Design of Stabilizing State Feedback for Delay Systems via Convex Optimization

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Abstract

For linear systems with delays, we define a new class of Lyapunov-like functionals that may be used to prove stability. We also show how we may design a stabilizing (delayed) state feedback for delay systems using these functionals and convex optimization techniques.

1 Introduction

We consider linear systems with delays, described by the state equation

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + Bu(t), \qquad (1)$$

where the state $x(t) \in \mathbf{R}^n$, the input $u(t) \in \mathbf{R}^p$, and $0 < \tau_1 < \tau_2 < \cdots < \tau_m$ are the *delays* in the system. We assume that the full state of the system is available with a delay $\tau > 0$. Our objective is to design a constant, delayed state feedback $u(t) = -Kx(t-\tau)$ that stabilizes the system. We remark that proving stability of system (1) (with $u(t) = 0$) is in itself a hard problem. Our approach towards designing K combines a Lyapunov-like method with some recent advances in convex optimization.

Note that (1) is *not* a finite dimensional system, and therefore Lyapunov *functionals* rather than the more conventional Lyapunov *functions* are needed. In §2, we will describe one such functional, which we will call the Modified Lyapunov-Krasovskii (MLK) functional. We then show how we may pose the problem of design of a stabilizing (delayed) state-feedback as a convex feasibility problem.

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2 Stabilizing state feedback

With the delayed state feedback $u(t) = -Kx(t-\tau)$, the state equation is

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) - BKx(t-\tau). \qquad (2)$$

In the sequel, we assume that $0 < \tau < \tau_1$; the case $\tau_1 \le \tau$ may be dealt with similarly.

Motivated by the work of Krasovskii [4] (see also [6]), we propose a class of functionals for system (2), which we will refer to as Modified Lyapunov-Krasovskii (MLK) functionals:

$$\begin{aligned} V(x,t) = {} & x(t)^T L_0 x(t) + \\ & \textstyle\sum_{i=1}^{m} \int_{-\tau_i}^{-\tau_{i-1}} x(t+s)^T L_i x(t+s)\,ds + \\ & \textstyle\int_{-\tau}^{0} x(t+s)^T L x(t+s)\,ds, \end{aligned} \qquad (3)$$

where $L, L_0, \ldots, L_m$ are symmetric positive definite matrices and $\tau_0 = \tau$. The derivative $\dfrac{d}{dt}V(x,t)$, computed using (2) is

$$\begin{aligned} & 2x(t)^T L_0 \begin{pmatrix} A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) \\ -BKx(t-\tau) \end{pmatrix} \\ & + \textstyle\sum_{i=1}^{m} \begin{pmatrix} x(t-\tau_{i-1})^T L_i x(t-\tau_{i-1}) \\ -x(t-\tau_i)^T L_i x(t-\tau_i) \end{pmatrix} \\ & + \left(x(t)^T L x(t) - x(t-\tau)^T L x(t-\tau) \right). \end{aligned}$$

This can be rewritten as $d/dt\ V(x,t) = y^T W y$, where W and y^T are given by

$$\begin{bmatrix} N & -L_0BK & L_0A_1 & \cdots & L_0A_m \\ -K^TB^TL_0 & L_1 - L & 0 & \cdots & 0 \\ A_1^TL_0 & 0 & L_2 - L_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m^TL_0 & 0 & 0 & \cdots & -L_m \end{bmatrix},$$

and

$$[x(t)^T, x(t-\tau)^T, x(t-\tau_1)^T, \cdots, x(t-\tau_m)^T],$$

respectively, with $N = L_0A_0 + A_0^TL_0 + L$.

We then have:

If there exist $L_0, L, L_1, \ldots, L_m$ and K such that W as above is negative definite, then system (2) is stable.

The proof is along the lines of the one for Lyapunov-Krasovskii functionals in reference [4].

We now show that finding $L_0, L, L_1, \ldots, L_m$ and K such that W as above is negative definite can be posed as a convex feasibility problem. Our manipulations are based on a recent result on the parametrization of state-feedback controllers [3].

We multiply every block entry of W on the left and on the right by L_0^{-1} and set $M_0 = L_0^{-1}$, $M_i = L_0^{-1}L_iL_0^{-1}$, $i = 1, \ldots, m$, $M = L_0^{-1}LL_0^{-1}$ and $Y = KL_0^{-1}$, to obtain a new matrix X given by

$$\begin{bmatrix} \tilde{N} & -BY & A_1M_0 & \cdots & A_mM_0 \\ -YB^T & M_1 - M & 0 & \cdots & 0 \\ M_0A_1^T & 0 & M_2 - M_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_0A_m^T & 0 & 0 & \cdots & -M_m \end{bmatrix},$$

where $\tilde{N} = A_0M_0 + M_0A_0^T + M$.

We then have: $W < 0$ *if and only if* $X < 0$.

X is a linear function of $M_0, M_1, \cdots, M_m$, M and Y, and therefore therefore the set

$$\Psi = \{X \mid X < 0\}$$

is convex in these variables. Checking its nonemptiness can then be done via a convex feasibility program.

There exist several methods for solving this convex feasibility problem. In [6], Skorodinskii proposes the use of the ellipsoid algorithm [1]. There have been recent advances in convex programming which promise much faster algorithms [5, 2].

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