Constructive Convex Analysis and Disciplined Convex Programming

Stephen Boyd   Steven Diamond
Akshay Agrawal   Junzi Zhang

EE & CS Departments

Stanford University

Outline

Outline

Convex optimization problem — standard form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

with variable $x \in \mathbf{R}^n$

- objective and inequality constraints $f_0, \ldots, f_m$ are convex

  for all x, y, $\theta \in [0, 1]$,

  $$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

  i.e., graphs of f_i curve upward
- equality constraints are linear

Convex optimization problem — conic form

cone program:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \in \mathcal{K} \end{array}$$

with variable $x \in \mathbf{R}^n$

- linear objective, equality constraints; $\mathcal{K}$ is convex cone
- special cases:
  - linear program (LP)
  - semidefinite program (SDP)

- the modern canonical form
- *there are well developed solvers for cone programs*

Other canonical forms

- quadratic program (QP):

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & l \leq Ax \leq u \end{array}$$

- smooth optimization:

$$\text{minimize} \quad f(x)$$

  where $f : \mathbf{R}^n \to \mathbf{R}$ is smooth
- linearly constrained least squares:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Fx = g \end{array}$$

- prox-affine:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^N f_i(H_i x_i) \\ \text{subject to} & \sum_{i=1}^N A_i x_i = b. \end{array}$$

Why convex optimization?

- beautiful, fairly complete, and useful theory
- solution algorithms that work well in theory and practice
  - convex optimization is **actionable**
- **many applications** in
  - control
  - combinatorial optimization
  - signal and image processing
  - communications, networks
  - circuit design
  - machine learning, statistics
  - finance

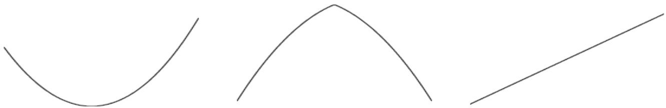
. . . and many more

How do you solve a convex problem?

- use an existing custom solver for your specific problem
- develop a new solver for your problem using a currently fashionable method
  - requires work
  - but (with luck) will scale to large problems
- transform your problem into a cone program, and use a standard cone program solver
  - can be *automated* using *domain specific languages*

Outline

Curvature: Convex, concave, and affine functions



- f is *concave* if $-f$ is convex, *i.e.*, for any x, y, $\theta \in [0, 1]$,

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y)$$

- f is *affine* if it is convex and concave, *i.e.*,

$$f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta)f(y)$$

  for any x, y, $\theta \in [0, 1]$
- f is affine $\iff$ it has form $f(x) = a^T x + b$

Verifying a function is convex or concave

(verifying affine is easy)

approaches:

- via basic definition (inequality)
- via first or second order conditions, *e.g.*, $\nabla^2 f(x) \succeq 0$

- via convex calculus: construct f using
  - library of basic functions that are convex or concave
  - calculus rules or transformations that preserve convexity

Convex functions: Basic examples

- x^p ($p \geq 1$ or $p \leq 0$), *e.g.*, x^2, $1/x$ ($x > 0$)
- e^x
- $x \log x$
- $a^T x + b$
- $x^T P x$ ($P \succeq 0$)
- $\|x\|$ (any norm)
- $\max(x_1, \ldots, x_n)$

Concave functions: Basic examples

- x^p $(0 \leq p \leq 1)$, *e.g.*, $\sqrt{x}$
- $\log x$
- $\sqrt{xy}$
- $x^T P x$ $(P \preceq 0)$
- $\min(x_1, \ldots, x_n)$

Convex functions: Less basic examples

- x^2/y $(y > 0)$, $x^T x/y$ $(y > 0)$, $x^T Y^{-1} x$ $(Y \succ 0)$
- $\log(e^{x_1} + \cdots + e^{x_n})$
- $f(x) = x_{[1]} + \cdots + x_{[k]}$ (sum of largest k entries)
- $f(x, y) = x \log(x/y)$ $(x, y > 0)$
- $\lambda_{\max}(X)$ $(X = X^T)$

