduces the fading memory concept, and §3 and §4 contain the main approximation theorems. In §5 we give discrete-time approximation results, one of which concerns approximation by *nonlinear moving-average (NLMA) operators*. In §6 we consider a simple illustrative example, and in §7 we give two other applications of the notion of fading memory.

1. Notation, Definitions and Preliminary Discussion

In this chapter it will be convenient to consider only continuous input signals. $\mathbf{C}(\mathbf{R})$ will denote the space of bounded continuous functions $:\mathbf{R} \rightarrow \mathbf{R}$, with the usual norm $||u|| \triangleq \sup_{t \in \mathbf{R}} |u(t)|$. $\mathbf{R}_-$ will denote $\{ t \mid t \leq 0 \}$, and $\mathbf{C}(\mathbf{R}_-)$ will denote the space of bounded continuous functions on $\mathbf{R}_-$, with the usual norm $||u|| \triangleq \sup_{t \leq 0} |u(t)|$. A function F from $\mathbf{C}(\mathbf{R}_-)$ into $\mathbf{R}$ is called a *functional* on $\mathbf{C}(\mathbf{R}_-)$, and a function N from $\mathbf{C}(\mathbf{R})$ into $\mathbf{C}(\mathbf{R})$ is called an *operator*. As in chapter 2 we will usually drop the parentheses around the arguments of functionals and operators, writing e.g. Fu for $F(u)$ and $Nu(t)$ for $N(u)(t)$.

U_τ will denote the τ-second *delay operator* defined by

$$(U_\tau u)(t) \triangleq u(t-\tau)$$

We say an operator N is *time-invariant* (TI) if $U_\tau N = NU_\tau$ for all $\tau \in \mathbf{R}$.

N is *causal* if $u(\tau) = v(\tau)$ for $\tau \le t$ implies $Nu(t) = Nv(t)$.

N is *continuous* if it is a continuous function $:\mathbf{C}(\mathbf{R}) \to \mathbf{C}(\mathbf{R})$.

With each TI causal operator N we associate a functional F on $\mathbf{C}(\mathbf{R}_-)$ defined by

$$Fu \triangleq Nu_e(0) \tag{1.1}$$

for $u \in \mathbf{C}(\mathbf{R}_-)$, where

$$u_e(t) \triangleq \begin{cases} u(t) & t \le 0 \\ u(0) & t > 0 \end{cases}$$

is just a continuous extension of u to $\mathbf{C}(\mathbf{R})$ (any other would do). In words, F maps the *past input* to N (which is an element of $\mathbf{C}(\mathbf{R}_-)$) into the *present output* of N (which is in $\mathbf{R}$). N can be recovered from its associated functional F via:

$$Nu(t) = FPU_{-t}u \tag{1.2}$$

where $P:\mathbf{C}(\mathbf{R}) \to \mathbf{C}(\mathbf{R}_-)$ truncates an element $u \in \mathbf{C}(\mathbf{R})$ into an element of $\mathbf{C}(\mathbf{R}_-)$:

$$Pu(t) \triangleq u(t) \quad \text{for} \quad t \le 0 \tag{1.3}$$

It's easy to see that N is continuous if and only if F is, so equations (1.1) and (2.1.2) establish a one-to-one correspondence between TI causal continuous operators N and continuous functionals F on $\mathbf{C}(\mathbf{R}_-)$. For this reason we often see nonlinear *functionals* studied, where we are really interested in their associated TI *operators*. This has caused some confusion; some authors have mistakenly used the word *functional* to refer to what are really *operators*.

We can reexpress causality and continuity as follows:

A TI operator N is causal and continuous iff for each $u \in \mathbf{C}(\mathbf{R})$ and $\epsilon > 0$ there is a $\delta > 0$ such that for all v

$$\sup_{t \le 0} |u(t) - v(t)| < \delta \longrightarrow |Nu(0) - Nv(0)| < \epsilon \tag{1.4}$$

That is, a TI operator N satisfying (1.4) is causal and continuous, and a TI causal continuous operator satisfies (1.4).

A Volterra series operator N is TI causal continuous on B_ρ, where ρ is its radius of convergence.

2. The Fading Memory Concept

Roughly speaking, an operator is *continuous* if input signals which are *close* (meaning, the *peak deviation* of the signals *over all past time* is small) have present outputs which are close. We will see that a slight strengthening of continuity is much more useful. Intuitively, an operator has *fading memory* if two input signals which are close in the *recent past*, but not necessarily close in the *remote past* yield present outputs which are close. In §1 of chapter 3 we encountered a similar notion: the Steady State theorem for Volterra series. For dynamical systems, fading memory is related to the notion of a *unique steady-state* (see §9).

The concept of fading memory has a history at least as long as Volterra series themselves. Indeed we find it in Volterra[28, p188]:

> *A first extremely natural postulate is to suppose that the influence of the (input) a long time before the given moment gradually fades out.*

and in Wiener[30, p89]:

> *We are assuming (the output) of the network does not depend on the infinite past. If the response of this apparatus depends on the remote past, then the Brownian motion is not a good approximation because we shall always have to consider the remote past. So we are considering networks in which the output is asymptotically independent of the remote past input...*

and in various other work over the years.[12, 5] The fading memory assumption, then, is by no means a new stronger restriction on the operators to be approximated. It is simply an old assumption whose full power has not, until now, been used.

How should we define fading memory? The problem is that in (1.4) we want $Nu(0)$ to depend less and less on the input when elapsed time $-t$ is large. To do this we simply introduce a weight in (1.4).

Definition: N has *Fading Memory* (FM) on a subset K of $\mathbf{C}(\mathbf{R})$ if there is a decreasing function $w:\mathbf{R}_+ \to (0,1]$, $\lim_{t\to\infty} w(t) = 0$, such that for each $u \in K$ and $\epsilon > 0$ there is a $\delta > 0$ such that for all

$v \in K$

$$\sup_{t \leq 0} |u(t) - v(t)| w(-t) < \delta \longrightarrow |Nu(0) - Nv(0)| < \epsilon \tag{2.1}$$

(This should be compared to (1.4)).

w will be called the weighting function; we will say that N has a w-fading memory, for example if $w(t) = e^{-\lambda t}$ then we might say N has a λ-exponentially fading memory on K. Note that since $w(t) \leq 1$, an operator with FM is continuous, so FM is indeed stronger than continuity.†

The FM property can be clearly expressed in terms of the functional F associated with N as follows: On $\mathbf{C}(\mathbf{R}_-)$ define the *weighted norm*

$$||u||_w \triangleq ||u(t)w(-t)|| = \sup_{t \leq 0} |u(t)w(-t)| \tag{2.2}$$

Then N has FM on K if and only if F is *continuous* with respect to the weighted norm $||\cdot||_w$ on $PK \triangleq \{Pu \mid u \in K\}$.

Remark 1: As in (1.4) above, if a TI N has fading memory, then N is causal.

Remark 2: It is interesting to note that this is very close to Volterra's "definition" of fading memory given on p.188 of [28] (which unfortunately is not clear enough to be a real definition).

Remark 3: For LTI operators, having a fading memory is equivalent to having a convolution representation; see §8.1.

Remark 4: By modifying the proof of the Steady State theorem it can be shown that all finite Volterra series operators have fading memory on any ball in $\mathbf{C}(\mathbf{R})$.

Remark 5: If N has FM, then the steady state theorem holds for N. We may conclude, for example, that if N has FM and $u(t) = \alpha 1(t)$, then $Nu(t) \rightarrow N\alpha$ which is a constant (by time-invariance). To cite another property, N can have no subharmonic response.

Perhaps the best way to appreciate the notion of fading memory is to consider an example of a continuous operator which does *not* have fading memory.

† Our requirements on the weighting function w are more stringent than necessary. All we really need is $w \geq 0$ and $\lim_{t \to \infty} w(t) = 0$; our additional assumptions simplify some of the proofs in the sequel.

Example (Peak-Hold Operator): Define $N_{pk}:\mathbf{C}(\mathbf{R})\rightarrow\mathbf{C}(\mathbf{R})$ by

$$N_{pk}u(t) \triangleq \sup_{\tau\le t} u(\tau)$$

that is, N_{pk} is a *peak-hold operator.* N_{pk} is continuous, since for all $u,\ v\in\mathbf{C}(\mathbf{R})$

$$\|N_{pk}u - N_{pk}v\| \le \|u - v\|$$

Nevertheless N_{pk} does not have a fading memory.†

Let us consider the problem of approximating N_{pk} by a Volterra series operator $\hat{N}$. Consider the signal

$$u_0(t) \triangleq \begin{cases} 1-|t| & |t|\le 1 \\ 0 & |t|>1 \end{cases}$$

Then

$$N_{pk}u_0(t) = \begin{cases} 0 & t\le -1 \\ 1 & t\ge 0 \end{cases}$$

Now for *any* Volterra series operator $\hat{N}$ we have

$$\hat{N}u_0(t) = h_0 \quad \text{for } t < -1$$

and

$$\lim_{t\to\infty}\hat{N}u_0(t) = h_0$$

(This is a consequence of the Steady-state theorem). Hence for *any* Volterra series operator $\hat{N}$

$$\|N_{pk}u_0 - \hat{N}u_0\| \ge \max\{|h_0|, |1-h_0|\} \ge \frac{1}{2}$$

Thus we may conclude *no Volterra series operator* can approximate N_0 within 0.1 over all time, even for the single input u_0. In fact the same argument holds for *any* operator $\hat{N}$ with fading memory, if we substitute $\hat{N}0$ (which must be a constant) for h_0. In particular, N_{pk} itself does not have fading memory.

This example suggests that approximation results which rely only on the continuity of the operator, and no fading memory assumption, will be very weak. In particular, the approxima-

† There are also continuous LTI operators which don't have fading memory, but they are quite pathological; see §A8.

tions need not hold for all time, even on compact sets of signals (in this example, the signal set has only one element, u_0, and so is compact). And yet a very strong approximation is possible for operators with fading memory.

3. Approximation by Volterra Series

Theorem 3.1 (Approximation by Volterra series): Let $\epsilon > 0$ and

$$K \triangleq \left\{ u \in \mathbf{C}(\mathbf{R}) \,\middle|\, ||u|| \leq M_1,\ ||U_\tau u - u|| \leq M_2\tau \text{ for } \tau \geq 0 \right\} \tag{3.1}$$

Suppose that N is *any* TI operator with fading memory on K. Then there is a finite Volterra series operator $\hat{N}$ such that for all $u \in K$

$$||Nu - \hat{N}u|| \leq \epsilon \tag{3.2}$$

Remark 1: The assumption on N is extremely weak. As mentioned earlier, it does not in any way concern the internal structure or realization of N. For example N could arise from a nonlinear PDE, but even this is not necessary.

Remark 2: We can reexpress K as

$$K = \left\{ u \in \mathbf{C}(\mathbf{R}) \,\middle|\, |u(t)| \leq M_1,\ |u(s) - u(t)| \leq M_2(s-t) \text{ for } t \leq s \right\}$$

Thus K can be described as those signals *bounded* by M_1 and having Lipschitz constant M_2, that is, *slew-rate* bounded by M_2.†

Remark 3: The signals in K are not "time-limited" (i.e. zero outside of some interval such as $[0,T]$), and the approximation $|Nu(t) - \hat{N}u(t)| \leq \epsilon$ holds *for all* $t \in \mathbf{R}$, not just in some interval $[0,T]$ (cf Rugh's theorem, (0.1)).

Remark 4: K is *not* a compact subset of $\mathbf{C}(\mathbf{R})$!

Before starting the proof of theorem 3.1, we state the Stone-Weierstrass theorem in a convenient form (see, e.g. Dieudonne[18]).

† In fact K can be any bounded equicontinuous set in $\mathbf{C}(\mathbf{R})$; see §12. The K defined in (3.1), while far from the most general, has a nice engineering description.

Suppose E is a compact metric space and $\mathbf{G}$ a set of continuous functionals on E which separate points, that is, for any distinct u, $v \in E$ there is a $G \in \mathbf{G}$ such that $Gu \neq Gv$. Let F be any continuous functional on E and $\epsilon > 0$. Then there is a polynomial $p:\mathbf{R}^M \rightarrow \mathbf{R}$ and $G_1, \ldots, G_M \in \mathbf{G}$ such that for all $u \in E$

$$|Fu - p(G_1 u, \ldots, G_M u)| < \epsilon$$

Proof of theorem 3.1: Suppose K is given by (3.1) and N has fading memory on K, with weighting function w. Let F be the functional associated with N, given by (1.1), and define $K_- \triangleq PK$, that is

$$K_- = \left\{ Pu \;\middle|\; u \in K \right\}$$

(P is the projection (1.3)).

Lemma 3.2: Consider the weighted norm $\|\cdot\|_w$ on $\mathbf{C}(\mathbf{R}_-)$ defined above in equation (2.2). K_- is compact with the weighted norm $\|\cdot\|_w$.

The proof uses the Arzela-Ascoli theorem and a diagonal argument and is in the appendix, §A6. Since lemma 3.2 is the key to obtaining approximations valid for all time and on noncompact sets, some discussion is in order. Note that K_- is not compact with the standard norm $\|\cdot\|$. To see this, let

$$u_0(t) \triangleq \max\{0, M_1 - M_2|t|\}$$

and consider the sequence $v_n \triangleq PU_{-n} u_0$ in K_- (see figure 1).

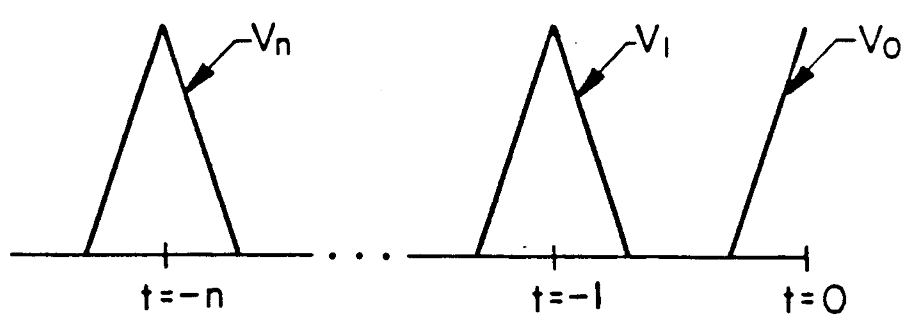

figure 1

With the standard norm, this sequence has no convergent subsequence, and hence K_- is not compact in $\mathbf{C}(\mathbf{R}_-)$. Yet intuitively, *to a device with fading memory* the sequence v_n should appear to be *converging to zero*, and this is indeed true: $||v_n||_w \to 0$ as $n \to \infty$. The idea of lemma 3.2 is that the fading memory makes K_- "appear" compact to our functional F.

Continuing our proof, we define a set of functionals $\mathbf{G}$ on K_- which are continuous with respect to the weighted norm $||\cdot||_w$.

$$\mathbf{G} \triangleq \left\{ G \;\middle|\; Gu = \int_0^\infty g(\tau)u(-\tau)d\tau,\; \int_0^\infty |g(\tau)|w(\tau)^{-1}d\tau < \infty \right\} \tag{3.3}$$

Note that since $0 < w(t) \leq 1$, the condition $g/w \in \mathbf{L}^1(\mathbf{R}_+)$ implies $g \in \mathbf{L}^1(\mathbf{R}_+)$. The fact that any $G \in \mathbf{G}$ is continuous with respect to the weighted norm $||\cdot||_w$ follows from

$$|Gu - Gv| \leq \int_0^\infty \left(|g(t)|w(t)^{-1}\right)\left(|u(-t) - v(-t)|w(t)\right)dt$$

$$\leq \sup_{t\geq 0}|u(-t) - v(-t)|w(t)\int_0^\infty |g(t)|w(t)^{-1}dt = ||u - v||_w \int_0^\infty |g(t)|w(t)^{-1}dt$$

Lemma 3.3: The functionals $\mathbf{G}$ separate points in K_-.

Proof: Let $u,\ v \in K_-$, $u \neq v$. Define

$$g_0(t) \triangleq (u(-t) - v(-t))w(t)e^{-t}$$

Then

$$\int_0^\infty |g_0(t)|w(t)^{-1}dt \leq ||u|| + ||v|| < \infty$$

so let G_0 be the functional in $\mathbf{G}$ associated with g_0 as in (3.3). Then

$$G_0u - G_0v = \int_0^\infty (u(-t) - v(-t))^2 w(t)e^{-t}dt > 0$$

since u and v are continuous and $u \neq v$. This proves lemma 3.3. □

Now by lemmas 3.2 and 3.3 and the Stone-Weierstrass theorem, we conclude that there is a polynomial $p: \mathbf{R}^M \to \mathbf{R}$ and $G_1, \ldots, G_M \in \mathbf{G}$ such that for all $u \in K_-$

$$|Fu - p(G_1u,\ldots,G_Mu)| < \epsilon \tag{3.4}$$

Explicitly writing out p:

$$p(G_1u,\ldots,G_Mu) = \alpha_0 + \sum_{n=1}^{K} \sum_{i_1,\ldots,i_n \leq M} \alpha_{i_1 \ldots i_n} G_{i_1}u \ldots G_{i_n}u$$

$$= h_0 + \sum_{n=1}^{K} \int \cdots \int h_n(\tau_1,\ldots,\tau_n) u(-\tau_1)\ldots u(-\tau_n) d\tau_1 \ldots d\tau_n$$

where $h_0 \triangleq \alpha_0$ and

$$h_n(\tau_1,\ldots,\tau_n) \triangleq \sum_{n=1}^{K} \sum_{i_1,\ldots,i_n \le M} \alpha_{i_1 \ldots i_n} g_{i_1}(\tau_1)\ldots g_{i_n}(\tau_n)$$

and the g_i are the kernels of the functionals G_i as in (3.3).

