Convex Optimization Overview

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Outline

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Optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & g_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $x \in \mathbf{R}^n$ is (vector) variable to be chosen
- f_0 is the *objective function*, to be minimized
- $f_1, \ldots, f_m$ are the *inequality constraint functions*
- $g_1, \ldots, g_p$ are the *equality constraint functions*

- variations: maximize objective, multiple objectives, ...

Finding good (or best) actions

- x represents some *action*, *e.g.*,
  - trades in a portfolio
  - airplane control surface deflections
  - schedule or assignment
  - resource allocation
  - transmitted signal
- constraints limit actions or impose conditions on outcome
- the smaller the objective $f_0(x)$, the better
  - total cost (or negative profit)
  - deviation from desired or target outcome
  - fuel use
  - risk

Engineering design

- x represents a design (of a circuit, device, structure, ...)
- constraints come from
  - manufacturing process
  - performance requirements
- objective $f_0(x)$ is combination of cost, weight, power, ...

Finding good models

- x represents the *parameters* in a model
- constraints impose requirements on model parameters (*e.g.*, nonnegativity)
- objective $f_0(x)$ is the prediction error on some observed data (and possibly a term that penalizes model complexity)

Inversion

- x is something we want to estimate/reconstruct, given some measurement y
- constraints come from prior knowledge about x
- objective $f_0(x)$ measures deviation between predicted and actual measurements

Worst-case analysis (pessimization)

- variables are actions or parameters out of our control (and possibly under the control of an adversary)
- constraints limit the possible values of the parameters
- minimizing $-f_0(x)$ finds *worst possible parameter values*

- if the worst possible value of $f_0(x)$ is tolerable, you're OK
- it's good to know what the worst possible scenario can be

Optimization-based models

- model an entity as taking actions that solve an optimization problem
  - an individual makes choices that maximize expected utility
  - an organism acts to maximize its reproductive success
  - reaction rates in a cell maximize growth
  - currents in an electric circuit minimize total power

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- (except the last) these are *very crude* models
- and yet, they often work very well

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- **the bad news**: most optimization problems are *intractable*
  i.e., *we cannot solve them*

- **an exception**: *convex optimization problems are tractable*
  i.e., we (generally) *can* solve them

Outline

Convex optimization problem

convex optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- equality constraints are linear
- $f_0, \ldots, f_m$ are **convex**: for $\theta \in [0, 1]$,

$$f_i(\theta x + (1 - \theta)y) \leq \theta f_i(x) + (1 - \theta) f_i(y)$$

i.e., f_i have nonnegative (upward) curvature

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  - duality, optimality conditions, . . .
- effective algorithms, methods (in theory and practice)
  - get **global solution** (and optimality certificate)
  - polynomial complexity
- conceptual unification of many methods
- **lots of applications** (many more than previously thought)

Application areas

- machine learning, statistics
- finance
- supply chain, revenue management, advertising
- control
- signal and image processing, vision
- networking
- circuit design
- combinatorial optimization
- quantum mechanics
- flux-based analysis

The approach

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  - using generic software if your problem is not really big
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- some tricks:
  - change of variables
  - approximation of true objective, constraints
  - *relaxation*: ignore terms or constraints you can't handle

Outline

Medium-scale solvers

- 1000s–10000s variables, constraints
- reliably solved by interior-point methods on single machine (especially for problems in standard cone form)
- exploit problem sparsity
- not quite a technology, but getting there
- used in control, finance, engineering design, ...

Large-scale solvers

- 100k – 1B variables, constraints
- solved using custom (often problem specific) methods
  - limited memory BFGS
  - stochastic subgradient
  - block coordinate descent
  - operator splitting methods
- require custom implementation, tuning for each problem
- used in machine learning, image processing, ...

Modeling languages

- (new) high level language support for convex optimization
  - describe problem in high level language
  - description automatically transformed to a standard form
  - solved by standard solver, transformed back to original form
- implementations:
  - YALMIP, CVX (Matlab)
  - CVXPY (Python)
  - Convex.jl (Julia)
  - CVXR (R)

CVX

(*Grant & Boyd, 2005*)

```
cvx_begin
  variable x(n)   % declare vector variable
  minimize sum(square(A*x-b)) + gamma*norm(x,1)
  subject to norm(x,inf) <= 1
cvx_end
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- after `cvx_end`
  - problem is converted to standard form and solved
  - variable `x` is over-written with (numerical) solution

CVXPY

(*Diamond & Boyd, 2013*); (*Agrawal et al., 2018*)

```
import cvxpy as cp

x = cp.Variable(n)
cost = cp.sum_squares(A@x-b) + gamma*cp.norm(x,1)
prob = cp.Problem(cp.Minimize(cost),
                  [cp.norm(x,"inf") <= 1])
opt_val = prob.solve()
solution = x.value
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- `solve` method converts problem to standard form, solves, assigns `value` attributes

