Control System Analysis and Synthesis via Linear Matrix Inequalities

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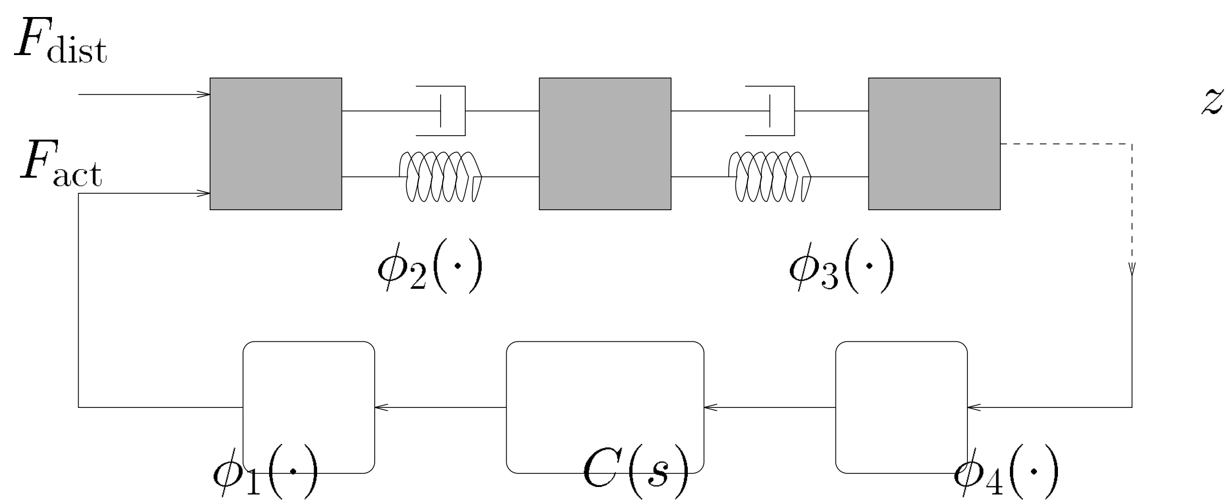
Basic idea

Many problems in control can be cast in terms of **linear matrix inequalities** (LMIs)

LMI problems can be solved **extremely efficiently**

Example problem



Question: how large can RMS(z) be?

- ignoring nonlinearities, RMS(z) ≤ 4.1
- Monte Carlo / simulation **suggests** RMS(z) ≤ 5.6 or so
- LMI-based analysis **guarantees** RMS(z) ≤ 7.0

Outline

- What is an LMI?
- Examples from control
- Solving LMI problems
- History of LMIs
- Conclusions & challenges

Linear matrix inequality

LMI has form:

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m \geq 0$$

$F_i = F_i^T$ are given; $x \in \mathbf{R}^m$ is the variable

- a set of **polynomial inequalities** in x
- multiple LMIs can be combined into one

Optimization problems over LMIs:

- **Feasibility:** find x s.t. $F(x) \geq 0$, or show that none exists
- **Linear objective:** minimize $c^T x$ subject to $F(x) \geq 0$

LMI Problems

LMI problems

- appear difficult (nonlinear, nondifferentiable)
- have no "analytic solution"

but:

LMI problems are **readily solved** by computer

Why? . . . convexity

What “readily solved” means

There are algorithms for LMI problems that

- are globally convergent (good initial guess not needed)
- compute the **global optimum** within a given tolerance, or find proof of infeasibility
- computational effort required is small and grows slowly with problem size

…more details later

LMI problems: two examples

Symmetric matrix P is the variable:

Lyapunov inequality:

$$P > 0, \qquad A^T P + PA < 0$$

analytic solution (1890)

Two Lyapunov inequalities:

$$P > 0, \qquad A^T P + PA < 0, \qquad \tilde{A}^T P + P\tilde{A} < 0$$

no analytic solution, but readily solved

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- What is an LMI?
- **Examples from control**
  - **multimodel LQR synthesis**
  - Popov analysis
- Solving LMI problems
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A time-varying system

Consider time-varying linear system

$$\dot{x}(t) = A(t)x(t), \qquad A(t) \in \mathbf{Co}\{A_1, \ldots, A_L\}$$

Question: is it stable?

Note: implies stability of nonlinear time-varying system

$$\dot{x} = f(x(t), t)$$

if for all x, t, $\frac{\partial f}{\partial x} \in \mathbf{Co}\{A_1, \ldots, A_L\}$ ("global linearization")

Quadratic Lyapunov function search

Is there a quadratic Lyapunov function $V(z) \triangleq z^T P z$ that proves stability?

Equivalent to: is there P s.t.

$$P > 0, \qquad A_i^T P + P A_i < 0, \quad i = 1, \ldots, L?$$

- an LMI in P; no analytic solution but **readily solved**
- looks simple, but **more powerful** than many well known methods (multivariable circle criteria, . . .)