We mentioned above that the g_i's are in $\mathbf{L}^1(\mathbf{R}_+)$, so $h_n \in \mathbf{L}^1(\mathbf{R}_+^n)$, and thus they are the kernels of a finite Volterra series operator which we call $\hat{N}$. We finally show that $\hat{N}$ is the desired finite Volterra series approximator of N. Let $u \in K$ and $t \in \mathbf{R}$. Then $PU_{-t}u \in K_-$, hence by (3.4)

$$|FPU_{-t}u - p(G_1PU_{-t}u,\ldots,G_MPU_{-t}u)| = |Nu(t) - \hat{N}u(t)| < \epsilon \tag{3.5}$$

Since (3.5) is true for all $t \in \mathbf{R}$, we conclude for all $u \in K$

$$\|Nu - \hat{N}u\| \le \epsilon$$

which proves theorem 3.1. □

4. Approximation by Dynamical Systems

The block diagram of $\hat{N}$ is shown in figure 2. Note that it consists of a single-input multi-output *linear time-invariant operator* followed by a multi-input single-output *memoryless non-linearity*. One question arises immediately: can the LTI block be realized as a finite dimensional linear dynamical system? We will now show that it can.

In the proof of the approximation theorem we used only two properties of the set $\mathbf{G}$ of functionals: first, that each $G \in \mathbf{G}$ has a w-fading memory, and second, that $\mathbf{G}$ separates points in K_-.

Let's examine the first property. For a functional G on $\mathbf{C}(\mathbf{R}_-)$ given by

$$Gu = \int_0^\infty g(\tau) u(-\tau) d\tau \tag{4.1}$$

(where $g \in \mathbf{L}^1(\mathbf{R}_+)$) the necessary and sufficient condition that it have w-fading memory, that is, be continuous with respect to the the w-weighted norm, is

$$\int_0^\infty |g(\tau)| w(\tau)^{-1} d\tau < \infty \tag{4.2}$$

Now we make the observation that if a TI operator N has a w-fading memory, then it has a $\tilde{w}$-

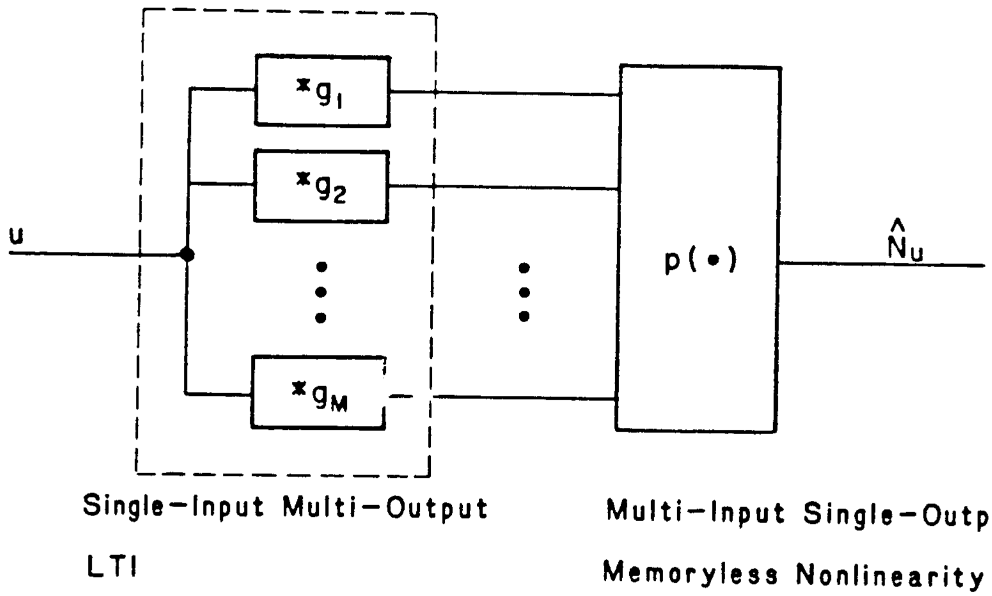

figure 2

fading memory for any weighting function $\tilde{w}$ which dominates w (i.e. $\tilde{w}(t) \geq w(t)$). By using the weight

$$\tilde{w}(t) \triangleq \max\{\, w(t),\, (1+t)^{-1} \}$$

(and relabeling it w) we may simply assume that the weight satisfies $w(t)^{-1} \leq 1+t$. Under this assumption it follows that *every G which comes from a finite dimensional (exponentially stable) linear dynamical system has a w-fading memory,* since the integrand on left hand side of (4.2) is exponentially decaying, that is

$$\int_0^\infty |g(\tau)|\, w(\tau)^{-1} d\tau \leq \int_0^\infty M e^{-\lambda t}(1+t)\, dt < \infty$$

if $|g(t)| \leq M e^{-\lambda t}$. In the next subsection we will show that the G's which come from finite-dimensional linear dynamical systems separate points in $\mathbf{C}(\mathbf{R}_-)$. From this discussion we conclude:

Theorem 4.1 (Approximation by finite-dimensional dynamical systems): Let $\epsilon > 0$ and K be given by (3.1). Suppose that N is *any* TI operator with fading memory on K. Then there is a finite Volterra series operator $\hat{N}$ such that for all $u \in K$

$$||Nu - \hat{N}u|| \leq \epsilon$$

where $\hat{N}$ is the I/O operator of the dynamical system

$$\dot{x} = Ax + bu \qquad y = p(x) \tag{4.3}$$

where A is an exponentially stable $M \times M$ matrix and $p: \mathbf{R}^M \to \mathbf{R}$ is a polynomial.

We have shown that under one extremely weak condition on a TI operator, namely that it have fading memory, it can be approximated in the strong sense of theorem 3.1 by the I/O operator of a finite-dimensional *linear* dynamical system with a *nonlinear* (indeed, polynomial) readout map, as shown in figure 3.

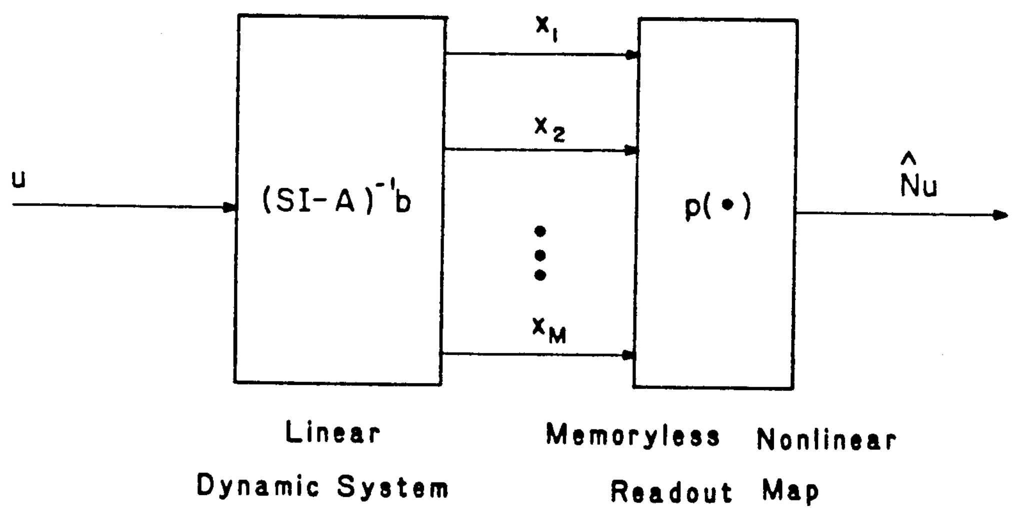

figure 3

In principle, then, a dynamical system of the form (4.3) can always be used as a *macro-model*[35] of a complicated or large-scale nonlinear system, as long as the system has a fading memory. Whether an acceptable approximation is possible with M reasonably small is, of course, a harder question.

The idea that a system of the form (4.3), shown in figure 3, could be used to approximate a very wide class of TI operators is not new. Wiener considered the case where the LTI block in figure 3 consists of a set of Laguerre filters, that is,

$$(sI - A)^{-1}b = \sqrt{2}\left[\frac{1}{1+s}, \frac{1-s}{(1+s)^2}, \ldots, \frac{(1-s)^{M-1}}{(1+s)^M}\right]^T \tag{4.4}$$

which Lee realized with the lattice filter shown in figure 4 (see Wiener[30,p92]). †

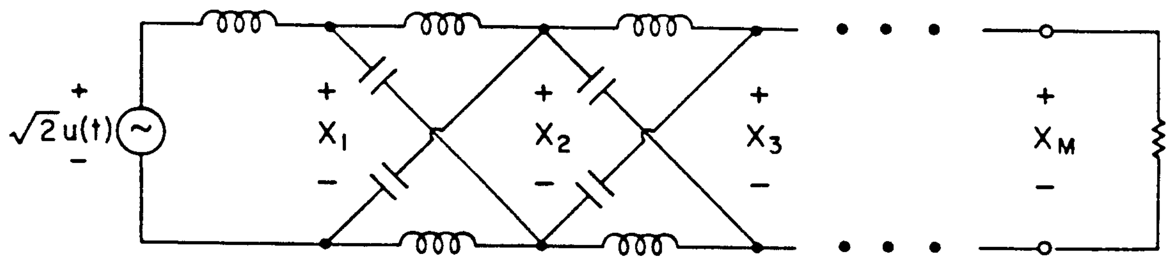

figure 4

To see that Wiener's Laguerre system can approximate any TI causal operator with fading memory in the strong sense of theorem 3.1 or 4.1 (a result evidently unknown to Wiener and his coworkers), we need to establish that the Laguerre functionals $\{L_1, L_2, \ldots\}$ given by

$$L_k u \triangleq \int_0^\infty l_k(t) u(-t)\,dt$$

where $\hat{l}_k(s) = \sqrt{2}(1-s)^{k-1}(1+s)^{-k}$, separate points in $\mathbf{C}(\mathbf{R}_-)$. If not, there are $u_1 \neq u_2 \in \mathbf{C}(\mathbf{R}_-)$ such that for all k $L_k u_1 = L_k u_2$. Let $u = u_1 - u_2$, so that $L_k u = 0$ for all k. We will show that $u = 0$, which will prove that the Laguerre functionals separate points in $\mathbf{C}(\mathbf{R}_-)$. Note that $l_k(t)e^{t/2} \in \mathbf{L}^2(\mathbf{R}_+)$ and $u(-t)e^{-t/2} \in \mathbf{L}^2(\mathbf{R}_+)$ and

$$L_k u = \int_0^\infty \left(l_k(t)e^{t/2}\right)\left(u(-t)e^{-t/2}\right)dt = 0$$

for all k. But the span of the functions $l_k(t)e^{t/2}$ is dense in $\mathbf{L}^2(\mathbf{R}_+)$,†† so we conclude $u(-t)e^{-t/2} = 0$ and hence $u = 0$. This proves that the Laguerre functionals separate points in $\mathbf{C}(\mathbf{R}_-)$; since they are a subset of the functionals which come from finite-dimensional linear dynamical systems, *a fortiori* these functionals separate points, a fact used in the previous subsection. Of course there are many other sequences of functionals which separate points in $\mathbf{C}(\mathbf{R}_-)$.

† The only real difference between (4.4) and (4.3) is that in (4.4) we require the minimal polynomial of A to be $(s+1)^M$, since a change of coordinates can change the numerator polynomials. See e.g. §8.2.

†† See §A11 for a self-contained proof of this fact.

5. A Note on Approximation by Bilinear Systems

The dynamical system approximator (4.3) can be realized as a bilinear system, that is, one of the form

$$\dot{z} = Ez + Fzu + Gu \tag{5.1}$$

$$y = Hz \tag{5.2}$$

where $z \in \mathbb{R}^r$ (usually r is much larger than M of theorem 4.1) and $z(0)=0$. In fact this is a special case of an exercise in Rugh's book[6,p130]; here is a simple way to see it: Suppose the polynomial p in (4.3) is of degree n.

Let z be a vector consisting of all $r = \sum_{k=0}^{n} \binom{M+k-1}{k}$ monomials of degree $\leq n$ formed from $x_1, \ldots, x_M$. Clearly we can write $y = p(x)$ in the form (5.2), where H contains the coefficients of p.

We will now verify that z satisfies an equation of the form (5.1). Consider the lth component of z, say $z_l = x_1^{i_1} \cdots x_M^{i_M}$, where $i_1 + \ldots + i_M \leq n$. Then

$$\dot{z}_l = \sum_{m=1}^{M} i_m \dot{x}_m x_1^{i_1} \cdots x_m^{i_m - 1} \cdots x_M^{i_M} \tag{5.3}$$

$$= \sum_{m,k=1}^{M} i_m a_{mk} x_m x_1^{i_1} \cdots x_m^{i_m-1} \cdots x_M^{i_M} x_k + \tag{5.4}$$

$$+ \sum_{m=1}^{M} i_m x_m x_1^{i_1} \cdots x_m^{i_m-1} \cdots x_M^{i_M} b_m u \tag{5.5}$$

using (4.3). Since each monomial in (5.4) and (5.3.5) has degree (in **x**) $\leq n$, we can reexpress this as:

$$\dot{z}_l = \sum_{p=1}^{r} E_{lp} z_p + \sum_{p=1}^{r} F_{lp} z_p u + G_l u$$

which is of the form (5.1).

In (5.1) the readout map is *linear*, but the vector field contains the product term Fzu (cf (4.3)).

Approximation by bilinear systems has received much attention, but in a context different from that considered here. Usually (but not always) the systems to be approximated are dynamical systems with analytic vector fields. The approximation is generally not in an I/O sense, but rather in the sense of a perturbational expansion of x in u, meaning the input-to-state maps agree to order r in u. See, for example, Fliess,[36] Sussman,[37] or Brockett.[13]

The discrete-time analog of bilinear systems are *state-affine systems*, which have been used to model complicated processes, e.g. in [38].

6. Approximation by Discrete-Time Volterra Series

In this section we present analogous results for discrete-time systems. $\mathbb{Z}$ will denote the integers, $\mathbb{Z}_+$ ($\mathbb{Z}_-$) the nonnegative (nonpositive) integers. Our signal space $\mathbf{C}(\mathbf{R})$ is replaced by $\mathbf{l}^\infty$, the space of bounded sequences (i.e. functions $:\mathbb{Z} \to \mathbf{R}$) with norm

$$\|u\| \triangleq \sup_k |u(k)|$$

The definitions of time-invariance, causality, and fading memory for discrete-time systems require only notational changes. For example a TI operator $N:\mathbf{l}^\infty \to \mathbf{l}^\infty$ has fading memory on a subset K of $\mathbf{l}^\infty$ if there is a decreasing sequence $w:\mathbb{Z}_+ \to (0,1]$, $\lim_{k\to\infty} w(k) = 0$, such that for each $u \in K$ and $\epsilon > 0$ there is a $\delta > 0$ such that for all $v \in K$

$$\sup_{k\le 0} |u(k) - v(k)| w(-k) < \delta \longrightarrow |Nu(0) - Nv(0)| < \epsilon$$

(cf. (2.1)).

A (finite) discrete-time Volterra series operator $N:\mathbf{l}^\infty \to \mathbf{l}^\infty$ is one of the form

$$Nu(k) = h_0 + \sum_{n=1}^{K} \sum_{i_1,\ldots,i_n \ge 0} h_n(i_1,\ldots,i_n) u(k-i_1) \ldots u(k-i_n)$$

where $h_n \in \mathbf{l}^1(\mathbb{Z}_+{}^n)$, that is,

$$\sum_{i_1,\ldots,i_n \ge 0} |h_n(i_1,\ldots,i_n)| < \infty$$

(cf. (2.1)).

Theorem 6.1 (Discrete-time approximation theorem): Let $\epsilon > 0$ and

$$K \triangleq \left\{ u \in \mathbf{l}^\infty \,\middle|\, \|u\| \le M_1 \right\}$$

Suppose that N is *any* TI operator $:\mathbf{l}^\infty \to \mathbf{l}^\infty$ with fading memory on K. Then there is a finite Volterra series operator $\hat{N}$ such that for all $u \in K$

$$||Nu - \hat{N}u|| \leq \epsilon$$

Remark: In the discrete-time theorem there is no "slew-rate" limit on the signals in K; K here is just the ball of radius M_1 in l^∞.

In the next section we will see a stronger form of theorem 6.1, so we omit the proof.