Convex.jl

(*Udell et al., 2014*)

```
using Convex
x = Variable(n);
cost = sum_squares(A*x-b) + gamma*norm(x,1);
prob = minimize(cost, [norm(x,Inf) <= 1]);
opt_val = solve!(prob);
solution = x.value;
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- `solve!` method converts problem to standard form, solves, assigns `value` attributes

Modeling languages

- enable rapid prototyping (for small and medium problems)
- ideal for teaching (can do a lot with short scripts)

- slower than custom methods, but often not much
- current work focuses on extension to large problems

Outline

Radiation treatment planning

- radiation beams with intensities $x_j \geq 0$ directed at patient
- radiation dose y_i received in voxel i
- $y = Ax$
- $A \in \mathbf{R}^{m \times n}$ comes from beam geometry, physics
- goal is to choose x to deliver prescribed radiation dose d_i
  - $d_i = 0$ for non-tumor voxels
  - $d_i > 0$ for tumor voxels
- $y = d$ not possible, so we'll need to compromise
- typical problem has $n = 10^3$ beams, $m = 10^6$ voxels

Radiation treatment planning via convex optimization

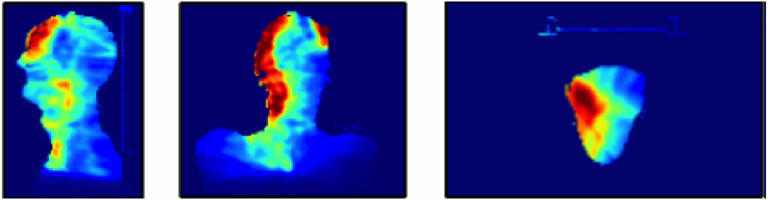
$$\begin{array}{ll} \text{minimize} & \sum_i f_i(y_i) \\ \text{subject to} & x \geq 0, \quad y = Ax \end{array}$$

- variables $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$
- objective terms are

$$f_i(y_i) = w_i^{\text{over}}(y_i - d_i)_+ + w_i^{\text{under}}(d_i - y_i)_+$$

- w_i^{over} and w_i^{under} are positive weights
- *i.e.*, we charge linearly for over- and under-dosing
- a convex problem

Example



- real patient case with $n = 360$ beams, $m = 360000$ voxels
- optimization-based plan essentially the same as plan used

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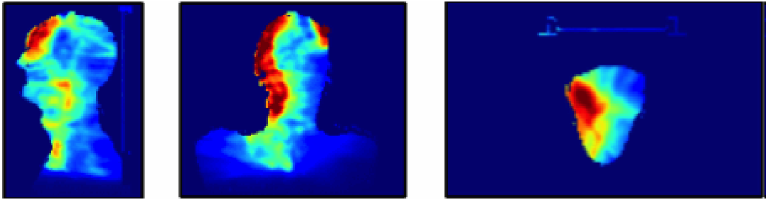


- real patient case with $n = 360$ beams, $m = 360000$ voxels
- optimization-based plan essentially the same as plan used
  - but we computed the plan in a few seconds on a GPU
  - original plan took hours of least-squares weight tweaking

Image in-painting

- guess pixel values in obscured/corrupted parts of image
- *total variation in-painting*: choose pixel values $x_{ij} \in \mathbf{R}^3$ to minimize *total variation*

$$\mathrm{TV}(x) = \sum_{i,j} \left\| \begin{bmatrix} x_{i+1,j} - x_{ij} \\ x_{i,j+1} - x_{ij} \end{bmatrix} \right\|_2$$

- a convex problem

Example

512 × 512 grayscale image ($n \approx 300000$ variables)

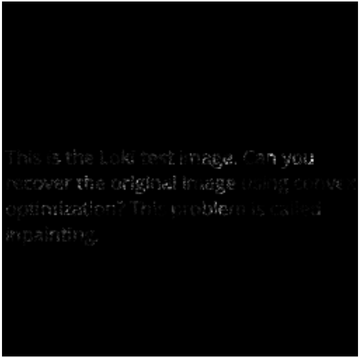
Original Image

Corrupted Image

Example

In-Painted Image

Difference Image