Concave functions: Less basic examples

- $\log \det X$, $(\det X)^{1/n}$ $(X \succ 0)$
- $\log \Phi(x)$ (Φ is Gaussian CDF)
- $\lambda_{\min}(X)$ $(X = X^T)$

Calculus rules

- **nonnegative scaling**: f convex, $\alpha \geq 0 \implies \alpha f$ convex
- **sum**: f, g convex $\implies f + g$ convex
- **affine composition**: f convex $\implies f(Ax + b)$ convex
- **pointwise maximum**: $f_1, \ldots, f_m$ convex $\implies \max_i f_i(x)$ convex
- **composition**: h convex increasing, f convex $\implies h(f(x))$ convex

. . . and similar rules for concave functions

(there are other more advanced rules)

Examples

from basic functions and calculus rules, we can show convexity of ...

- piecewise-linear function: $\max_{i=1,\ldots,k}(a_i^T x + b_i)$
- ℓ_1-regularized least-squares cost: $\|Ax - b\|_2^2 + \lambda\|x\|_1$, with $\lambda \geq 0$
- sum of largest k elements of x: $x_{[1]} + \cdots + x_{[k]}$
- log-barrier: $-\sum_{i=1}^m \log(-f_i(x))$ (on $\{x \mid f_i(x) < 0\}$, f_i convex)
- KL divergence: $D(u,v) = \sum_i (u_i \log(u_i/v_i) - u_i + v_i)$ $(u, v > 0)$

A general composition rule

$h(f_1(x), \ldots, f_k(x))$ is convex when h is convex and for each i

- h is increasing in argument i, and f_i is convex, or
- h is decreasing in argument i, and f_i is concave, or
- f_i is affine

- there's a similar rule for concave compositions (just swap convex and concave above)
- this one rule subsumes all of the others
- *this is pretty much the only rule you need to know*

Example

let's show that

$$f(u, v) = (u + 1)\log((u + 1)/\min(u, v))$$

is convex

- u, v are variables with $u, v > 0$
- $u + 1$ is affine; $\min(u, v)$ is concave
- since $x\log(x/y)$ is convex in (x, y), decreasing in y,

  $$f(u, v) = (u + 1)\log((u + 1)/\min(u, v))$$

  is convex

Example

- $\log(e^{u_1} + \cdots + e^{u_k})$ is convex, increasing
- so if $f(x, \omega)$ is convex in x for each ω and $\gamma > 0$,

$$\log\left(\left(e^{\gamma f(x,\omega_1)} + \cdots + e^{\gamma f(x,\omega_k)}\right)/k\right)$$

  is convex
- this is $\log \mathbf{E}\, e^{\gamma f(x,\omega)}$, where $\omega \sim \mathcal{U}\left(\{\omega_1, \ldots, \omega_k\}\right)$
- arises in stochastic optimization via bound

$$\log \mathbf{Prob}(f(x, \omega) \geq 0) \;\leq\; \log \mathbf{E}\, e^{\gamma f(x,\omega)}$$

Constructive convexity verification

- start with function given as **expression**
- build parse tree for expression
  - leaves are variables or constants/parameters
  - nodes are functions of children, following general rule
- tag each subexpression as convex, concave, affine, constant
  - variation: tag subexpression signs, use for monotonicity
    e.g., $(\cdot)^2$ is increasing if its argument is nonnegative
- sufficient (but not necessary) for convexity