7. Approximation by Nonlinear Moving-Average Operators

As in §4, the discrete time Volterra series approximator $\hat{N}$ can be realized as a finite-dimensional LTI dynamical system with a polynomial readout map. But for discrete-time systems we can choose the LTI dynamical system to have a particularly simple form: its transfer function can be simply

$$H_{su}(z) = \left[1, z^{-1}, \ldots, z^{-M+1}\right]^T$$

(This should be compared to the Laguerre system described in §5.2) The approximator has the block diagram shown in figure 5; $\hat{N}$ is simply a *nonlinear moving-average operator*.

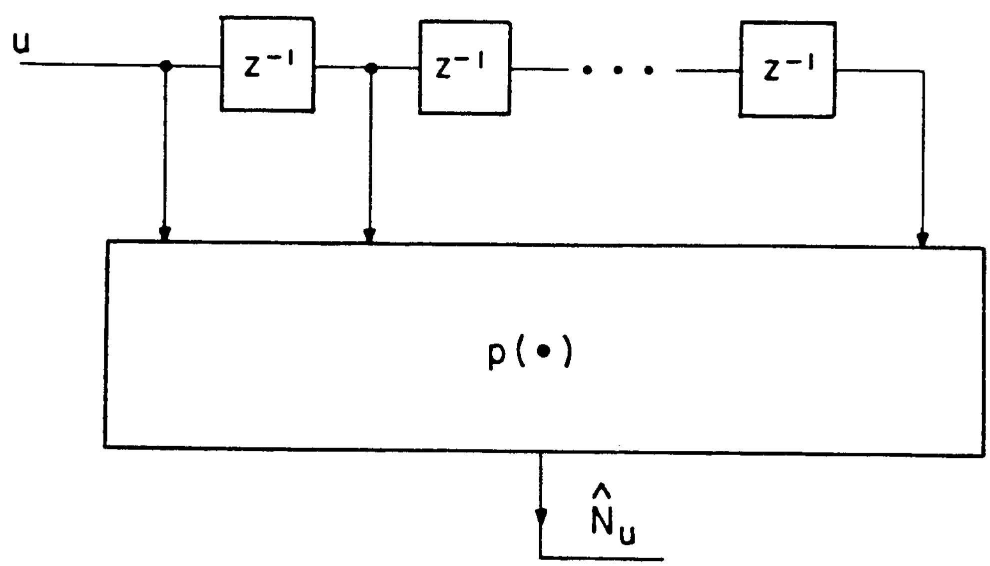

figure 5

To summarize:

Theorem 7.1 (NLMA approximation theorem): Let $\epsilon > 0$, K be any ball in l^∞, and suppose N is any TI operator $:l^\infty \to l^\infty$ with fading memory on K.

Then there is a polynomial $p:\mathbf{R}^M \to \mathbf{R}$ such that for all $u \in K$

$$||Nu - \hat{N}u|| \leq \epsilon$$

where $\hat{N}$ is the NLMA operator given by

$$\hat{N}u(k) \triangleq p(u(k), u(k-1), \ldots, u(k-M+1))$$

The proof is in §A7. Note that this theorem implies theorem 6.1, since every NLMA operator with polynomial nonlinearity is also a finite Volterra series operator.

8. A Simple Example

In this section we consider a simple example, one which illustrates some of the previous ideas and results. We consider the simple RMS detector N shown in figure 6a, and show how a Volterra series approximation and a Laguerre system approximation can be found. More precisely, N is given by

$$Nu(t) \triangleq \left\{ 0.1 \int_0^\infty e^{-0.1(t-\tau)} \left[\int_0^\infty e^{-(\tau-s)} u(s) ds \right]^2 d\tau \right\}^{1/2}$$

We chose this example for several reasons. First, N has no Volterra series representation. To see this, suppose N were a Volterra series operator with kernels h_n. Let $u(t) = \alpha$, a constant. For any Volterra series operator N, $N\alpha$ is also a constant, in fact an analytic function of α (see §5 of chapter 2). But in this case $N\alpha = |\alpha|$, which is not even differentiable at $\alpha = 0$, let alone analytic. So our RMS detector N is not given (exactly) by a Volterra series. Yet it can be shown to have a fading memory on any set K of the form (4.1), and hence our approximation theorems hold for this N.

Another reason for choosing this example is that it is typical of the operators for which the Laguerre system approximation requires very many terms, that is, N is hard to approximate with a Laguerre system. Roughly speaking, this is because N has its nonlinearity near the *input*, and we seek to approximate N with a system with nonlinearity at the *output*.

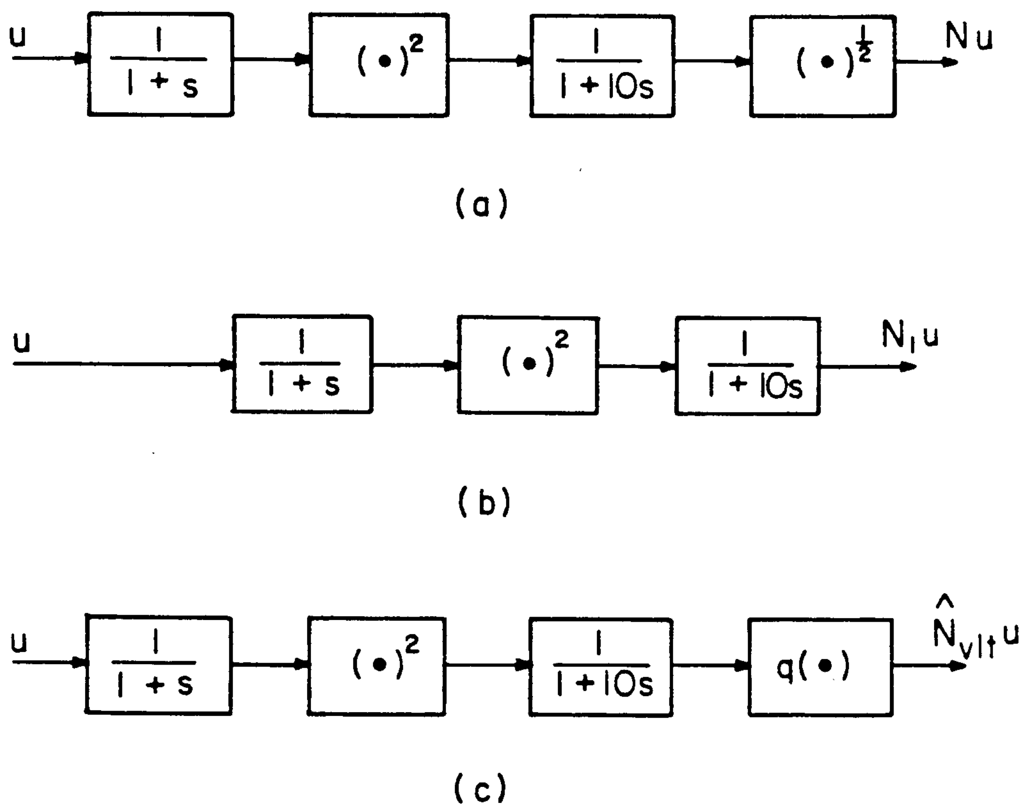

figure 6

8.1. Finding a Volterra Series Approximation

To find a Volterra series approximation of N on the set K given by (4.1), we find a polynomial $q(x)$ such that $|\, q(x) - \sqrt{|x|}\,| < \epsilon$ for $|\, x \,| \le M_1^2$.†

The mean-square operator N_1 shown in figure 6b is a Volterra series operator, its only nonzero kernel

$$h_2(\tau_1,\tau_2) = 19^{-1}\left(e^{1.9\min(\tau_1,\tau_2)} - 1\right)e^{-(\tau_1+\tau_2)} \tag{7.1.1}$$

It follows that the operator $\hat{N}_{vlt}$ shown in figure 6c is a Volterra series operator, whose kernels could be computed, if desired, from (7.1.1) and the composition formula. For $u \in K$ we have

† For example, let q_M be the even polynomial of degree $2M$ which agrees with $\sqrt{|x|}$ at the points $0, M_1^2/M, \ldots, M_1^2$. Then for M large enough, q_M will work.

$0 \leq N_1 u \leq M_1^2$ and hence:

$$||Nu - \hat{N}_{ut} u|| \leq \epsilon \quad \text{for } u \in K$$

8.2. Finding a Laguerre System Approximation

We will now show how a Laguerre approximation to N can be found. It will suffice to find a Laguerre system approximation to the mean-square operator N_1 shown in figure 6b, since passing its output through a polynomial $q(\cdot)$ which approximates the squareroot operator will yield a Laguerre system approximation of the overall operator N, as in the previous subsection.

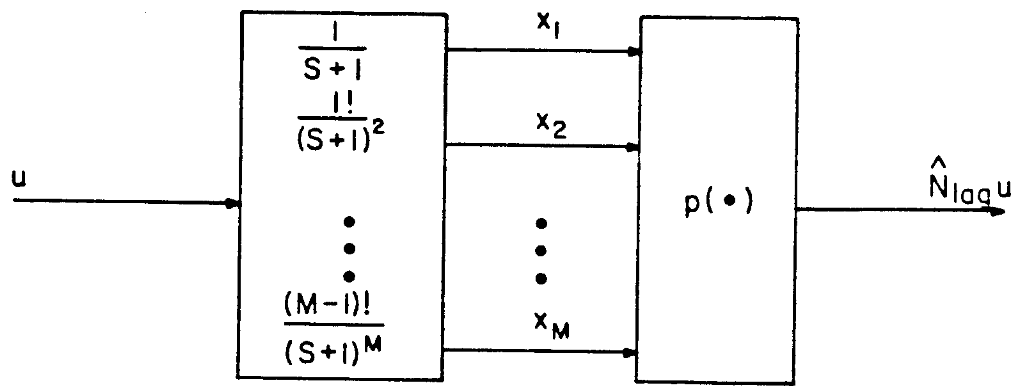

figure 7

Consider the system $\hat{N}_{lag}$ shown in figure 7, where the readout polynomial p is homogeneous of degree two, that is

$$p(x_1, \ldots, x_M) = \sum_{i,j=1}^{M} \beta_{ij} x_i x_j$$

This $\hat{N}_{lag}$ can be transformed to a Laguerre system via the change of coordinates $\bar{x} = Tx$, where T is the (constant, invertible) matrix such that

$$T \begin{bmatrix} (1+s)^{-1} \\ \vdots \\ (M-1)!(1+s)^{-M} \end{bmatrix} = \sqrt{2} \begin{bmatrix} (1+s)^{-1} \\ \vdots \\ (1-s)^{M-1}(1+s)^{-M} \end{bmatrix}$$

$\hat{N}_{lag}$ is a Volterra series operator whose only nonzero kernel is

$$\hat{h}_2(\tau_1,\tau_2) = \sum_{i,j=1}^{M} \beta_{ij}\tau_1^{i-1}\tau_2^{j-1}e^{-(\tau_1+\tau_2)}$$

We will now show that by proper choice of p (that is, M and the β_{ij}'s) $\hat{N}_{lag}$ approximates N on K. Define

$$q(\tau_1,\tau_2) = 19^{-1}\left(e^{1.9\min\{\tau_1,\tau_2\}} - 1\right)e^{-(\tau_1+\tau_2)/2}$$

so that $h_2(\tau_1,\tau_2) = q(\tau_1,\tau_2)\exp-(\tau_1+\tau_2)/2$. Since $q \in \mathbf{L}^2(\mathbf{R}_+^2)$ and the span of the functions $\tau_1^i\tau_2^j\exp-(\tau_1+\tau_2)/2$ is dense in $\mathbf{L}^2(\mathbf{R}_+^2)$,† we can find M and β_{ij}'s such that

$$\left|\left| q(\tau_1,\tau_2) - \sum_{i,j=1}^{M} \beta_{ij}\tau_1^{i-1}\tau_2^{j-1}e^{-(\tau_1+\tau_2)/2} \right|\right|_2 \le \frac{\epsilon}{M_1} \tag{7.2.1}$$

Now we claim that for $u \in K$ we have $||N_1u - \hat{N}_{lag}u|| \le \epsilon$. To see this:

$$N_1u(t) - \hat{N}_{lag}u(t) = \int_0^\infty\int_0^\infty \left(h_2(\tau_1,\tau_2) - \hat{h}_2(\tau_1,\tau_2)\right)u(t-\tau_1)u(t-\tau_2)\,d\tau_1 d\tau_2$$

$$= \int_0^\infty\int_0^\infty \left[q(\tau_1,\tau_2) - \sum_{i,j=1}^{M} \beta_{ij}\tau_1^{i-1}\tau_2^{j-1}e^{-(\tau_1+\tau_2)/2}\right]\left(e^{-(\tau_1+\tau_2)/2}u(t-\tau_1)u(t-\tau_2)\right)d\tau_1 d\tau_2$$

so by (7.2.1) and the Cauchy-Schwarz inequality:

$$|N_1u(t) - \hat{N}_{lag}u(t)| \le \frac{\epsilon}{M_1}\left|\left|e^{-(\tau_1+\tau_2)/2}u(t-\tau_1)u(t-\tau_2)\right|\right|_2 \le \epsilon$$

since

$$\left|\left|e^{-(\tau_1+\tau_2)/2}u(t-\tau_1)u(t-\tau_2)\right|\right|_2 \le M_1$$

Thus for $u \in K$ we have $||N_1u - \hat{N}_{lag}u|| \le \epsilon$.

9. Linear Time-Invariant Operators and Fading Memory

We have seen that the notion of fading memory is quite useful in establishing various approximation theorems. In this section and the next, we discuss briefly two other topics which involve fading memory.

There is a folk theorem that every LTI causal continuous operator has a convolution

† See §A11 for a simple proof of this fact.

representation. Unfortunately this folk theorem is *false,* since there are LTI causal continuous operators which have no convolution representation. But in fact these operators are unlikely to occur in engineering; for example they do not have fading memory (see §A8 for an example of such an operator).

However if "continuous" is strengthened to "FM", our folk theorem becomes true.

Theorem 9.1 (Convolution theorem):

(I) $A:\mathbf{C}(\mathbb{R}) \to \mathbf{C}(\mathbb{R})$ is LTI FM iff A has a convolution representation

$$Au(t) = \int_0^\infty u(t-\tau)h(d\tau) \tag{9.1}$$

where h is a bounded measure on $\mathbb{R}_+$.

(II) $A:\mathbf{l}^\infty \to \mathbf{l}^\infty$ is LTI FM iff A has a convolution representation

$$Au(n) = \sum_0^\infty h(k)u(n-k) \tag{9.2}$$

where $h \in \mathbf{l}^1(\mathbb{Z}_+)$.

Remark: (9.1) may be more familiar to the reader in the form

$$Au(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau$$

where in this equation h is to be interpreted as a measure, e.g. may contain δ-functions.

The proof of theorem 9.1 is in §A9. Theorem 9.1 shows that for LTI causal systems, having a fading memory is equivalent to having a convolution representation.

10. Fading Memory and Unique Steady-State in Dynamical Systems

The notion of fading memory is strictly an input/output property, that is, it refers only to the operator N which maps inputs into outputs; the realization of N (there need not even be one) is irrelevant. But if N does have a realization as a dynamical system, then the fading memory property is related to the *unique steady state property*[39] for dynamical systems. In this section we elaborate this point.

Consider the system

$$\dot{x} = f(x,u) \tag{10.1}$$

$$x(0) = 0 \tag{10.2}$$

where $x(t) \in \mathbf{R}^n$, $u \in \mathbf{C}(\mathbf{R}_+)$, and $f: \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$. Suppose f is such that (10.1) and (10.2) define an operator $N: \mathbf{C}(\mathbf{R}_+) \to \mathbf{C}(\mathbf{R}_+)^n$ given by $x = Nu$.

Theorem 10.1: Suppose N has FM on $\mathbf{C}(\mathbf{R}_+)$, and f is such that all of $\mathbf{R}^n$ is reachable from the origin.

Then the system (10.1), (10.2) has a *unique steady-state.*

More precisely, let x_0, $\tilde{x}_0 \in \mathbf{R}^n$, and let x and $\tilde{x}$ denote the solutions of (10.1), but with initial conditions x_0 and $\tilde{x}_0$, respectively. Then

$$\lim_{t \to \infty} \|x(t) - \tilde{x}(t)\| = 0$$

Thus the fading memory assumption implies that the state will be "asymptotically independent" of the initial condition, to use Wiener's phrase.

The proof of theorem 10.1 is in §A10. We have presented theorem 10.1 only to demonstrate that there is a connection between the ideas of fading memory and unique steady state; far stronger theorems can be proved.

The conditions under which a dynamical system has a fading memory is a very important topic itself. To mention perhaps the simplest condition, if an equilibrium point is well-behaved (meaning, the vector field is continuously differentiable there and the linearized system is exponentially stable and controllable) then for inputs small enough the input-to-state map will have a fading memory.