Some extensions

- **performance specifications,**
  e.g., actuator effort, RMS gain, LQR cost
- synthesis of linear **state feedback**
- synthesis of gain-scheduled controllers

In these cases we get LMI problems that

- look more complicated
- also have no analytic solutions
- are just as readily solved

Multimodel LQR example

$$y''' + a_1(t)y'' + a_2(t)y' + a_3(t)y = a_4(t)u$$

$$y''(0) = y'(0) = y(0) = 1$$

Nominal LQR cost:

$$J_{\mathrm{nom}} = \int_0^\infty \left(u(t)^2 + y(t)^2\right) dt \quad \text{with} \quad a_i(t) = 1$$

Worst-case LQR cost:

$$J_{\mathrm{wc}} = \max_{1/2 \le a_i(t) \le 2} \int_0^\infty \left(u(t)^2 + y(t)^2\right) dt$$

We'll explore linear state feedback . . .

LQR vs. multi-model LQR

Two state feedback gains:

- K_{lqr} minimizes J_{nom}
  (requires solution of Riccati equation)
- K_{mmlqr} minimizes upper bound on J_{wc}
  (requires solution of an LMI problem)

	$\sqrt{J_{\mathrm{nom}}}$	$\sqrt{J_{\mathrm{wc}}}$
K_{lqr}	1.00	∞
K_{mmlqr}	1.70	≤ 4.00

- K_{mmlqr} has traded off nominal performance for improved robustness
- K_{mmlqr} performance guaranteed with **nonlinear system**

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Search for Lyap fct with Popov terms

Consider system with scalar nonlinearities $\phi_1(\cdot), \ldots, \phi_r(\cdot)$

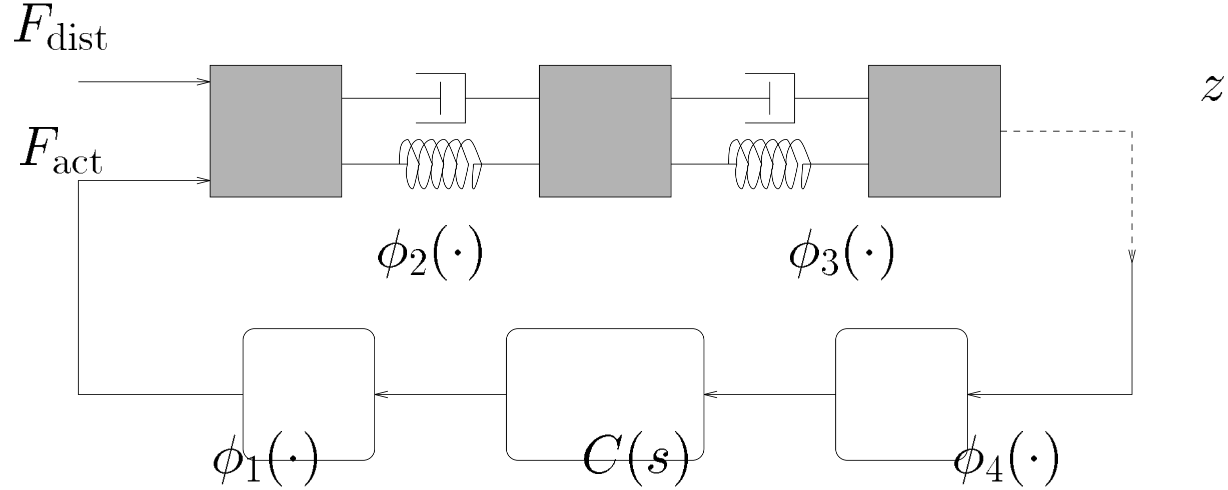
Lyapunov function with Popov terms:

$$V(x) = x^T P x + \lambda_1 \int_0^{y_1} \phi_1(a)da + \cdots + \lambda_r \int_0^{y_r} \phi_r(a)da$$

We seek V that proves performance specification
(stability, LQR cost, RMS gain, ...)

- can reduce to LMI problem in several ways (Skorodinsky, Pyatnitsky, Hall)
- hence **readily solved**

Popov analysis example



- actuator, sensor, springs are 10% **nonlinear**:
  $0.9 \leq \dfrac{\phi_i(a)}{a} \leq 1.1$
- C is simple LQG controller
- RMS(F_{dist})≤ 1

Question: how large can RMS(z) be?