Example

for $x < 1$, $y < 1$

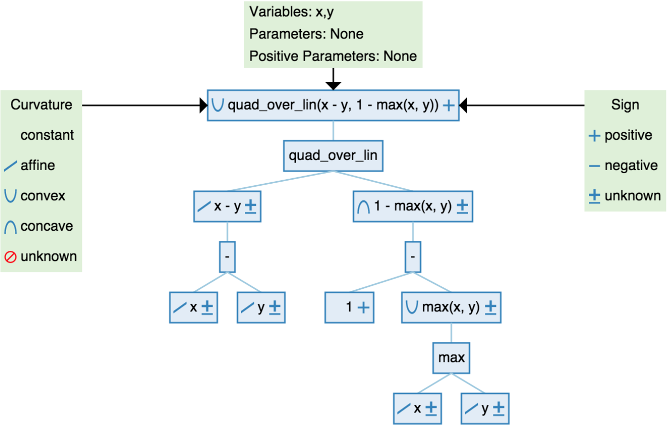
$$\frac{(x-y)^2}{1-\max(x,y)}$$

is convex

- (leaves) x, y, and 1 are affine expressions
- $\max(x,y)$ is convex; $x - y$ is affine
- $1 - \max(x,y)$ is concave
- function u^2/v is convex, monotone decreasing in v for $v > 0$
  hence, convex with $u = x - y$, $v = 1 - \max(x,y)$

Example

analyzed by `dcp.stanford.edu` (*Diamond 2014*)



Example

- $f(x) = \sqrt{1 + x^2}$ is convex
- but cannot show this using constructive convex analysis
  - (leaves) 1 is constant, x is affine
  - x^2 is convex
  - $1 + x^2$ is convex
  - but $\sqrt{1 + x^2}$ **doesn't match general rule**
- writing $f(x) = \|(1, x)\|_2$, however, works
  - $(1, x)$ is affine
  - $\|(1, x)\|_2$ is convex

Outline

Disciplined convex programming (DCP)

(*Grant, Boyd, Ye, 2006*)

- framework for describing convex optimization problems
- based on constructive convex analysis
- sufficient but not necessary for convexity
- basis for several domain specific languages and tools for convex optimization

Disciplined convex program: Structure

a DCP has

- zero or one **objective**, with form
  - minimize {scalar convex expression} or
  - maximize {scalar concave expression}
- zero or more **constraints**, with form
  - {convex expression} <= {concave expression} or
  - {concave expression} >= {convex expression} or
  - {affine expression} == {affine expression}

Disciplined convex program: Expressions

- expressions formed from
  - **variables**,
  - **constants/parameters**,
  - and **functions** from a library
- library functions have known convexity, monotonicity, and sign properties
- all subexpressions match general composition rule

Disciplined convex program

- a valid DCP is
  - convex-by-construction (cf. posterior convexity analysis)
  - 'syntactically' convex (can be checked 'locally')

- convexity depends only on *attributes* of library functions, and not their meanings
  - *e.g.*, could swap $\sqrt{\cdot}$ and $\sqrt[4]{\cdot}$, or $\exp \cdot$ and $(\cdot)_+$, since their attributes match

Canonicalization

- easy to build a DCP parser/analyzer
- not much harder to implement a *canonicalizer*, which transforms DCP to equivalent cone program
- then solve the cone program using a generic solver

- yields a *modeling framework* for convex optimization

Outline

Optimization modeling languages

- domain specific language (DSL) for optimization
- express optimization problem in high level language
  - declare variables
  - form constraints and objective
  - solve
- long history: AMPL, GAMS, ...
  - no special support for convex problems
  - very limited syntax
  - callable from, but not embedded in other languages

Modeling languages for convex optimization

all based on DCP

YALMIP	Matlab	Löfberg	2004
CVX	Matlab	Grant, Boyd	2005
CVXPY	Python	Diamond, Boyd; Agrawal et al.	2013; 2018
Convex.jl	Julia	Udell et al.	2014
CVXR	R	Fu, Narasimhan, Boyd	2017

some precursors

- SDPSOL (*Wu, Boyd, 2000*)
- LMITOOL (*El Ghaoui et al., 1995*)