11. Conclusion

We have shown that any operator with fading memory can be approximated in a strong sense by a (finite) Volterra series operator which can be realized as a finite dimensional linear dynamical system with a polynomial readout map. For discrete-time systems, the approximating operator can simply be a *nonlinear moving-average operator*. The approximation holds over any *bounded* set of signals K; in the continuous-time case we must add a *slew-rate* limitation as well. The approximation is in the sense of *peak error, worst case* for all signals in K.

Since the original work of Volterra there has been much research on this topic, but none has yielded the strong approximations presented here. The reason is related to a remark in §2.1 concerning the difference between TI causal operators and functionals on $\mathbf{C}(\mathbf{R}_-)$. Intuitively it would seem that this correspondence implies that an approximation of a *functional* (perhaps, via the Stone-Weierstrass theorem) should also yield an approximation of the corresponding *TI causal operator*. This is true, if the set of signals $K \subset \mathbf{C}(\mathbf{R}_-)$ over which the approximation holds is also time-invariant, i.e. $U_t K = K$ for all $t \geq 0$. But here's the catch: TI subsets of $\mathbf{C}(\mathbf{R}_-)$ are generally *not compact*,† and hence the Stone-Weierstrass theorem can't be used to approximate the functional. Our solution to this problem was to observe that while a set such as K_-, although not compact, should "appear" compact to an operator whose memory fades with elapsed time.

We close with some remarks concerning the practical application of the material presented in this chapter. While the approximations are certainly strong enough to be useful in applications like macro-modeling of complicated systems or in universal nonlinear system identifiers, we know of no general procedure, based only on input/output measurements, by which an approximation can be found. Perhaps an adaptive scheme can be made to work in practice.

12. A Mathematical Formulation

It is possible to generalize the results of this chapter to a clean and simple mathematical form, at the cost of some engineering intuition. First, we extend our definition of fading memory to:

> *N has fading memory if its associated functional F is continuous with respect to the compact-open topology.*

For continuous-time systems, this is the topology of uniform convergence on compact sets; for discrete-time systems, this is the topology of pointwise convergence. The definition of fading

† For example if K contains at least one compactly supported element, then it is not compact. There are TI compact subsets of $\mathbf{C}(\mathbf{R}_-)$, for example $\{ U_t f \mid t \geq 0 \}$, where f is almost periodic.

memory given in this chapter, in terms of a weighting function $w(\cdot)$, implies fading memory in this sense. Our lemma 3.2 can be generalized to:

> *A closed bounded equicontinuous subset of* $\mathbf{C}(\mathbf{R}_{-})$ *is compact in the compact-open topology.*

For the discrete-time case:

> *A closed bounded subset of* $\mathbf{l}^{\infty}$ *is compact in the compact-open topology.*

Since in $\mathbf{l}^{\infty}$ the compact-open topology is the weak-* topology, this last assertion is just an instance of a classic theorem of functional analysis: the closed unit ball is weak-* compact.[40]

With these extended definitions, all of the approximation theorems presented still hold.

Chapter 5

Measuring Volterra Kernels

Volterra series have appeared in the engineering literature for forty years now, and yet relatively few attempts have been made, outside the biological areas, to actually *measure* Volterra kernels. In this chapter we will discuss practical techniques for measuring the Volterra kernels of a weakly nonlinear system. In this context, a *weakly nonlinear system* simply means a system which is well described by its first few Volterra kernels; more precisely, the error bound function (see corollary 2.2 of chapter 2) for the Volterra series truncated after a few terms should be small for the maximum signal amplitudes of interest.

We assume that the nonlinearities may be subtle (i.e. distortion products 40db or more down) and that the measurement noise is low (or that the necessary signal averaging has been done). Examples of such systems are some high quality transformers, electromechanical and electroacoustic transducers, simple communications systems; not included are e.g. devices with dead zone, hard saturation or hysteretic nonlinearities. While the problems of kernel measurement in biology are quite different, involving stronger nonlinearities and very poor S/N ratios, much of the following is still relevant.

Related work includes that of Narayanan and Meyer et al.[44,45,46,47] who have studied IM distortion in transistor circuits; Weiner and others have done similar work for simple communications systems.[48,49] In these studies a model of a transistor or modulator is assumed and expressions derived for the various kernels; then certain distortions such as $2f_1-f_2$ are measured at a few frequencies and input levels and checked against the model's predictions. Certain recent work by Ewen and Weiner [48] assumes a specific (but important) form for the Volterra kernels and gives methods to solve the resulting parameter identification problem. In contrast to these studies we make no assumption about the form of the kernels. These measurements are thus useful in systems of such complexity that no simple model is obvious, and for model validation when one is.

We have chosen frequency domain Volterra kernels over time domain Volterra kernels and Wiener kernels for two reasons. The first is that it is easier to accurately measure frequency domain kernels than time domain Volterra kernels when the nonlinearities are subtle. Second and more important, we are usually interested in frequency domain Volterra kernels precisely because they have an *intuitive interpretation:* for example $H_2(j\omega_1,-j\omega_2)$ is a measure of the second order difference intermodulation of ω_1 and w_2. While a similar interpretation exists for time domain Volterra kernels, *no such simple interpretation* can be given to the Wiener kernels, whose apparent advantage is "ease" of measurement with white noise[50,51,52,53] Concerning this last "advantage", we feel that in many applications the advent of microcomputers, D/As and A/Ds has outmoded the use of white noise/correlation techniques. With only a few inexpensive components it is now possible to generate very complicated multitone signals with all distortion products near the noise floor, often 70db or more down. Signal processing too has gone far beyond Y. F. Lee's Laguerre lattice filter (see §4 and figure 4 of chapter 4). These practical considerations allow us to make a more direct attack on the measurement problem than was possible twenty five years ago.

The organization of this chapter is as follows: in §1 we discuss resolving the output into its homogeneous components, in §2 we cover the basic multitone method of measuring the kernels, in §3 we introduces a new quick method of measuring the second kernel, and in §4 we describe a simple experimental example.

We will use trigonometric polynomial input signals, that is, input signals of the form:

$$u(t) = 1(t)\sum_{k=-M}^{M}\alpha_k e^{j\omega kt}$$

where $\alpha_{-k} = \overline{\alpha}_k$.

Recall from chapter 3 that for input signals of this form the output y approaches a steady-state y_s as $t\to\infty$, $y_s(t) = \sum_{n=1} y_{sn}(t)$, where

$$y_{sn}(t) = \left\{\sum_{-M\le k_1,\ldots,k_n\le M}\right\}\alpha_{k_1}\ldots\alpha_{k_n}H_n(j\omega k_1,\ldots,j\omega k_n)e^{j(\omega k_1+\cdots+\omega k_n)t}$$

Note that the nth order component of y_s is a sum of exponentials whose frequencies are sums of n input frequencies, negative frequencies included.

We will assume that the steady state output spectrum (i.e. the Fourier coefficients of $y_s(t)$)

is measured. For notational convenience we will assume $\omega=1$ and drop the qualifier "steady-state" in the sequel.

1. The Problem of Kernel Separation

In general the output y due to u has components of all degrees, though in the systems we consider their amplitudes fall off quickly, that is, only a few are significant. One step in measuring the kernels is to estimate the components $y_1,...$ of y. What we need is a stable method of estimating

$$y_n=\frac{1}{n!}\,\frac{\partial^n}{\partial\alpha^n}\,N(\alpha u)\Big|_{\alpha=0}$$

(see chapter 2). While $N(\alpha u)(t)$ is in general an analytic function of α, for the systems we consider it is close to a low order *polynomial* in α, with coefficients y_i. Thus the problem of estimating the different order components is *in practice* one of estimating the coefficients of a noisy polynomial. There are many ways to do this. We'll first describe the simplest, which we call the interpolation method.

Consider the fact that y_n is homogeneous of degree n in u. Thus if our input is reduced 6db, y_1 falls 6db, y_2 12db and so on; if $-u$ is applied, the odd degree components change sign while the even ones do not. Suppose that the components of degree five and higher are negligible, i.e. buried in the measurement/quantization noise. Let us apply the signals $\alpha_i u(t)$ to the device and call the resulting responses $r_i(t)$, where α_i, $i=1,..4$ are some wisely chosen nonzero distinct constants. Then we have

$$\begin{bmatrix} r_1\\ r_2\\ r_3\\ r_4\end{bmatrix}=\begin{bmatrix}\alpha_1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4\\ \alpha_2 & \alpha_2^2 & \alpha_2^3 & \alpha_2^4\\ \alpha_3 & \alpha_3^2 & \alpha_3^3 & \alpha_3^4\\ \alpha_4 & \alpha_4^2 & \alpha_4^3 & \alpha_4^4\end{bmatrix}\begin{bmatrix} y_1\\ y_2\\ y_3\\ y_4\end{bmatrix}+\begin{bmatrix} e_1\\ e_2\\ e_3\\ e_4\end{bmatrix}$$

where the e_i contain measurement noise and terms of degree five and higher. The matrix A above is a Vandermonde matrix, and is invertible since the α_i are distinct and nonzero. Approximating $e=0$ and solving this equation gives us an estimate of the components y_i in terms of the measurements r_i. This is just a simple polynomial interpolation and is mentioned in Simpson and

Power [53] and Halme.[19]

Sometimes we know apriori that only certain y_i appear; the other y_i's may then be dropped from the y vector and the corresponding columns from the A matrix. For example if we know only even order responses occur, the equations above can be replaced with a two by two system involving just y_2 and y_4. This is of course equivalent to interpolating with an even polynomial.

The α_i must be chosen carefully. Choosing the α_i large has the advantage of making $||A^{-1}||$ small, so the error in our resulting estimates is small. The disadvantage is that to estimate the components at some reference level we apply a larger signal, perhaps overloading the device (that is, operating the device where it is not weakly nonlinear in our strict sense). The α_i should alternate in sign and not be too close, to keep $||A^{-1}||$ small.

But even with careful choice of the α_i, the interpolation method is in general sensitive to measurement error. To see this consider estimating y_1 and y_2 with $\alpha_1=1$, $\alpha_2=-1$. We average r_1 and r_2 to get y_2, and since r_1 is very nearly $-r_2$ (y_2 is generally much smaller than y_1) we have committed the cardinal sin of subtracting nearly equal quantities. Of course this example is oversimplified, but it conveys the basic idea. A more formal explanation is that the absolute error in y is bounded by $||A^{-1}e||$, but y_2, y_3 ... are generally much smaller than y_1 so the relative error in these entries may be huge. Rescaling the equations, perhaps using $y_1, 10y_2, 100y_3 \cdots$ instead of y_1, y_2 ... simply makes A^{-1} blow up.

One improvement is to take additional measurements and use the least squares solution of the resulting overdetermined equations as our estimate of y. This is the method we used, and although it is an improvement over the simplest interpolation method, it still gives poor estimates of the higher order components: estimating the rapidly decreasing coefficients of a noisy polynomial is inherently difficult. What we can say is this: we can get a good estimate of the first coefficient appearing, a poorer estimate of the next, and a very poor estimate of the small high order coefficients. This observation suggests that our measurement method should arrange for the component we need to estimate at some frequency to be the *first* (i.e. lowest order) component appearing at that frequency. We call this *frequency separation*.

The simplest and oldest use of frequency separation is as follows: suppose the input frequencies are all odd. Then the odd and even order responses occur at odd and even order frequencies, respectively. To isolate a second order response at some even frequency we need only remove the 4th, 6th, etc. order responses, that is, estimate the x^2 coefficient of an *even* polynomial. We could use the interpolation method, modifying the matrix and y, but the estimate will be very accurate since the second order response we seek is not swamped by a larger first order response; it is the first large response occurring at that frequency. Moreover by applying the signal at three levels we can approximately remove the effects of the components through degree six, as opposed to degree three for the general case. This trick is widely known, the requirement is simply that the input signal be odd, i.e. have the inverse-repeat property as it is sometimes called.

It should be mentioned that complete separation of the components of different order by frequency separation is impossible. For whenever w is an nth order response frequency, it is also an $n+2, n+4$, ... order response frequency, at least.

2. The Multitone Method ("Harmonic Probing")

In this section we discuss the actual measurement of the kernels. Suppose we apply a two tone signal $u(t)=\cos(n_1t)+\cos(n_2t)$, $n_1>n_2>0$. Then $\hat{y}(n_1\pm n_2)=1/4H_2(jn_1,\pm jn_2)+$ terms of order 4, 6,... and for certain values of n_1 and n_2, additional terms of order 3, 5,... Applying the signal at two or three levels and using the interpolation method to estimate the second degree component of $\hat{y}(n_1\pm n_2)$ yields an accurate measurement of $H_2(jn_1,jn_2)$ and $H_2(jn_1,-jn_2)$. At the same time we can measure $H_2(jn_1,jn_1)$ and $H_2(jn_2,jn_2)$ but these are of less interest since they always lie on the line $\omega_1=\omega_2$. We simply repeat this procedure until a sufficient number of points has been measured.

A variant of this method can be used to measure the third and higher order kernels. Suppose a three tone signal is applied. Third degree responses occur at up to 22 different (positive) frequencies, three of which are the input frequencies n_1, n_2, n_3.† If we choose integer triplets such

† They are:

$$n_1,\quad n_2,\quad n_3,\quad 3n_1,\quad 3n_2,\quad 3n_3,\quad |n_1\pm n_2\pm n_3|,$$

that the full 22 frequencies appear (the "general" triplet has this property), estimation of y_3 yields a good estimate of 19 points of H_3. The four points $H_3(jn_1, \pm jn_2, \pm jn_3)$ are of more importance than the remaining 15 which lie on planes where two frequencies are equal. Note that 12 points of H_2 can be measured from the same experiment.

3. A New Method

Unfortunately, measuring kernels by the multitone method can be quite slow. For example to measure H_2 at only 100 points (relatively few) requires at least 100 experiments, each experiment consisting of generation of a signal, waiting for steady state, sampling the output, and then computation (FFT, kernel separation). One may have to wait through half of these before deciding the input level is too low or high or that another frequency range might be more interesting. We've developed a method for getting a quick estimate of the second kernel. We use this method to make decisions about input level, frequency range, etc. before using the slower but more robust multitone method.

It is perhaps surprising that many points of $H_2(j\omega_1, j\omega_2)$ can be simultaneously measured since methods for simultaneously measuring many points of $H(j\omega)$ for a linear device (pseudonoise, impulse testing) rely very heavily on linearity. The idea is simple: arrange the second order IM tones to lie on distinct frequencies which don't include the input frequencies.

We start with two relatively prime integers p and q, q odd. The probing signal will have two parts: one with frequencies $p, 2p, \ldots p(q-1)/2$ and the other with frequencies $q, 2q, \ldots q(p-1)$. We claim that the part one- part two intermodulation tones are distinct. These IM tones occur at frequencies $np+mq$, $0<|n|\leq(q-1)/2$, $0<m\leq p-1$; the input tones are precisely the $n=0$ or $m=0$ cases. Suppose that $\bar{n}p+\bar{m}q = np+mq$, where $0\leq\bar{n}, n\leq(q-1)/2$ and $0\leq\bar{m}, m\leq p-1$. Taking residues mod q, we have $\bar{n}=n \bmod q$, and thus $\bar{n}=n$ considering the inequality in $\bar{n}$, n above. Hence $\bar{m}=m$ as well. This shows that the part one -part two IM tones are distinct and do not include any input frequencies. They also do not include any part one(two) -part one(two)

$|2n_1 \pm n_2|$, $|2n_1 \pm n_3|$, $|2n_2 \pm n_1|$, $|2n_2 \pm n_3|$, $|2n_3 \pm n_1|$, $|2n_3 \pm n_2|$

intermodulation tones since these are all $0 \bmod p$ $(\bmod q)$; here we use the inequality in $\overline{m}$, m. The conclusion is that at the part one -part two IM frequencies, there is no first order component and only one second order contribution. Let us take $p=7$, $q=5$ as an example. We make a table as follows:

14	19	24	29	34	39	44
7	12	17	22	27	32	37
0	**5**	**10**	**15**	**20**	**25**	**30**
-7	2	3	8	13	18	23
-14	9	4	1	6	11	16

The left column and center row (in bold) are input frequencies; the other entries are the part one -part two IM frequencies and it is easily checked that at these frequencies there is no first order and just one second order contribution.

A quick estimate of H_2 is now easy: we apply this multitone signal u at, say, six different levels and use a least squares interpolation to estimate $\hat{y}_2$. Almost every entry of $\hat{y}_2$ gives us a value of H_2 : in our example above $\hat{y}_2(8)=H_2(15j,-7j)\alpha\overline{\beta}$ where α and β are the complex amplitudes at 15 and 7 in u. This should be compared to the multitone method where only two or four of the entries of $\hat{y}_2$ are used and in fact the efficiency of using the FFT is questionable.