Linearized vs. Popov performance analysis

- For **nominal linear** system: compute $\mathbf{H}_\infty$ norm to find

$$\text{RMS}(z) \leq 4.13$$

- For **nonlinear** system, solve LMI problem to find

$$\text{RMS}(z) \leq 7.00$$

- Extensive simulation only allows **guess** of performance degradation
- This analysis **guarantees** a maximum performance degradation

Other problems that reduce to LMIs

- optimal filter / controller realization
- linear controller design via Q-parametrization
- multi-criterion LQG
- inverse problem of optimal control
- synthesis of multipliers for analysis of systems with unknown constant parameters
- analysis & design for stochastic systems

i.e., **many other problems**

Some problems that (probably) **won't** reduce to LMIs:

- **output** feedback synthesis
- state feedback synthesis for Popov, multipliers

The V–K iteration

repeat {

fix V; design K (synthesis)

fix K; find good V (analysis)

}

- handles constraints on controller structure
- each step is LMI problem
- controller may not be globally optimal, but
- controller comes with **certificate of performance**
- appears to work in practice (Safonov, Skelton)

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- **Solving LMI problems**
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Solving LMI problems

In theory:

- fundamental complexity is low (polynomial)

In practice:

- classical optimization methods **do not** work
- ellipsoid algorithm (1970s) is robust but slow
- new interior-point methods (1988) **extremely efficient**
  - can exploit problem structure (Lyapunov, etc)

Example: multiple Lyapunov inequalities

(L. Vandenberghe)

Problem: minimize linear fct of P subject to:

$$A_i^T P + P A_i + Q_i \leq 0, \qquad i = 1, \ldots, L$$

(A_i, Q_i given; $P \in \mathbf{R}^{n \times n}$ is the variable)

- number of variables: $n(n+1)/2 = O(n^2)$
- cost of solving L independent Lyapunov equations: $O(Ln^3)$
- new primal-dual method solves above problem in $O(L^{1.2}n^4)$
- ratio: $O(L^{0.2}n)$

solving 100 50×50 **independent Lyapunov equations**

≈ solving 100 16×16 **coupled Lyapunov inequalities**

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LMIs in control: key events

1890s: Lyapunov's theory

1940s: Lur'e, Postnikov apply to real problems

Early 1960s: Yakubovich recognizes important role of LMIs

Late 1960s: graphical, ARE methods for solving **some** LMIs

Early 1980s: LMIs recognized as **convex programs** (Skorodinsky, Pyatnitsky)

Late 1980s: Interior-point algorithms for LMIs (Nesterov & Nemirovsky)

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Conclusion

Interesting and useful control problems can be cast as LMI problems, **hence solved**

- robust **performance** analysis
- some robust synthesis
- many other problems

LMI-based analysis / design will **complement,** not replace

- highly nonlinear feedback synthesis (*e.g.*, geometric . . .)
- simulation, Monte Carlo analysis

What constitutes a solution?

- reduction to an Algebraic Riccati equation?
- reduction to an LMI problem?

Challenges: theory

- **output** feedback?
- relation between **system theory** duality and **convex optimization** duality?
- what control problems can be cast as LMIs?

Challenges: applications

- Develop **useful** computer-aided robustness analysis / synthesis tools
  - user interface
  - automated translation, block diagram $\longrightarrow$ LMI
  - efficient specialized codes

- Try LMI-based analysis / synthesis on **real problems**
  - are worst-case performance bounds too conservative?
  - is multi-model robust synthesis useful?
  - does V–K iteration work in practice?

LMI problem from Popov example

$$\begin{bmatrix} -A^TP - PA - C^TC & -P\tilde{B} + A^T\tilde{C}^TL + \tilde{C}^TT & -PB \\ -\tilde{B}^TP + L\tilde{C}A + T\tilde{C} & L\tilde{C}\tilde{B} + \tilde{B}^T\tilde{C}^TL + 2T\tilde{D} & L\tilde{C}B \\ -B^TP & B^T\tilde{C}^TL & \mu I \end{bmatrix} > 0$$

variables:

$$\mu, \quad P > 0, \quad L = \mathrm{diag}(\lambda_1, \ldots, \lambda_4) \geq 0, \quad T = \mathrm{diag}(\tau_1, \ldots, \tau_4) \geq 0$$

objective: μ

- looks ugly (to people)
- impossible to solve analytically or by hand
- easily solved numerically
- $\Longrightarrow$ should be hidden from user

(Some) recent work on LMIs

- Yakubovich, Megretsky: *S-procedure*
- Geromel, Peres & Bernussou: *robust control*
- Khargonekar, Rotea: *mixed* $\mathbf{H}_2/\mathbf{H}_\infty$
- Barmish, Hollot: *quadratic stabilizability*
- Doyle, Packard & associates: *LMIs, LFTs, μ*
- Bernstein & Haddad, Hall & How: *multipliers, Popov*
- Safonov: *K_m-synthesis*
- Skelton et al.: *covariance control*
- Boyd, El Ghaoui, Feron & Balakrishnan: *monograph*
- Nesterov & Nemirovsky, Vandenberghe & Boyd, Overton & Haeberly, Fan: *algorithms*
- Gahinet: *LMIlab*

warning: advertisement

Boyd, El Ghaoui, Feron & Balakrishnan,

Linear Matrix Inequalities in System and Control Theory

- monograph in preparation, late 1993 (?)
- rough draft via ftp: email `boyd@isl.stanford.edu`
- ACC paper: end of Proceedings vol 2
  & anonymous ftp into `isl.stanford.edu`