CVX

```
cvx_begin
  variable x(n)   % declare vector variable
  minimize sum(square(A*x-b)) + gamma*norm(x,1)
  subject to norm(x,inf) <= 1
cvx_end
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- variables, expressions, constraints exist inside problem
- after `cvx_end`
  - problem is canonicalized to cone program
  - then solved

Some functions in the CVX library

function	meaning	attributes
`norm(x, p)`	$\|x\|_p$, $p \geq 1$	cvx
`square(x)`	x^2	cvx
`pos(x)`	x_+	cvx, nondecr
`sum_largest(x,k)`	$x_{[1]} + \cdots + x_{[k]}$	cvx, nondecr
`sqrt(x)`	$\sqrt{x}$, $x \geq 0$	ccv, nondecr
`inv_pos(x)`	$1/x$, $x > 0$	cvx, nonincr
`max(x)`	$\max\{x_1, \ldots, x_n\}$	cvx, nondecr
`quad_over_lin(x,y)`	x^2/y, $y > 0$	cvx, nonincr in y
`lambda_max(X)`	$\lambda_{\max}(X)$, $X = X^T$	cvx

DCP analysis in CVX

```
cvx_begin
    variables x y
    square(x+1) <= sqrt(y) % accepted
    max(x,y) == 1 % not DCP
    ...

Disciplined convex programming error:
   Invalid constraint: {convex} == {real constant}
```

CVXPY

```
import cvxpy as cp
x = cp.Variable(n)
cost = cp.sum_squares(A@x-b) + gamma*cp.norm(x,1)
prob = cp.Problem(cp.Minimize(cost),
                  [cp.norm(x,"inf") <= 1])
opt_val = prob.solve()
solution = x.value
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- variables, expressions, constraints exist outside of problem
- `solve` method canonicalizes, solves, assigns `value` attributes

Signed DCP in CVXPY

function	meaning	attributes
`abs(x)`	$\lvert x \rvert$	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$
`huber(x)`	$\begin{cases} x^2, & \lvert x \rvert \leq 1 \\ 2\lvert x \rvert - 1, & \lvert x \rvert > 1 \end{cases}$	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$
`norm(x, p)`	$\lVert x \rVert_p$, $p \geq 1$	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$
`square(x)`	x^2	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$

DCP analysis in CVXPY

$$\text{expr} = \frac{(x-y)^2}{1-\max(x,y)}$$

```
x = cp.Variable()
y = cp.Variable()
expr = cp.quad_over_lin(x - y, 1 - cp.maximum(x,y))
expr.curvature # CONVEX
expr.sign # POSITIVE
expr.is_dcp() # True
```

Parameters in CVXPY

- symbolic representations of constants
- can specify sign
- change value of constant without re-parsing problem

- for-loop style trade-off curve:

```
x_values = []
for val in numpy.logspace(-4, 2, 100):
    gamma.value = val
    prob.solve()
    x_values.append(x.value)
```

Parallel style trade-off curve

```
# Use tools for parallelism in standard library.
from multiprocessing import Pool

# Function maps gamma value to optimal x.
def get_x(gamma_value):
    gamma.value = gamma_value
    result = prob.solve()
    return x.value

# Parallel computation with N processes.
pool = Pool(processes = N)
x_values = pool.map(get_x, numpy.logspace(-4, 2, 100))
```

Convex.jl

```
using Convex
x = Variable(n);
cost = sum_squares(A*x-b) + gamma*norm(x,1);
prob = minimize(cost, [norm(x,Inf) <= 1]);
opt_val = solve!(prob);
solution = x.value;
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- similar structure to CVXPY
- `solve!` method canonicalizes, solves, assigns `value` attributes

Outline

Conclusions

- DCP is a formalization of constructive convex analysis
  - simple method to certify problem as convex (sufficient, but not necessary)
  - basis of several DSLs/modeling frameworks for convex optimization

- modeling frameworks make rapid prototyping of convex optimization based methods easy

References

- *Disciplined Convex Programming* (Grant, Boyd, Ye)
- *Graph Implementations for Nonsmooth Convex Programs* (Grant, Boyd)
- *Matrix-Free Convex Optimization Modeling* (Diamond, Boyd)
- *A Rewriting System for Convex Optimization Problems* (Agrawal, Verschueren, Diamond, Boyd)

- CVX: `http://cvxr.com/`
- CVXPY: `https://www.cvxpy.org/`
- Convex.jl: `http://convexjl.readthedocs.org/`
- CVXR: `https://cvxr.rbind.io/`
- DCP tools: `https://dcp.stanford.edu/`