We should make one comment concerning the choice of the complex amplitudes of the frequencies in the probing signal. While it is tempting to make them all one, this is the worst choice possible. This results in $\sin(\theta N/2)/\sin(\theta/2)$ type signals with very high crest factors; the signals spend most of their time down where the quantization step is significant. For a given peak level (to keep from clipping the device, perhaps) the amplitudes are small, and the second order distortions we are trying to measure are extremely small (i.e. small squared). To avoid these problems we simply let an optimization routine adjust the phases to minimize the peak.[54] The practical result of this is to pack as much probing signal (L^2) as possible into a given peak. For signals with frequencies f, $2f$, $\ldots Kf$ near optimal phases are $\delta_k=(\pi k^2)/(K+1)$;† our optimization routine used these as starting points. For the quick tests we used $(7<p,q<25)$, we were able to reduce the peak by more than 10db and thus realize a 20db gain in measurement sensitivity.

† D. J. Newman, personal communication.

This is not far from the bound peak$>\sqrt{K/2}$ for a K tone unit amplitude signal:††

$$\frac{K}{2} = \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{k=1}^{K}\cos(kt + \delta_k)\right\}^2 dt = ||u||_2^2$$
$$< \sup_{0\le t\le 2\pi}\left\{\sum_{k=1}^{K}\cos(kt + \delta_k)\right\}^2 = ||u||_\infty^2$$

To illustrate this figure 1 shows two 7-5 quick test signals: the first (darker) with optimized phases and a peak of about 4, the second with all phases zero and a peak of 8. In this case the peak has only been reduced about 6db (representing a 12db gain in second kernel measurement sensitivity), but in more realistic cases the improvement is greater.

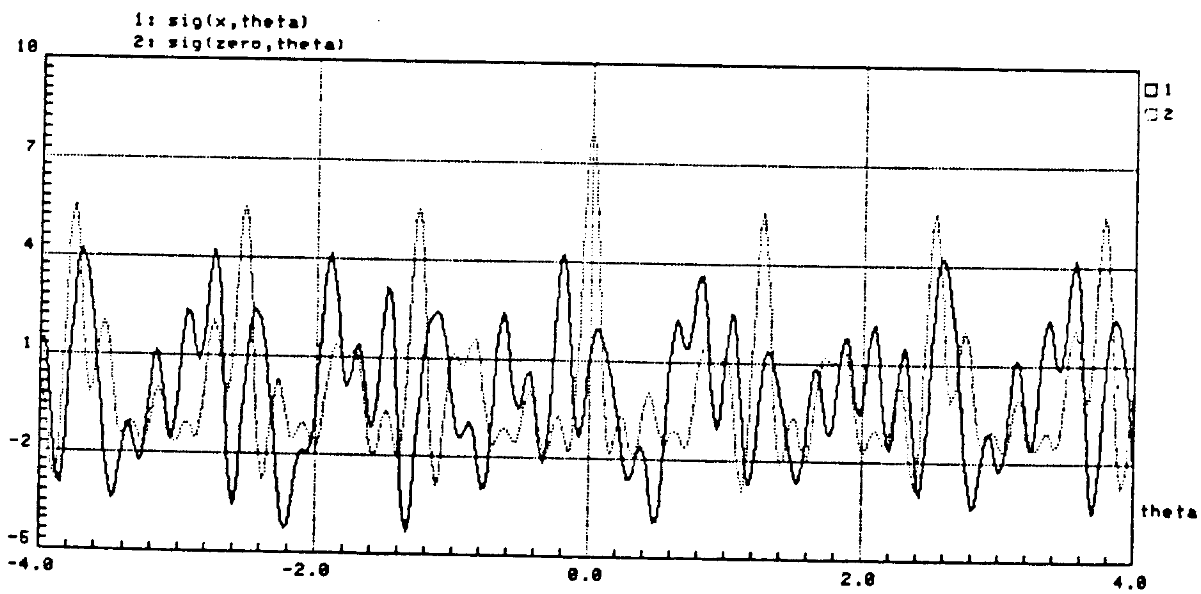

figure 1

We have now arrived at probing signals which at first glance resemble the white noise we complained about in section one, but we hope the reader will appreciate the difference.

†† Incidently this technique of choosing the phases of a multitone signal to minimize the crest factor of the probing signal is readily applicable to *linear* multitone testing, and as far as the author knows, has never been used before.

4. An Example

In this section we briefly describe our test set up and illustrate some of the above with an example. We used a small 8085 based microcomputer to generate the probing and trigger signals and do all computation except the FFT; an HP3582A spectrum analyzer collected and transformed the responses. We built several reference nonlinear devices with known kernels like

$$H_1(s)=\frac{6.4}{1+s/s_0} \qquad H_2(s_1,s_2)=\frac{0.064}{(1+s_1/s_0)(1+s_2/s_0)}$$

$$H_n=0, \quad n>2 \qquad s_0=2\pi 350 \qquad |V_{in}|<1V$$

and used them to check the algorithms above. Note that the distortion is at most 1%, i.e. at least 40db under y_1. The values of H_2 measured by the multitone and quick methods were within 2% and 7%, respectively, of the predicted values. Figure 2 shows the magnitude of H_2 measured by the multitone method; it is indistinguishable from the graphs based on either the quick test measurement or the expression above.

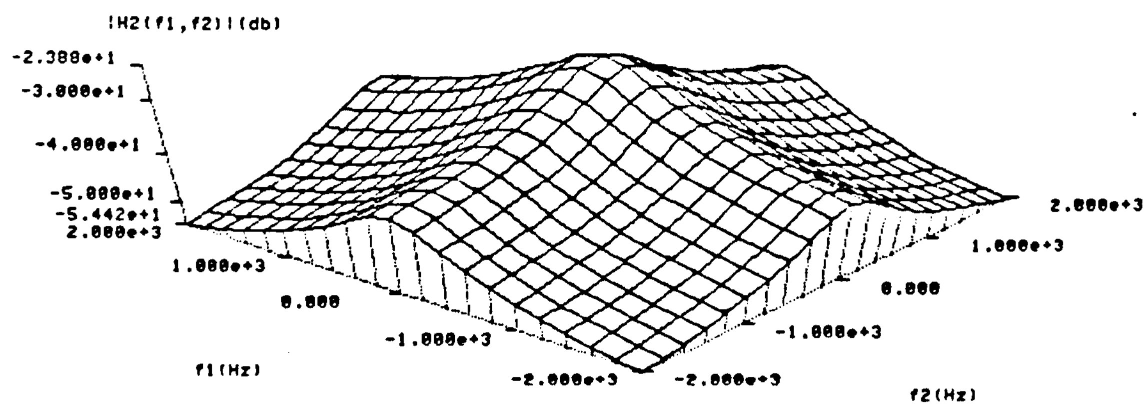

figure 2

The example we give is an electro-acoustic transducer, a JBL 2441 compression driver on a Northwest Sound 90 degree radial horn, measured 0.5m on axis. We chose this example because it has no simple model† and as far as we know these measurements have never been made before.

† An accurate model would involve at least: nonlinear flux-coil linkage, nonlinear support compliance, and thermodynamic nonlinearity (distributed!).

To illustrate frequency separation and the fact that $N(\alpha u)$ is indeed close to a low order (even) polynomial in α, figure 3 shows the real part of the output at 800Hz versus the input amplitude of a 400Hz signal. The interpolation method correctly estimates a large second order, small fourth order, and nearly zero first and third order components at 800Hz. A plot like figure 3 can warn us that a device is not well-described by its first few Volterra kernels if it is not close to a low order polynomial.

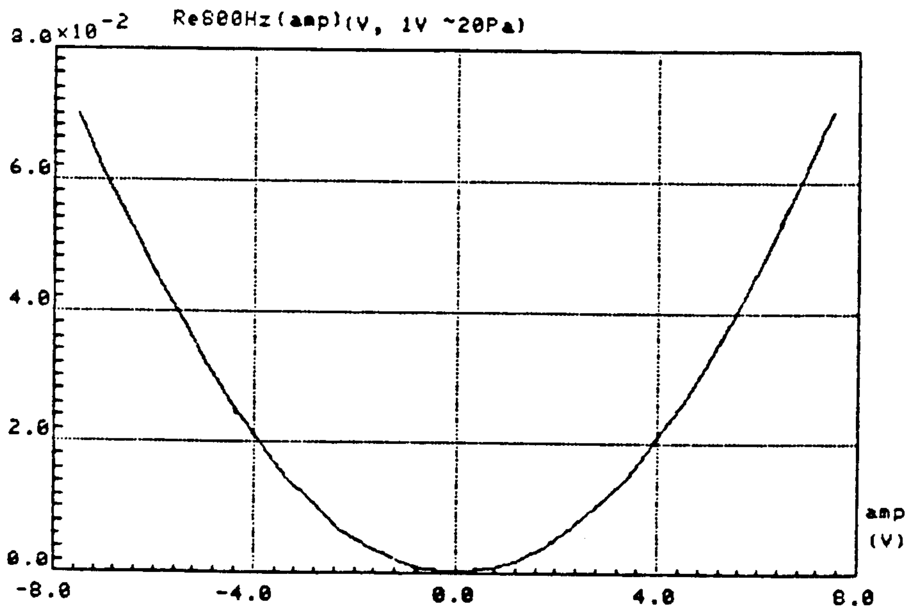

figure 3

Figures 4 and 5 show typical input and output spectra for this transducer during a 13-11 quick test.

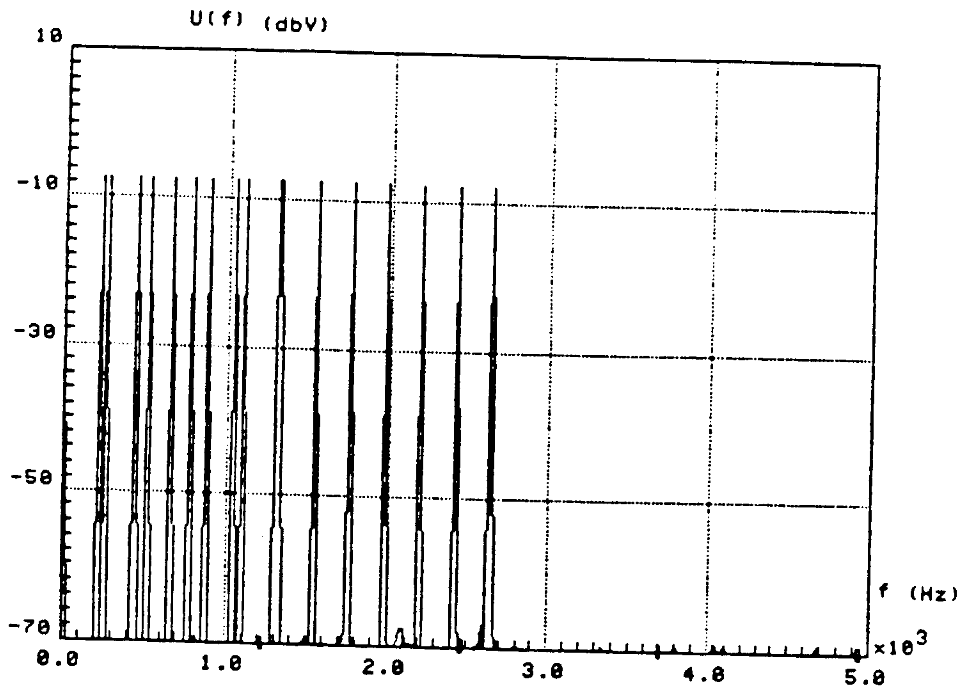

figure 4

In figure 5 one can see clearly the large first order responses at the input frequencies and the smaller higher order responses.

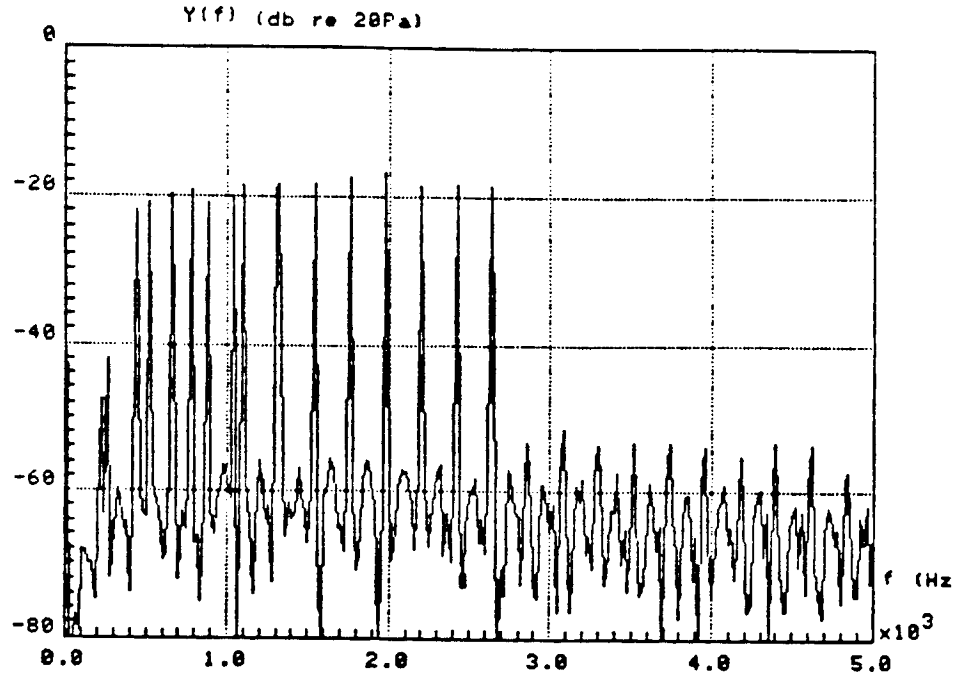

figure 5

The responses on the right which are about 8db higher are mostly second order part II -part II intermodulation. Measurements of the second kernel of the transducer by the quick method and the two-tone method agreed within 5%. Figure 6 shows the magnitude of the second kernel measured by the quick method. The peak distortion here is only 2%. Some features are recognizable, for example the "trough" along the line $f_1 + f_2 = 0$ suggests a linear high pass filter (horn cut-off) following a nonlinear operator.

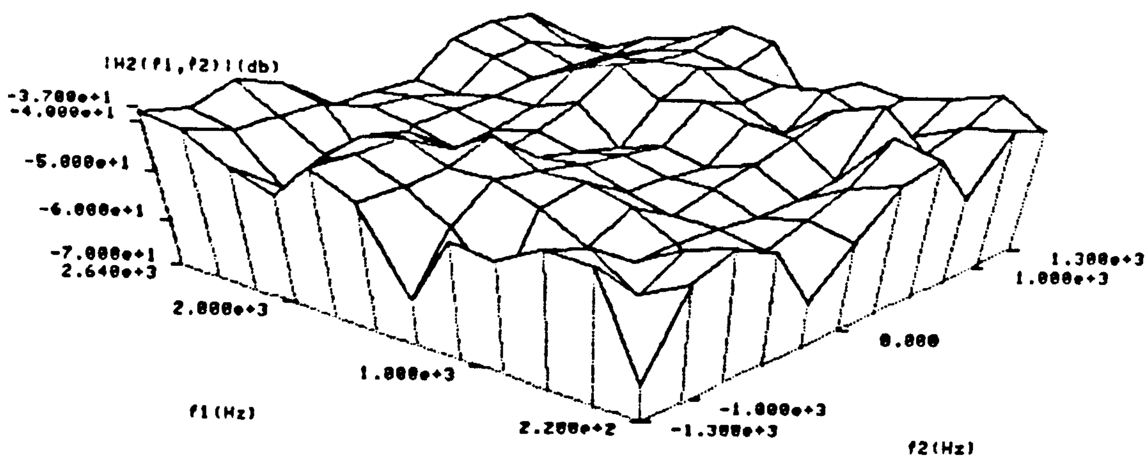
|H2(f1,f2)|(db)
-3.700e+1
-4.000e+1
-5.000e+1
-6.000e+1
-7.000e+1
2.640e+3
2.000e+3
1.000e+3
f1(Hz)
2.200e+2
-1.300e+3
-1.000e+3
0.000
1.000e+3
1.300e+3
f2(Hz)

figure 6

Appendices

Chapter 2 Appendices

A1. Volterra-Like Series

In the study of (linear) convolution operators in engineering it is common to consider only a subalgebra of the bounded measures, for example the subalgebra of measures lacking singular continuous part.[16] This algebra is large enough to capture all of the commonly occurring distributed systems such as distributed transmission lines, transport delays in control systems, etc. Similarly in the study of Volterra series operators only certain types of measures occur in practice; the singular kernel $1(\tau_1)1(\tau_2)e^{-\tau_1}\delta(\tau_1-\tau_2)$ of example 2 of §1 is typical. Sandberg calls series with kernels of this form *Volterra-Like;*[3] the idea occurs as early as 1953 in L. Zadeh's paper.[55]

In a Volterra-like series we index the series not by the order n but by a multi-index $\vec{n}=(n_1,\ldots n_k)$ $(n_j>0)$. k is called the *length* of $\vec{n}$; the *degree* of $\vec{n}$ is defined by $\partial\vec{n} = n_1+\ldots+n_k$.

$$Nu(t) = \sum_{\vec{n}} y_{\vec{n}}(t)$$

$$y_{\vec{n}}(t) = \int\cdots\int h_{\vec{n}}(\tau_1,\ldots\tau_k)u(t-\tau_1)^{n_1}\ldots u(t-\tau_k)^{n_k}\,d\tau_1\ldots d\tau_k$$

where now the kernels $h_{\vec{n}}$ are ordinary $\mathbf{L}^1$ *functions* instead of bounded measures. Each Volterra-like kernel $h_{\vec{n}}$ can be turned into an equivalent Volterra kernel $h_{[\vec{n}]}$ by:

$$h_{[\vec{n}]}(\tau_1,\ldots\tau_n) \triangleq \mathbf{SYM}\, h_{\vec{n}}(\tau_1,\tau_{n_1+1},\ldots\tau_{n-n_k+1})\delta(\tau_1-\tau_2)\ldots\delta(\tau_{n_1-1}-\tau_{n_1})\ldots\delta(\tau_{n-1}-\tau_n)$$

We call $h_{[\vec{n}]}$ the *associated Volterra kernel* of the Volterra-like kernel $h_{\vec{n}}$. Collecting the associated Volterra kernels by degree

$$h_n = \sum_{\partial\vec{n}=n} h_{[\vec{n}]} \tag{A1.1}$$

yields a Volterra series equivalent to the Volterra-like series. Via this associated Volterra series, Volterra-like series inherit the concepts of gain bound function and radius of convergence.