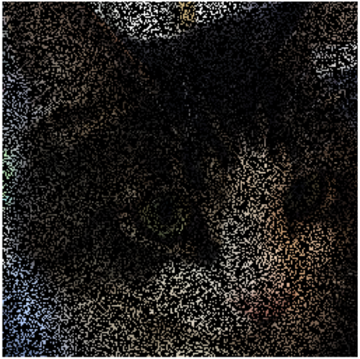


Example

512×512 color image ($n \approx 800000$), 80% of pixels removed

Original Image

Corrupted Image

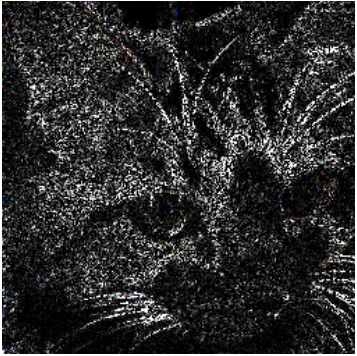


Example

80% of pixels removed

In-Painted Image

Difference Image



Control

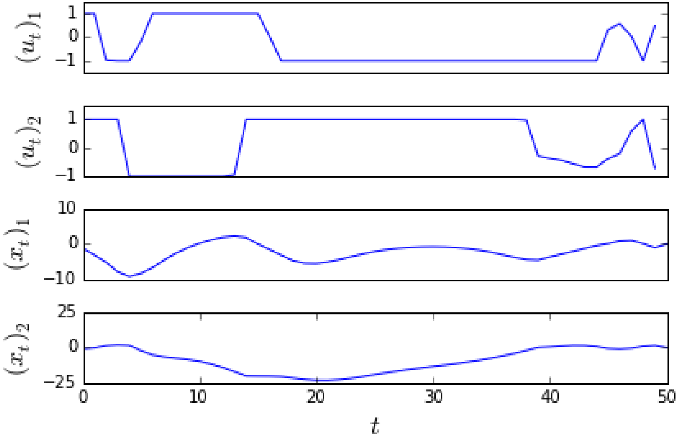
$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{T-1} \ell(x_t, u_t) + \ell_T(x_T) \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t \\ & (x_t, u_t) \in \mathcal{C}, \quad x_T \in \mathcal{C}_T \end{array}$$

- variables are
  - system states $x_1, \ldots, x_T \in \mathbf{R}^n$
  - inputs or actions $u_0, \ldots, u_{T-1} \in \mathbf{R}^m$
- ℓ is stage cost, ℓ_T is terminal cost
- $\mathcal{C}$ is state/action constraints; $\mathcal{C}_T$ is terminal constraint
- convex problem when costs, constraints are convex
- applications in many fields

Example

- $n = 8$ states, $m = 2$ inputs, horizon $T = 50$
- randomly chosen A, B (with $A \approx I$)
- input constraint $\|u_t\|_\infty \leq 1$
- terminal constraint $x_T = 0$ ('regulator')
- $\ell(x, u) = \|x\|_2^2 + \|u\|_2^2$ (traditional)
- random initial state x_0

Example



Support vector machine classifier with ℓ_1-regularization

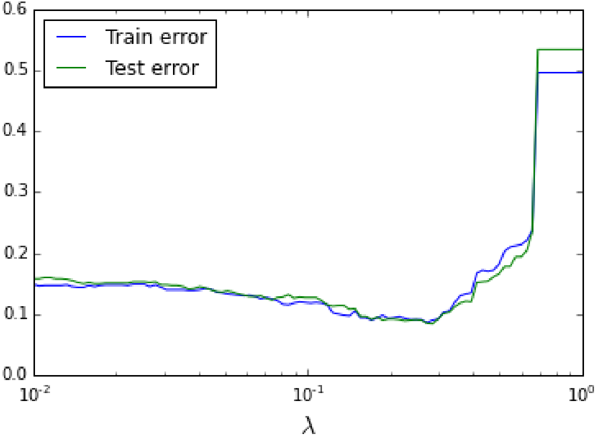
- given data (x_i, y_i), $i = 1, \ldots, m$
  - $x_i \in \mathbf{R}^n$ are feature vectors
  - $y \in \{\pm 1\}$ are associated boolean outcomes
- linear classifier $\hat{y} = \mathrm{sign}(\beta^T x - v)$
- find parameters β, v by minimizing (convex function)

$$(1/m) \sum_i \left(1 - y_i(\beta^T x_i - v)\right)_+ + \lambda \|\beta\|_1$$

  - first term is average hinge loss
  - second term shrinks coefficients in β and encourages sparsity
  - $\lambda \geq 0$ is (regularization) parameter
- simultaneously selects features and fits classifier

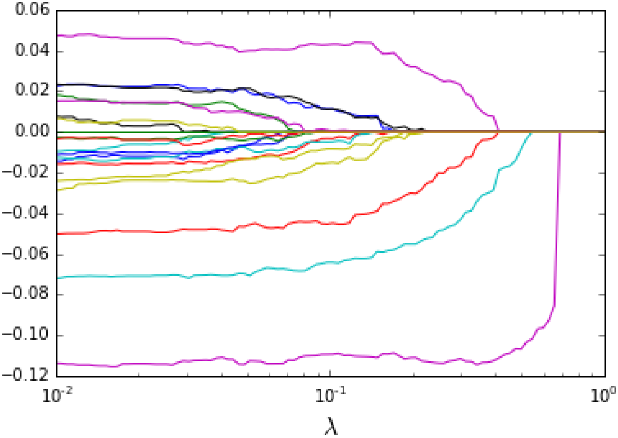
Example

- $n = 20$ features
- trained and tested on $m = 1000$ examples each



Example

β_i vs. λ (regularization path)



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Summary

- convex optimization problems **arise in many applications**
- convex optimization problems **can be solved effectively**
  - using generic methods for not huge problems
  - by developing custom methods for huge problems
- high level language support (CVX/CVXPY/Convex.jl/CVXR) makes prototyping easy

Resources

many researchers have worked on the topics covered

- *Convex Optimization* (book)
- *EE364a* (course slides, videos, code, homework, ...)
- software CVX, CVXPY, Convex.jl, CVXR

all available online