Chapter 1

Entropy and random feedback

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1.1 γ-entropy of a matrix

Let $C \in \mathbf{C}^{m\times n}$. The Frobenius norm of C is defined as $\|C\|_F = \sqrt{\mathbf{Tr} CC^*}$, where C^* denotes the complex-conjugate transpose and $\mathbf{Tr}$ denotes trace. The spectral norm of C is defined as $\|C\| = \sqrt{\lambda_{\max}(CC^*)}$, where $\lambda_{\max}$ denotes the maximum eigenvalue.

The problem described in this chapter involves the γ-entropy, which is a convex function of C closely related to these two norms. For $\gamma > 0$ we define the γ-entropy of C as

$$I_\gamma(C) = \begin{cases} -\gamma^2 \log\det\left(I - \gamma^{-2} CC^*\right) & \text{if } \|C\| < \gamma \\ \infty & \text{if } \|C\| \geq \gamma \end{cases}$$

The Frobenius and spectral norms, and the γ-entropy, are unitarily invariant, and so can be expressed in terms of the singular values σ_i of C (*i.e.*, the squareroots of the eigenvalues of CC^*, ordered as $\sigma_1 \geq \sigma_2 \geq \cdots$):

$$\|C\|_F = \sqrt{\sum_i \sigma_i^2}, \qquad \|C\| = \sigma_1,$$

and

$$I_\gamma(C) = \begin{cases} \sum_i -\gamma^2 \log\left(1 - (\sigma_i/\gamma)^2\right) & \text{if } \sigma_1 < \gamma \\ \infty & \text{if } \sigma_1 \geq \gamma \end{cases} .$$

The following facts are readily shown (see [1, §5.3.5]). The squareroot of the γ-entropy always exceeds the Frobenius norm:

$$\sqrt{I_\gamma(C)} \geq \|C\|_F,$$

and as γ becomes large, it converges to the Frobenius norm:

$$\lim_{\gamma \to \infty} \sqrt{I_\gamma(C)} = \|C\|_F.$$

We also have the more complicated converse inequality

$$\sqrt{I_\gamma(C)} \leq \frac{1}{\alpha}\sqrt{-\log(1-\alpha^2)}\,\|C\|_F,$$

where $\alpha = \|C\|/\gamma < 1$. Thus, the relative increase in the squareroot of the γ-entropy over the Frobenius norm can be bounded by an expression that only depends on how close the spectral norm is to the critical value γ. For example, if $\|C\| \leq \gamma/2$ ($\alpha \leq 0.5$), we have $\|C\|_F \leq \sqrt{I_\gamma(C)} \leq 1.073\|C\|_F$.

The γ-entropy arises in several contexts, for example $\mathbf{H}_\infty$ control, in which the so-called central controller minimizes the integral over frequency of the γ-entropy of the closed-loop transfer matrix (see, *e.g.*, [2, 1]). It arises as the natural self-concordant barrier for the convex set $\{ C \mid \|C\| \leq \gamma \}$, in interior-point optimization methods (see [4, 3]). The γ-entropy also arises in other applications, *e.g.*, contractive matrix completion problems [5].

1.2 Stochastic interpretation

The Frobenius norm can be interpreted as the root-mean-square gain of the matrix C, as follows. Suppose $w \in \mathbf{C}^n$ is a random vector with zero mean and covariance I, *i.e.*,

$$\mathbf{E}w = 0, \qquad \mathbf{E}ww^T = I,$$

and let $z = Cw$. Then we have

$$\mathbf{E}\|z\|^2 = \|C\|_F^2.$$

We now connect a feedback gain $\Delta \in \mathbf{C}^{n\times m}$ around C, *i.e.*, we consider

$$z = Cu, \qquad u = w + \Delta z.$$

Eliminating u yields the familiar formula for the 'closed-loop gain':

$$z = C(I - \Delta C)^{-1}w.$$

(The inverse exists if the 'small-gain' condition $\|\Delta\| < \gamma$ holds.) Evidently the root-mean-square value of z is given by $\|C(I-\Delta C)^{-1}\|_F$.

Now we assume that Δ is a random matrix, independent of w, such that $\|\Delta\| < \gamma$ with probability one. The mean-square value of z is then

$$\mathbf{E}\|z\|^2 = \mathbf{E}\|C(I-\Delta C)^{-1}\|_F^2.$$

where the expectation is over the random feedback gain Δ.

Our open problem can now be stated:

Find a distribution for Δ *(if one exists) such that the mean-square value of* z *is given by* $I_\gamma(C)$, i.e., $I_\gamma(C) = \mathbf{E}\|C(I-\Delta C)^{-1}\|_F^2$.

Evidently the distribution should be unitarily invariant, and must satisfy $\|\Delta\| < \gamma$ with probability one.

If such a distribution can be found we will have a nice interpretation of the entropy as the mean-square value of the output of a system, with a random input and a random feedback connected around it. The inequalities above would then show that the random feedback has little effect unless the norm of the feedback is significant compared to the norm of C.

1.3 The scalar case

The problem has been solved for the scalar case $m = n = 1$ in [1]. If Δ is uniformly distributed on the disk of radius $1/\gamma$ in the complex plane, then we have

$$I_\gamma(C) = \mathbf{E}\|C(I-\Delta C)^{-1}\|_F^2 = \mathbf{E}\left|\frac{C}{1-\Delta C}\right|^2.$$

This can be shown as follows.

$$\begin{aligned}
\mathbf{E}\left|\frac{C}{1-\Delta C}\right|^2 &= \frac{\gamma^2}{\pi}\int_0^{1/\gamma}\int_0^{2\pi}\left|\frac{C}{1-re^{i\theta}C}\right|^2 r\,d\theta\,dr \\
&= \begin{cases} -\gamma^2\log(1-|C|^2/\gamma^2) & |C|<\gamma \\ \infty & |C|\geq\gamma \end{cases}
\end{aligned}$$

(the integration over θ can be evaluated by residues).

Bibliography

[1] S. Boyd and C. Barratt. *Linear Controller Design: Limits of Performance*. Prentice-Hall, 1991.

[2] D. Mustafa and K. Glover. *Minimum Entropy* $\mathbf{H}_\infty$ *Control*. Lecture Notes in Control and Information Sciences. Springer-Verlag, 1990.

[3] Yu. Nesterov and A. Nemirovsky. *Interior-point polynomial methods in convex programming*, volume 13 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.

[4] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, March 1996.

[5] L. Vandenberghe, S. Boyd, and S.-P. Wu. Determinant maximization with linear matrix inequality constraints. *SIAM J. on Matrix Analysis and Applications*, April 1998. To appear.