Note that $h_{[\vec{n}]}$ is supported on the k-dimensional set given by†

† We appeal to the reader's intuitive notion of dimension.

$$C_{\vec{n}} = \left\{ (\tau_1,...\tau_n)^T \;\middle|\; n_1 \text{ of the } \tau'\text{s are } x_1, \cdots n_k \text{ of the } \tau'\text{s are } x_k \right\}$$

Thus the associated kernel is singular (with respect to Lebesgue measure) unless $\vec{n} = (1,...1)$.

We extend the notion of **SYM** to Volterra-like series by:

$$\mathbf{SYM} h_{\vec{n}}(\tau_1,...\tau_k) = \frac{1}{k!} \sum_{\sigma \in S^k} h_{\sigma\vec{n}}(\tau_{\sigma 1},...\tau_{\sigma k})$$

where $\sigma\vec{n} = (n_{\sigma 1},...n_{\sigma k})$; we say $h_{\vec{n}}$ is symmetric if $\mathbf{SYM} h_{\vec{n}} = h_{\vec{n}}$. This agrees with our earlier notation if we think of the old order n as the n-long multi-index $(1,...1)$, since $\sigma(1,...1)=(1,...1)$. Note that $\mathbf{SYM} h_{\vec{n}}$ involves not just the Volterra-like kernel $h_{\vec{n}}$ but all Volterra-like kernels of the form $\vec{m} = \sigma\vec{n}$. We say $\vec{m}$ and $\vec{n}$ have the same *type* in this case. A Volterra-like series thus has $\mathbf{P}(n)$ different types of kernels of degree n, where $\mathbf{P}(n)$ is the number of partitions of n.† If the Volterra-like series is symmetric then the kernels of the same type have identical associated kernels and are simply related by:

$$h_{\vec{m}}(\tau_1,...\tau_k) = h_{\sigma\vec{n}}(\tau_1,...\tau_k) = h_{\vec{n}}(\tau_{\sigma 1},...\tau_{\sigma k})$$

This extension of **SYM** will also be useful in the study of multi-input Volterra series.

Theorem A1.1 (Uniqueness theorem for Volterra-like series):

Suppose N and M are Volterra-Like series operators with kernels $h_{\vec{n}}$ and $g_{\vec{n}}$, respectively. Then $N = M$ iff $\mathbf{SYM} h_{\vec{n}} = \mathbf{SYM} g_{\vec{n}}$ for all $\vec{n}$.

Proof: The "if" part is clear. By the Uniqueness theorem (theorem 5.2) we know $h_n = g_n$, where h_n and g_n are the kernels of the associated Volterra series (given in (A1.1) above). We will finish the proof by showing that h_n determines the Volterra-like kernels $\mathbf{SYM} h_{\vec{n}}$.

Theorem A1.2 (Decomposition theorem for Volterra-like series):

Suppose h_n are the kernels of the Volterra series associated with a Volterra-like series with kernels $h_{\vec{n}}$. Then h_n uniquely determines the Volterra-like kernels $\mathbf{SYM} h_{\vec{n}}$.

Thus if a Volterra series comes from a Volterra-like series, then each kernel can be uniquely decomposed into the 2^{n-1} symmetric Volterra-like kernels with which it is associated. Another

† There is no nice formula for $\mathbf{P}(n)$. For those interested it is asymptotic to $(4\sqrt{3}n)^{-1} \exp \pi\sqrt{2n/3}$.

way to think of the Decomposition theorem is: the (linear) map of the symmetric Volterra-like kernels into the associated Volterra kernels (given by formula (A1.1)) is *injective.*

Before starting the proof, let us consider a simple example which illustrates the idea. The second kernel of the associated Volterra series is:

$$h_2(\tau_1,\tau_2) = h_{[(1,1)]} + h_{[(2)]} = h_{(1,1)}(\tau_1,\tau_2) + \frac{1}{2}h_{(2)}(\tau_1)\delta(\tau_1-\tau_2) + \frac{1}{2}h_{(2)}(\tau_2)\delta(\tau_1-\tau_2)$$

Decomposing h_2 is easy: the terms $h_{[(2)]}$ and $h_{[(1,1)]}$ are *mutually singular measures* (The first is supported on the line $\{\tau_1=\tau_2\}$ and the second is absolutely continuous). To be quite explicit we have the formulas:

$$h_{(1,1)}(\tau_1,\tau_2) = h_2(\tau_1,\tau_2) \qquad \text{for} \quad \tau_1 \neq \tau_2$$

$$h_{(2)}(\tau) = \lim_{\epsilon\to 0}\frac{1}{2\sqrt{2}\epsilon}\int_{\tau-\epsilon}^{\tau+\epsilon}\int_{\tau-\epsilon}^{\tau+\epsilon} h_2(\tau_1,\tau_2)\,d\tau_1 d\tau_2$$

The proof of the Decomposition theorem uses the same idea: the associated kernels of $h_{\vec{n}}$ and $h_{\vec{m}}$ are mutually singular unless $\vec{n}$ and $\vec{m}$ are of the same type. To prove this, note that the associated kernel of $h_{\vec{n}}$ has all its mass in the set

$$C^*_{\vec{n}} = \left\{ (\tau_1,...\tau_n)^T \;\middle|\; n_1 \text{ of the } \tau'\text{s are } x_1,...n_k \text{ of the } \tau'\text{s are } x_k\,;\; x_i \text{ are distinct} \right\}$$

This is no more than the assertion that

$$\int\cdots\int h_{\vec{n}}(\tau_1,...\tau_k)u(t-\tau_1)^{n_1}...u(t-\tau_k)^{n_k}d\tau_1...d\tau_k = \int_{\tau_i \text{ distinct}}\cdots\int h_{\vec{n}}(\tau_1,...\tau_k)u(t-\tau_1)^{n_1}...u(t-\tau_k)^{n_k}d\tau_1...d\tau_k$$

(remember that $h_{\vec{n}}$ is an $\mathbf{L}^1$ function).

The sets $C^*_{\vec{n}}$ and $C^*_{\vec{m}}$ are disjoint if $\vec{n}$ and $\vec{m}$ are of different type, and equal if the types are the same. This establishes the claim that the associated kernels are mutually singular unless the multi-indices are of the same type. The $\mathbf{L}^1$ function $h_{\vec{n}}(\tau_1,...,\tau_k)$ is determined by the integrals

$$\int\cdots\int h_{\vec{n}}(\tau_1,...,\tau_k)u_1(\tau_1)...u_k(\tau_k)d\tau_1...d\tau_k \tag{A1.2}$$

where the u_i, $i=1,...,k$ are in $\mathbf{L}^\infty$. According to the discussion above we have

$$\int_{C^*_n}\cdots\int h_n(\tau_1,...,\tau_n)u_1(\tau_1)...u_1(\tau_{n_1})u_2(\tau_{n_1+1})...u_k(\tau_n)d\tau_1...d\tau_n$$

$$= K\int\cdots\int h_{\vec{n}}(\tau_1,...,\tau_k)u_1(\tau_1)...u_k(\tau_k)d\tau_1...d\tau_k$$

where K is the number of Volterra-like kernels with the same type as $\vec{n}$. Thus the integrals (A1.2), and hence the function $h_{\vec{n}}$, are determined by h_n. This proves the Decomposition theorem.

Remark: The Decomposition theorem is not so obvious as it might seem. For example consider the consequence that (nonzero) operators of the form

$$y(t) = \int\int h_{(2,2)}(\tau_1,\tau_2)u(t-\tau_1)^2 u(t-\tau_2)^2 d\tau_1 d\tau_2$$

can never be put in the form

$$y(t) = \int\int h_{(1,3)}(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)^3 d\tau_1 d\tau_2$$

This is so even though the associated kernels are both supported on two-dimensional sets. The frequency domain version of the example above is: Suppose $H_{(2,2)}(s_1,s_2)$ and $H_{(1,3)}(s_1,s_2)$ are the Laplace transforms of symmetric functions in $\mathbf{L}^1(\mathbf{R}_+^2)$. The Decomposition theorem says we can extract $H_{(2,2)}$ and $H_{(1,3)}$ from the fourth order frequency domain kernel

$$H_4(s_1,\ldots s_4) = \mathbf{SYM}\left\{H_{(2,2)}(s_1+s_2, s_3+s_4) + H_{(1,3)}(s_1, s_2+s_3+s_4)\right\}$$

(which has nine terms!) There are explicit formulas which effect this decomposition, but we will not give them here.

Corollary A1.3: If $h_{\vec{n}}$ are symmetric, then

$$||h_n|| = \sum_{\partial\vec{n}=n} ||h_{\vec{n}}||$$

Thus the gain bound function, which we originally defined via the associated Volterra series, is simply given by:

$$f(x) = \sum_{\vec{n}} ||h_{\vec{n}}|| x^{\partial\vec{n}}$$

A2. Incremental gain theorem for $\mathbf{L}^p$

To demonstrate the difficulty of a theory of Volterra series operators for $\mathbf{L}^p$, $p<\infty$, which is unadulterated by reference to $||u||_\infty$, consider just the memoryless operator $Nu(t) = f(u(t))$. If N is to be defined on any open subset of $\mathbf{L}^p$ then we must have $\mathrm{Rad}\,N = \rho = \infty$. It is not hard to show that N maps $\mathbf{L}^p$ back into $\mathbf{L}^p$ if and only if f is sector bounded, i.e. $|f(x)| \le K|x|$.

Sandberg has recently shown that if N has a Frechet derivative at 0 (as an operator from $\mathbf{L}^p$ into $\mathbf{L}^p$) then f is in fact *linear*![7]

We now give the proof of

lemma 3.4:

$$||N(u+v)-Nu||_p \leq ||v||_p \frac{f(||u||+||v||)-f(||u||)}{||v||} \leq ||v||_p f'(||u||+||v||)$$

(Remember that unmarked norms are ∞-norms).

Proof: The conclusion is, if anything, sharpened, if we assume the kernels are symmetric (see §5) so we will assume they are. Then:

$$(N(u+v)-Nu)(t) = \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1,\ldots,\tau_n)\left\{\prod_{i=1}^{n}(u+v)(t-\tau_i)-\prod_{i=1}^{n}u(t-\tau_i)\right\}d\tau_1\ldots d\tau_n =$$

$$= \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1,\ldots,\tau_n)\sum_{k=1}^{n}\binom{n}{k}\prod_{i=1}^{k}v(t-\tau_i)d\tau_i \prod_{i=k+1}^{n}u(t-\tau_i)d\tau_i$$

Thus

$$|N(u+v)-Nu|(t) \leq$$

$$\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{n}{k}||v||^{k-1}||u||^{n-k}\int\left\{\int\cdots\int|h_n(\tau_1,\ldots,\tau_n)|\,d\tau_2\ldots d\tau_n\right\}|v(t-\tau_1)|\,d\tau_1$$

As in theorem 3.3 the bracketed expression is a measure in τ_1 with norm $||h_n||$, so we have[16]

$$||N(u+v)-Nu||_p \leq ||v||_p\sum_{n=1}^{\infty}||h_n||\sum_{k=1}^{n}\binom{n}{k}||v||^{k-1}||u||^{n-k} = ||v||_p\frac{f(||u||+||v||)-f(||u||)}{||v||}$$

The last inequality in the conclusion of lemma 3.4 follows from the mean value theorem.

A3. Taylor Series Which Aren't Volterra Series

In §5 we showed that the Volterra series operators are simply Taylor series of TI operators $:\mathbf{L}^\infty \to \mathbf{L}^\infty$, but noted that the Volterra series are not *all* of the Taylor series. In this section we discuss this point in more detail.

Much of the theory of Volterra series holds for the more general Taylor series

$$Nu = \sum_{n=1}^{\infty} P_n(u) = \sum_{n=1}^{\infty} M_n(u,\ldots,u)$$

where M_n is the bounded TI n-linear map $:\mathbf{L}^\infty \to \mathbf{L}^\infty$ given by $M_n = (n!)^{-1}D^{(n)}N(0)$. With the gain bound function $f(x)=\sum||M_n||x^n$ only notational changes are required to prove all the results of §7-§9. For example, such an N has a Taylor series inverse near 0 if and only if M_1 is invertible.

The differences between our formulation of Volterra series and a more general formulation based on Taylor series are:

(I) Not all bounded TI n-linear maps $:\mathbf{L}^{\infty^n} \rightarrow \mathbf{L}^\infty$ have a convolution representation

$$M_n(u_1,...u_n) = \int \cdots \int h_n(\tau_1,...,\tau_n)u_1(t-\tau_1)...u_n(t-\tau_n)d\tau_1...d\tau_n \tag{A3.1}$$

with $h_n \in \mathbf{B}(\mathbf{R}_+^n)$.

(II) The norm we use, $||h_n||$, is *not* equivalent to the norm $||\ ||_{ML}$ on $L_n(\mathbf{L}^\infty,\mathbf{L}^\infty)$, it is stronger (larger). That is (with some abuse of notation)

$$||h_n||_{ML} \triangleq \sup_{||u_i||<1} ||\int...\int h_n(\tau_1,...,\tau_n)u(t-\tau_1)...u(t-\tau_n)d\tau_1...d\tau_n|| \leq ||h_n||$$

and the ratio of the two is not bounded away from zero. Indeed we will give an example where the ratio *is* zero.

(I) is true even for $n=1$. We now give an example. Consider the subspace of $\mathbf{L}^\infty$ of those u's with a limit at $t=-\infty$, that is

$$\left\{ u \in \mathbf{L}^\infty \;\middle|\; \lim_{t\rightarrow-\infty} u(t) \text{ exists} \right\}$$

On this subspace we define $F(u) \triangleq \lim_{t\rightarrow-\infty} u(t)$. F is clearly a LTI bounded functional on this subspace. Using the Hahn-Banach theorem and the Axiom of Choice F can be extended to a LTI bounded functional on all of $\mathbf{L}^\infty$, which we denote LIM (see Kantorovich[56,p58] or Rudin[11]; see also §A3). LIM can also be thought of as a bounded LTI operator $:\mathbf{L}^\infty \rightarrow \mathbf{L}^\infty$ (though its range is just the constants).

For any u which vanishes for $t<0$ we have LIM$u=0$. This establishes that LIM is *causal,* and that LIM has no representation as a convolution with a measure. It also shows that the Steady State theorem does not hold for LIM. To mention just one more bizarre property of LIM, it is a bounded LTI operator which maps *sinusoids* to *constants*!

Clearly this example is absurd from an engineering point of view. LIM's perfect memory of the *infinitely remote past* (and indeed, total amnesia for the *finite past*) contradicts our intuition that bounded LTI physical devices and systems should have a *fading memory* (see chapter 4).†

Let us now give an example of (II). For $n>1$ $\prod_{i=1}^{n}\text{LIM}u_i$ furnishes an example of a bounded multilinear operator not given by a convolution as in (A3.1). Less bizarre examples can also be given for $n>1$. For example we can have a convolution representation with h_n an *unbounded measure.*†† Consider the kernel

$$h_2(\tau_1,\tau_2) = 1(\tau_1)1(\tau_2)\frac{\cos(\tau_1\tau_2)}{(1+\tau_1)(1+\tau_2)}$$

Then $\|h_2\| = \int|h_2(\tau_1,\tau_2)|\ d\tau_1 d\tau_2 = \infty$. Nevertheless this kernel induces a bounded bilinear map $:\mathbf{L}^{\infty 2}\rightarrow\mathbf{L}^{\infty}$. First we have to say what we mean by the convolution since the integral in (A3.1) is not absolutely convergent with this h_2. We mean

$$M_2(u_1,u_2)(t) \triangleq \lim_{T\rightarrow\infty}\int_0^T\int_0^T h_2(\tau_1,\tau_2)u_1(t-\tau_1)u_2(t-\tau_2)d\tau_1 d\tau_2$$

To see that this limit exists and that M_2 is bounded, we rewrite this as

$$= \lim_{T\rightarrow\infty}\int_0^\infty\left\{\frac{u_2(t-\tau_2)}{1+\tau_2}1(T-\tau_2)\right\}\left\{\text{Re}\int_0^T e^{-i\tau_1\tau_2}\frac{u_1(t-\tau_1)}{1+\tau_1}d\tau_1\right\}d\tau_2 \tag{A3.2}$$

As $T\rightarrow\infty$ the lefthand bracketed expression in (A3.2) converges in $\mathbf{L}^2$ to the $\mathbf{L}^2$ function $1(\tau_2)u_2(t-\tau_2)/(1+\tau_2)$; by the Plancherel theorem the righthand bracketed expression in (A3.2) converges in $\mathbf{L}^2$ to the $\mathbf{L}^2$ function

$$\text{Re}\left[\frac{u_1(t-\cdot)}{1+(\cdot)}\right]^{\wedge}(\tau_2)$$

where by $\hat{f}$ we mean here the Plancherel transform of $f\in\mathbf{L}^2$. Consequently the limit in (A3.2) exists and is bounded by

$$|M_2(u_1,u_2)(t)| \le \left\|\frac{u_2(\cdot)}{1+(\cdot)}\right\|_2\left\|\text{Re}\left(\frac{u_1(\cdot)}{1+(\cdot)}\right)^{\wedge}\right\|_2$$

† Moral: don't fiddle with the Axiom of Choice.

†† In the literature this is often stated: "$\int\cdots\int|h_n(\tau_1,\ldots,\tau_n)|\,d\tau_1\ldots d\tau_n<\infty$ is a sufficient but not necessary

$$\leq \frac{2\sqrt{2\pi}}{3}||u_1||_\infty||u_2||_\infty$$

which establishes $||M_2||_{ML} \leq 2\sqrt{2\pi}/3$. This example was suggested by D. J. Newman. Like the first example LIM above, it is rather forced.

There are thus at least three costs associated with generalizing Volterra Series operators to arbitrary Taylor series:

(1) We lose the concrete convolution representation (A3.1);

(2) The norm $||h_n|| = \int \cdots \int |h_n| \, d\tau_1 \ldots d\tau_n$ is replaced by $||M_n||_{ML}$ which is nearly impossible to compute;

(3) We include clearly nonphysical operators such as LIM.

It is the authors' feeling, and we hope the examples above have convinced the reader, that the mathematical elegance and completeness of a general Taylor series formulation is not worth (1)-(3).

Chapter 3 Appendices

A4. Absolute Convergence of the Inner Sum

In §5 we established the Fundamental Frequency Domain Formula under the hypothesis that

$$\sum_{\{k_1+\cdots+k_n=m\}} |\hat{u}(k_1)\ldots\hat{u}(k_n)H_n(j\omega k_1,\ldots,j\omega k_n)| \tag{A4.1}$$

be finite. In this section we give two simple conditions which ensure that (A4.1) is finite, the first a condition on the input signal u, and the second a condition on the kernel H_n.

A4.1 Conditions on the Input Signal

We seek conditions which ensure that

$$\sum_{\{k_1+\cdots+k_n=m\}} |\hat{u}(k_1)\ldots\hat{u}(k_n)| \tag{A4.2}$$

condition for BIBO stability of a second order Volterra operator." An incorrect example is given in[57].

is finite. This of course implies that (A4.1) is finite, since $|H_n| \le ||h_n||$. Note that (A4.2) is simply (A4.1) when N is the simplest possible n-order operator: the memoryless n-power law device $Nu(t)=u(t)^n$.

Since $u \in \mathbf{L}^\infty[0,2\pi\omega^{-1}]$, $u \in \mathbf{L}^2[0,2\pi\omega^{-1}]$, so $\hat{u} \in \mathbf{l}^2$. Thus for $n=2$ (A4.2) is just a convolution of two sequences in $\mathbf{l}^2$ and thus is finite by the Cauchy-Schwarz inequality:

$$\sum_{k_1+k_2=m} |f(k_1)g(k_2)| = \sum_{k=1}^{\infty} |f(k)|\,|g(m-k)| \le ||f||_2 ||g||_2 \tag{A4.3}$$

Since the convolution of two $\mathbf{l}^2$ sequences is not, in general, in $\mathbf{l}^2$, the finiteness of (A4.2) already is dubious for $n=3$. On the other hand if $\hat{u} \in \mathbf{l}^1$, then convolution iterates of u make sense and are still in $\mathbf{l}^1$: (A4.2) is then bounded by $||\hat{u}||_1^n$.

It is a remarkable fact that for most u (A4.2) *is* finite, even when $\hat{u}$ is not in $\mathbf{l}^1$. It is not true for all $u \in \mathbf{L}^\infty[0,2\pi\omega^{-1}]$, $\cos(1/t)$ is a counterexample.†

Theorem A4.1.1 Suppose that $\hat{u}(k) = O(1/k)$. Then (A4.2) is finite, that is

$$\left\{\sum_{k_1+\cdots+k_n=m}\right\} |\hat{u}(k_1)\ldots\hat{u}(k_n)| < \infty$$

Proof: Suppose that $\hat{u}(k) = O(1/k)$. Then there is a constant β such that $|\hat{u}(k)| \le \beta\hat{v}(k)$ where

$$\hat{v}(k) \triangleq \begin{cases} 1 & k=0 \\ 1/|k| & k\neq 0 \end{cases}$$

Since $\hat{v} \in \mathbf{l}^2$, it is indeed the Fourier series of some $\mathbf{L}^2$ function which we will call, surprisingly enough, v. In fact

$$v(t) = 1 - \log 2 - \log(1-\cos t)$$

the verification of which we will spare the reader.

Now

$$\left\{\sum_{k_1+\cdots+k_n=m}\right\} |\hat{u}(k_1)\ldots\hat{u}(k_n)| \le \beta^n \left\{\sum_{k_1+\cdots+k_n=m}\right\} \hat{v}(k_1)\ldots\hat{v}(k_n) \tag{A4.4}$$

so it will suffice to show that the righthand side of (A4.4) is finite. We break up the proof of this

† D. J. Newman, personal communication.

into three lemmas:

Lemma 1: Suppose f and g are in $\mathbf{L}^2$. Then $(fg)^\wedge = \hat{f} * \hat{g}$.

Even though this is well known we give a short proof here for completeness.

Proof: We have already seen in equation (A4.3) that the convolution $\hat{f} * \hat{g}$ converges absolutely. Recall that (Plancherel theorem)

$$\| g - \sum_{k=-M}^{M} \hat{g}(k) e^{j\omega k t} \|_2 \to 0 \quad \text{as} \quad M \to \infty \tag{A4.5}$$

By the Cauchy-Schwarz inequality

$$\left| \frac{w}{2\pi} \int_0^{2\pi\omega^{-1}} f(t)(g(t) - \sum_{k=-M}^{M} \hat{g}(k) e^{j\omega k t}) e^{-j\omega m t} \, dt \right| \le \|f\|_2 \, \| g - \sum_{k=-M}^{M} \hat{g}(k) e^{j\omega k t} \|_2 \tag{A4.6}$$

By (A4.5) the righthand side of (A4.6), and therefore the lefthand side of (A4.6), converges to 0 as $M \to \infty$. But the lefthand side of (A4.6) is just

$$\left| (fg)^\wedge(m) - \sum_{k=-M}^{M} \hat{g}(k) \hat{f}(m-k) \right|$$

Letting $M \to \infty$ yields the conclusion. □

Lemma 2: $v(t)^n \in \mathbf{L}^1$ for all n. (That is, $v \in \mathbf{L}^p$ for all $p < \infty$).

Proof: Clearly we need only worry about the singularity at $t=0$, that is $v(t)^n \in \mathbf{L}^1$ if and only if $(\log(1-\cos t))^n$ is integrable near $t=0$. This is true iff $(\log t)^n$ is integrable near $t=0$, which is true since

$$\int_\epsilon^1 |\log t|^n \, dt = \int_0^{-\log \epsilon} e^{-x} x^n \, dx \le n!$$

which establishes lemma 2. □

Lemma 3:

$$\left\{ \sum_{k_1 + \cdots + k_n = m} \right\} \hat{v}(k_1) \ldots \hat{v}(k_n) = (v^n)^\wedge(m) \tag{A4.7}$$

Proof: By induction on n. Suppose we have established (A4.7) for n. By lemma 2 v^n and v are in $\mathbf{L}^2$, so applying lemma 3.1 we have $(v^{n+1})^\wedge = (v^n)^\wedge * \hat{v}$; using the inductive hypothesis

$$(v^{n+1})\hat{}(\overline{m}) = \sum_m \left\{ \sum_{k_1+ \cdots +k_n=m} \right\} \hat{v}(k_1)\ldots\hat{v}(k_n)\hat{v}(\overline{m}-m) = \left\{ \sum_{k_1+ \cdots +k_n=\overline{m}} \right\} \hat{v}(k_1)\ldots\hat{v}(k_{n+1})$$

the change of order valid *since the summand is positive* (Fubini theorem). This completes the proof of lemma 3. □

We can now finish the proof of theorem A4.1.1. From (A4.4), (A4.7), and lemma 2 we have

$$\left\{ \sum_{k_1+ \cdots +k_n=m} \right\} |\hat{u}(k_1)\ldots\hat{u}(k_n)| \le \beta^n (v^n)\hat{}(m) \le ||v^n||_1 < \infty$$

establishing theorem A4.1.1. □

One useful condition which implies $\hat{u}(n) = O(1/n)$ is that u have bounded variation over one period.

Lemma 5.1: Suppose u has bounded variation over one period. Then

$$\left\{ \sum_{k_1+ \cdots +k_n=m} \right\} |\hat{u}(k_1)\ldots\hat{u}(k_n)| < \infty$$

Proof: If u has bounded variation over one period then $\hat{u}(n) = O(1/n)$[24] (the proof is essentially integrating by parts the formula for $\hat{u}(n)$) and thus theorem A4.1.1 proves lemma 5.1. □

A4.2 Conditions on the Kernel H_n

Lemma 5.2: Suppose that $H_n(j\omega k_1,\ldots,j\omega k_n) = O(\frac{1}{k_1\ldots k_n})$. Then (A4.1) is finite, that is:

$$\left\{ \sum_{k_1+ \cdots +k_n=m} \right\} |\hat{u}(k_1)\ldots\hat{u}(k_n) H_n(j\omega k_1,\ldots,j\omega k_n)| < \infty$$

Proof: Suppose $H_n(j\omega k_1,\ldots,j\omega k_n) = O(1/k_1\ldots k_n)$. Then $H_n(j\omega k_1,\ldots,j\omega k_n) \in l^2(\mathbb{Z}^n)$. Since $\hat{u} \in l^2$, $\hat{u}(k_1)\ldots\hat{u}(k_n) \in l^2(\mathbb{Z}^n)$ with norm $||\hat{u}||_2^n$ so the Cauchy-Schwarz inequality yields

$$\left\{ \sum_{k_1+ \cdots +k_n=m} \right\} |\hat{u}(k_1)\ldots\hat{u}(k_n) H_n(j\omega k_1,\ldots,j\omega k_n)| \le \sum_{k_1,\ldots,k_n} |\hat{u}(k_1)\ldots\hat{u}(k_n) H_n(j\omega k_1,\ldots,j\omega k_n)|$$

$$\le ||\hat{u}(k_1)\ldots\hat{u}(k_n)||_2 ||H_n(j\omega k_1,\ldots,j\omega k_n)||_2 = ||\hat{u}||_2^n ||H_n(j\omega k_1,\ldots,j\omega k_n)||_2$$

which proves lemma 5.2. □

A5. Almost Periodic Inputs

Recall that τ is said to be an ϵ-translation number for u if $\|u(\cdot)-u(\cdot+\tau)\| \leq \epsilon$. u is *almost periodic* if for all $\epsilon>0$ there is an L such that all L-long intervals contain at least one ϵ-translation number for u. Formally

$$\forall\epsilon>0\,\exists L\ \forall a\ \exists\tau(\ a<\tau<a+L \ \text{ and } \ \|u(\cdot)-u(\cdot+\tau)\| \leq \epsilon\ )$$

These definitions and a concise discussion can be found in Wiener[58] or Corduneanu[59].

Theorem A5.1: Suppose u is almost periodic and $\|u\|<\rho=\mathrm{Rad}N$. Then Nu is almost periodic.

This extends some results of Sandberg, who established theorem A5.1 under the assumption that u has an absolutely convergent Fourier series with small enough coefficients.[7]

Proof: Let $\epsilon>0$. Choose r with $\|u\| < r < \rho$. By the Incremental Gain theorem (theorem 3.2 of chapter 2) there is a K such that on B_r $\|Nu-Nv\| \leq K\|u-v\|$. For any τ, $\|u(\cdot+\tau)\| \leq r$, hence if τ is an ϵ-translation number for u then

$$\|Nu(\cdot)-Nu(\cdot+\tau)\| \leq K\|u(\cdot)-u(\cdot+\tau)\| \leq K\epsilon$$

so τ is a $K\epsilon$-translation number for Nu.

Now to finish the proof: Since u is almost periodic find L such that all L-long intervals contain at least one ϵ/K-translation number for u. From the discussion above these translation numbers are ϵ-translation numbers for Nu, thus Nu is almost periodic. □

Remark: It is not hard to show that *any continuous time-invariant operator from* $\mathbf{L}^\infty$ *into* $\mathbf{L}^\infty$ *maps almost periodic functions into almost periodic functions.* To see this, we first give a modern (less concrete) definition of almost periodic functions: u is almost periodic iff it is continuous and the set of its translates $\{u(\cdot-t)\}$ is compact in $\mathbf{L}^\infty$.[59] Now if u is almost periodic and N is time-invariant and continuous $:\mathbf{L}^\infty \to \mathbf{L}^\infty$, the translates of Nu are $\{Nu(\cdot-t)\}$, which, being the continuous image of a compact set, is compact. Hence Nu is almost periodic.

We will now establish the analogous fundamental formula for almost periodic inputs.

Theorem A5.2 (Fundamental frequency domain formula for Almost Periodic Inputs):

Suppose that u is almost periodic and $||u|| < \rho = \mathrm{Rad}N$, and in addition

$$\sum_{\{\omega_{k_1}+\cdots+\omega_{k_n}=\omega\}} |\hat{u}(\omega_{k_1})\ldots\hat{u}(\omega_{k_n})H_n(j\omega_{k_1},\ldots,j\omega_{k_n})| < \infty \tag{A5.1}$$

Then for any $\omega \in \mathbf{R}$

$$(Nu)^\wedge(\omega) = \sum_{n=1}^{\infty} \sum_{\{\omega_{k_1}+\cdots+\omega_{k_n}=\omega\}} \hat{u}(\omega_{k_1})\ldots\hat{u}(\omega_{k_n})H_n(j\omega_{k_1},\ldots,j\omega_{k_n}) \tag{A5.2}$$

Proof: Due to the similarity to the case of periodic inputs, we give a shortened proof. As in §3 we first assume that the input has the form

$$u(t) = \sum_{k=-M}^{M} \alpha_k e^{j\omega_k t}$$

We will call such a u a *multitone signal.* It is easily verified that for multitone signals

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)e^{-j\nu t}\,dt = \begin{cases} \alpha_k & \nu = \omega_k \\ 0 & \text{otherwise} \end{cases} \tag{A5.3}$$

The limit in (A5.3), which can be shown to exist for any almost periodic function and any $\nu \in \mathbf{R}$, is denoted $\hat{u}(\nu)$. The same argument as in §3 establishes

$$(Nu)^\wedge(\omega) = \sum_{n=1}^{\infty} \sum_{\{\omega_{k_1}+\cdots+\omega_{k_n}=\omega\}} \hat{u}(\omega_{k_1})\ldots\hat{u}(\omega_{k_n})H_n(j\omega_{k_1},\ldots,j\omega_{k_n}) \tag{A5.4}$$

for the case of u a multitone signal. We now use the fact that almost periodic functions *are precisely the uniform limits of multitone signals.*[58] Thus there is a sequence of multitone signals u_M with $||u_M|| < \rho$ and $u_M \to u$ uniformly as $M \to \infty$. By the Incremental Gain theorem $Nu_M \to Nu$ uniformly as $M \to \infty$. Hence for any ν in $\mathbf{R}$ $(Nu_M)^\wedge(\nu) \to (Nu)^\wedge(\nu)$. Since formula (A5.4) above holds for multitone signals we have

$$(Nu)^\wedge(\omega) = \sum_{n=1}^{\infty} \lim_{M\to\infty} \sum_{\{\omega_{k_1}+\cdots+\omega_{k_n}=\omega\}} \hat{u_M}(\omega_{k_1})\ldots\hat{u_M}(\omega_{k_n})H_n(j\omega_{k_1},\ldots,j\omega_{k_n}) \tag{A5.5}$$

Since $u_M \to u$ uniformly, $\hat{u_M}(\omega) \to \hat{u}(\omega)$ uniformly. Dominated convergence and hypothesis (A5.1) yield

$$= \sum_{n=1}^{\infty} \sum_{\{\omega_{k_1}+\cdots+\omega_{k_n}=\omega\}} \hat{u}(\omega_{k_1})\ldots\hat{u}(\omega_{k_n})H_n(j\omega_{k_1},\ldots,j\omega_{k_n}) \tag{A5.6}$$

which is the conclusion of theorem A5.2. □

Chapter 4 Appendices

A6. Proof of Lemma 3.2

We must show that

$$K_- = \left\{ u \in \mathbf{C}(\mathbf{R}_-) \,\middle|\, |u(t)| \le M_1,\ |u(s) - u(t)| \le M_2(s-t) \text{ for } t \le s \le 0 \right\}$$

is compact with the weighted norm $||\cdot||_w$ in $\mathbf{C}(\mathbf{R}_-)$. Let u_n, $n=1,2,...$ be any sequence in K_-. We will find a $u_0 \in K_-$ and a subsequence of $\{u_n\}$ converging in the $||\cdot||_w$ norm to u_0, which will establish Lemma 3.2.

Let $K_-[-n,0]$ denote K_- restricted to $[-n,0]$, that is

$$K_-[-n,0] \triangleq \left\{ u \in \mathbf{C}[-n,0] \,\middle|\, |u(t)| \le M_1,\ |u(s) - u(t)| \le M_2(s-t) \text{ for } -n \le t \le s \le 0 \right\}$$

For each n, $K_-[-n,0]$ is uniformly bounded (by M_1) and equicontinuous (by the slew-rate limit M_2), hence compact in $\mathbf{C}[-n,0]$ by the Arzela-Ascoli theorem (see e.g. Dieudonne[18]). Since $K_-[-1,0]$ is compact in $\mathbf{C}[-1,0]$, we can find a $u_0^{(1)} \in K_-[-1,0]$ and an infinite subset $\mathbb{N}_1 \subset \mathbb{N}$ such that

$$\sup_{-1 \le t \le 0} |u_n(t) - u_0^{(1)}(t)| \to 0 \text{ as } n \to \infty,\ n \in \mathbb{N}_1$$

Viewing $\{u_n \mid n \in \mathbb{N}_1\}$ as a sequence in $K_-[-2,0]$, we conclude that there is a $u_0^{(2)} \in K_-[-2,0]$ and an infinite subset $\mathbb{N}_2 \subset \mathbb{N}_1$ such that

$$\sup_{-2 \le t \le 0} |u_n(t) - u_0^{(2)}(t)| \to 0 \text{ as } n \to \infty,\ n \in \mathbb{N}_2$$

Clearly $u_0^{(2)}$ extends $u_0^{(1)}$, that is, $u_0^{(1)}(t) = u_0^{(2)}(t)$ for $-1 \le t \le 0$.

Continuing in this way we find a $u_0 \in K_-$ and a decreasing sequence of infinite subsets $\mathbb{N} \supset \mathbb{N}_1 \supset \cdots$ such that for each k

$$\sup_{-k \le t \le 0} |u_n(t) - u_0(t)| \to 0 \text{ as } n \to \infty,\ n \in \mathbb{N}_k \tag{A6.1}$$

We now choose any increasing subsequence n_k such that $n_k \in \mathbb{N}_k$. Then from (A6.1) we have for each k_0

$$\sup_{-k_0 \leq t \leq 0} |u_{n_k}(t) - u_0(t)| \to 0 \text{ as } k \to \infty$$

that is, the sequence u_{n_k} converges to u_0 unformly on compact subsets.

Now we claim that u_{n_k} converges to u_0 in the weighted norm, that is, $\lim_{k\to\infty} ||u_{n_k} - u_0||_w = 0$. To prove our claim, let $\epsilon > 0$. Since $w(t) \to 0$ as $t \to \infty$ and w is nonincreasing, we can find $k_0 \in \mathbb{N}$ such that $w(k_0) < \epsilon/2M_1$; since u_{n_k}, $u_0 \in K_-$ we have

$$\sup_{t \leq -k_0} |u_{n_k}(t) - u_0(t)| w(-t) \leq 2M_1 w(k_0) < \epsilon \tag{A6.2}$$

Now use (A6.1) to find k_1 such that

$$\sup_{-k_0 \leq t \leq 0} |u_{n_k}(t) - u_0(t)| \leq \epsilon \quad \text{for } k \geq k_1 \tag{A6.3}$$

From (A6.2), (A6.3) and $w(t) \leq 1$ we conclude

$$||u_{n_k} - u_0||_w \leq \epsilon \quad \text{for } k \geq k_1$$

which concludes the proof of Lemma 3.2. □

A7. Proof of NLMA Approximation Theorem

We start with the discrete-time analog of Lemma 3.2:

Lemma:

$$K_- \triangleq \left\{ u \in l^\infty(\mathbb{Z}_-) \,\middle|\, ||u|| \leq M_1 \right\}$$

is compact with the weighted norm $||\cdot||_w$ given by

$$||u||_w \triangleq \sup_{k \leq 0} |u(k)| w(-k)$$

Proof: We give an abbreviated proof since it is similar to, and in fact simpler than, the proof of lemma 3.2 given in §A6.

Let $\{u^{(n)}\}$ be a sequence in K_-. Since $|u^{(n)}(0)| \leq M_1$, find a subsequence along which $u^{(n)}(0)$ converges; let us call the limit $u^{(0)}(0)$. Now find a subsequence of this subsequence along which $u^{(n)}(-1)$ converges; call this limit $u^{(0)}(-1)$.

Just as in the proof of lemma 3.2 we continue this process, defining the element $u^{(0)} \in K_-$ as we go. Take a diagonal subsequence n_k; $u^{(n_k)}$ converges pointwise to $u^{(0)}$ as $k \to \infty$, and exactly as in lemma 3.2 we can show

$$||u^{(n_k)} - u^{(0)}||_w \to 0 \quad \text{as } k \to \infty$$

which proves that K_- is compact. □

Now consider the set of functionals

$$\mathbf{G} \triangleq \left\{ G_0, G_1, \ldots \right\}$$

where $G_k u \triangleq u(-k)$, that is, G_k is the functional associated with the k-delay operator U_k (transfer function z^{-k}).

It is easy to verify that the G_k's are continuous with respect to the weighted norm $||\cdot||_w$ and that $\mathbf{G}$ separates points in $\mathbf{l}^\infty(\mathbb{Z}_-)$. Applying the Stone-Weierstrass theorem as in theorem 3.1 yields an approximation by a NLMA operator.

A8. Causal Continuous LTI Operator with no Convolution Representation

Here is a brief description of one such operator (see Kantorovich[56,p58] for details). It is possible to find a linear functional $LIM:\mathbf{l}^\infty \to \mathbf{R}$ such that

$$|\text{LIM}u| \leq ||u||$$

and if $\lim_{k \to -\infty} u(k)$ exists, then $\text{LIM}u = \lim_{k \to -\infty} u(k)$. Thus LIM assigns a "pseudo-limit" $LIMu$ to every element of $\mathbf{l}^\infty$ (the vast majority of which do not converge as $k \to -\infty$). Consider the operator $A:\mathbf{l}^\infty \to \mathbf{l}^\infty$ given by

$$Au(n) = \text{LIM}u$$

Thus for every $u \in \mathbf{l}^\infty$, Au is the constant sequence $LIMu$.

A is LTI causal continuous, but has no convolution representation since its response to a unit sample is zero, and yet it is not the zero operator. Note that A is a LTI causal operator which does not have fading memory. Of course, an operator like A is not likely to occur in engineering.

A9. Proof of Theorem 9.1 (Convolution theorem)

We will prove (II), and then indicate some of the changes necessary to prove the continuous-time version (I).

First suppose $Au = h*u$ where $h \in l^1(\mathbb{Z}_+)$. We will show that A has fading memory (that it is LTI causal is clear). Consider the weighting function

$$w(n) \triangleq ||h||_1^{-1/2}\left\{\sum_{k=n}^{\infty}|h(k)|\right\}^{1/2} \tag{A9.1}$$

We claim that A has a w-fading memory. As in (5.1.2) we need only establish

$$S \triangleq \sum_{n=0}^{\infty}|h(n)|w(n)^{-1} < \infty$$

In fact $S \leq 2$, which we now prove. Define

$$\theta(n) \triangleq \sum_{k=n}^{\infty}|h(k)|$$

so that

$$\begin{aligned} S &= \theta(0)^{-1/2}\sum_{n=0}^{\infty}\frac{\theta(n)-\theta(n+1)}{\theta(n)^{1/2}} \\ &= 1 + \theta(0)^{-1/2}\sum_{n=0}^{\infty}\theta(n+1)\left(\theta(n+1)^{-1/2}-\theta(n)^{-1/2}\right) \end{aligned} \tag{A9.2}$$

Since $0 \leq \theta(n+1) \leq \theta(n)$ we have

$$\theta(n+1)\left(\theta(n+1)^{-1/2}-\theta(n)^{-1/2}\right) \leq \theta(n)^{-1/2} - \theta(n+1)^{-1/2} \tag{A9.3}$$

(the ratio of the two is $\sqrt{\theta(n+1)/\theta(n)} \leq 1$). From (A9.2) and (A9.3)

$$S \leq 1 + \theta(0)^{-1/2}\sum_{n=0}^{\infty}\left(\theta(n)^{-1/2}-\theta(n+1)^{-1/2}\right) = 2$$

which proves that A has a w-fading memory.

Remark: If h happens to be exponentially decaying then we may use the weight $w(n) = (1+n)^{-1}$, but of course not all $h \in l^1(\mathbb{Z}_-)$ are exponentially decaying, and then the more complicated weight (A9.1) is necessary.

Now we prove the converse. Let A be any LTI operator with, say, a w-fading memory.†

† This w has nothing to do with the w defined in (A9.1).

Let h be the response of A to a unit sample, i.e. $h(n) \triangleq Ae(n)$ where $e(n) = \delta_{n0}$.

We will show (1) $h \in \mathbf{l}^1(\mathbb{Z}_+)$ (at the moment we know only $h \in \mathbf{l}^\infty(\mathbb{Z}_+)$), and (2) $Au = h*u$ for all $u \in \mathbf{l}^\infty$.

Let F be the functional associated with A via (1.1). Using linearity and FM we conclude there is an $M < \infty$ such that for all $u \in \mathbf{l}^\infty(\mathbb{Z}_-)$

$$|Fu| \leq M\|u\|_w \tag{A9.4}$$

Now for any $u : \mathbb{Z}_- \to \mathbf{R}$ define

$$u_N(k) = \begin{cases} u(k) & -N \leq k \leq 0 \\ 0 & k < -N \end{cases}$$

We now use a standard argument. From time-invariance and linearity we have

$$Fu_N = \sum_{k=0}^{N} h(k)u_N(-k) = \sum_{k=0}^{N} h(k)u(-k) \tag{A9.5}$$

Consider $u(k) \triangleq w(k)^{-1}\mathrm{sign}\,h(k)$; from (A9.4) and (A9.5) we conclude

$$\sum_{k=0}^{N} w(k)^{-1}|h(k)| \leq M$$

for all N and thus $hw^{-1} \in \mathbf{l}^1(\mathbb{Z}_+)$, which implies $h \in \mathbf{l}^1(\mathbb{Z}_+)$.

Now (2): for any $u \in \mathbf{l}^\infty(\mathbb{Z}_-)$ we have from (A9.4)

$$|Fu - Fu_N| \leq M\|u - u_N\|_w \leq Mw(N+1) \to 0 \quad \text{as } N \to \infty$$

Thus (noting that $h(\cdot)u(-\cdot) \in \mathbf{l}^1(\mathbb{Z}_+)$)

$$Fu = \lim_{N\to\infty} Fu_N = \sum_{k=0}^{\infty} h(k)u(-k)$$

which finishes our proof. □

To show that a LTI operator $A : \mathbf{C}(\mathbf{R}) \to \mathbf{C}(\mathbf{R})$ which has a convolution representation (8.1) has a fading memory, we use the weight

$$w(t) \triangleq \left\{\int |h(d\tau)|\right\}^{-1/2} \left\{\int_t^\infty |h(d\tau)|\right\}^{1/2}$$

Then by a change of variables we have

$$\int_0^\infty |h(dt)|w(t)^{-1} = 2$$

so that A has a w-fading memory.

To prove that a LTI FM operator has a convolution representation is technically more involved since we cannot directly apply an impulse input $\delta(t)$. But the idea is the same.

A10. Proof of Theorem 10.1

Assume the hypotheses of theorem 10.1. Since x_0 and $\tilde{x}_0$ are reachable from the origin, let $T \in \mathbb{R}$ and u_s, $\tilde{u}_s \in K$ be such that

$$Nu_s(T) = x_0 \quad N\tilde{u}_s(T) = \tilde{x}_0$$

Thus, u_s and $\tilde{u}_s$ steer x from 0 to x_0 and $\tilde{x}_0$, respectively, over the interval $[0, T]$.

Define

$$v(t) \triangleq \begin{cases} u_s(t) & 0 \le t \le T \\ u(t+T) & t > T \end{cases}$$

and similarly

$$\tilde{v}(t) \triangleq \begin{cases} \tilde{u}_s(t) & 0 \le t \le T \\ u(t+T) & t > T \end{cases}$$

In fact $x(t) = Nv(t+T)$ and $\tilde{x}(t) = N\tilde{v}(t+T)$, since the left-hand and right-hand sides satisfy the same differential equation and agree at $t=0$.

Let $\epsilon > 0$. Using our assumption that N has fading memory, there is a $\delta > 0$ such that for all $t \in \mathbb{R}$

$$\sup_{0 \le \tau \le t} |v(t) - \tilde{v}(t)| w(t-\tau) < \delta \longrightarrow \|Nv(t) - N\tilde{v}(t)\| < \epsilon \tag{A10.1}$$

Since $v(t) = \tilde{v}(t)$ for $t \ge T$,

$$\sup_{0 \le \tau \le t} |v(t) - \tilde{v}(t)| w(t-\tau) \le 2M_1 w(t-T)$$

Using $w(t) \to 0$ as $t \to \infty$, find $T_0 \ge T$ such that $w(T_0) < \delta/(3M_1)$. Then for $t \ge T_0$ the right hand side of (A10.1) is satisfied and hence

$$\|x(t) - \tilde{x}(t)\| < \epsilon \quad \text{for } t \ge T_0$$

which proves theorem 10.1. □

A11.

In this section we give self-contained proofs of two simple facts used in §4 and §8.

Fact 1: Span $\{l_k(t)e^{t/2}\}$ is dense in $\mathbf{L}^2(\mathbf{R}_+)$.

(Recall that $l_k(t)$ is the kth Laguerre function, $\hat{l}_k(s)=\sqrt{2}(1-s)^{k-1}(1+s)^{-k}$.)

Proof: Suppose $h \in \mathbf{L}^2(\mathbf{R}_+)$ and for all k

$$\int l_k(t)e^{t/2}h(t)dt = 0 \tag{A11.1}$$

Now

$$(l_k e^{t/2})^\wedge(s) = \hat{l}_k(s-1/2) = \sqrt{2}\frac{(3/2-s)^{k-1}}{(1/2+s)^k} \tag{A11.2}$$

Using (A11.1), (A11.2), and the Parseval equality we conclude for all k

$$\int \hat{h}(j\omega)\overline{\hat{l}_k(j\omega)}d\omega = \int \hat{h}(j\omega)\frac{(3/2+j\omega)^{k-1}}{(1/2-j\omega)^k}d\omega = 0$$

These integrals are easily evaluated; for example for $k=1$ we have $2\pi\hat{h}(1/2)=0$ and thus $\hat{h}(1/2)=0$ (recall that $h \in \mathbf{L}^2(\mathbf{R}_+)$ and so $\hat{h}$ is analytic in $\mathbf{C}_+$, in fact $\hat{h} \in \mathbf{H}^2(\mathbf{C}_+)$). In general

$$\sum_{n=0}^{k-1}\frac{2^n\hat{h}^{(n+1)}(1/2)}{(n!)^2(k-n-1)!} = 0$$

from which a simple inductive argument yields

$$\hat{h}^{(k)}(1/2) = 0$$

for all k and hence $\hat{h}=0$ in $\mathbf{C}_+$. Thus $h=0$ and Fact 1 is established. □

Fact 2: The span of

$$\{\, t_1^i t_2^j e^{-(t_1+t_2)/2} \mid i,\, j \in \mathbb{Z}_+ \,\} \tag{A11.3}$$

is dense in $\mathbf{L}^2(\mathbf{R}_+^2)$.

Proof: The set (A11.3) is just the tensor products of the functions

$$\{\, t^i e^{-t/2} \mid i \in \mathbb{Z}_+ \,\} \tag{A11.4}$$

so we need only show that the span of (A11.4) is dense in $\mathbf{L}^2(\mathbf{R}_+)$.

Note that for each k

$$l_k(t)e^{t/2} = \sum_{i=0}^{k-1} a_{ik}t^i e^{-t/2}$$

for some coefficients a_{ik}. By Fact 1, then, the span of (A11.4) is dense in $\mathbf{L}^2(\mathbf{R}_+)$. □

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