Optimal Allocation of Local Feedback in Multistage Amplifiers via Geometric Programming

Joel L. Dawson, Stephen P. Boyd, Thomas H. Lee
Stanford University
Stanford, CA 94305-4070

Maria del Mar Hershenson
Barcelona Design
192 South Murphy St.
Sunnyvale, CA 94086

***Abstract*—We consider the problem of optimally allocating local feedback to the stages of a multistage amplifier. The local feedback gains affect many performance indices in a complicated and nonlinear fashion, making optimization of the feedback gains a very challenging problem. We show that *geometric programming* provides a complete solution.**

I Introduction

The use of linear feedback around an amplifier stage was pioneered by Black [1], Bode [2], and others. The relation between the choice of feedback gain and the (closed-loop) gain, bandwidth, rise-time, sensitivity, noise, and distortion properties, is well understood (see, e.g., [3]). For a single stage amplifier, the choice of the (single) feedback gain is a simple problem.

In our work we consider the *multistage* amplifier shown in figure 1, consisting of n open-loop amplifier stages denoted $A_1, \ldots, A_n$, with local feedback gains $f_1, \ldots, f_n$ employed around the stages.

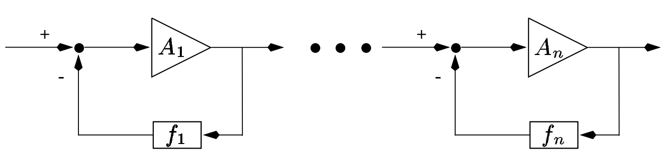

Fig. 1. Block diagram of multistage amplifier.

We assume that the amplifier stages are fixed, and consider the problem of choosing the feedback gains $f_1, \ldots, f_n$. The choice of these feedback gains affects a wide variety of performance measures for the overall amplifier, including gain, bandwidth, rise-time, delay, noise, distortion and sensitivity properties, maximum output swing, and dynamic range. These performance measures depend on the feedback gains in a complicated and nonlinear manner. It is thus far from clear, given a set of specifications, how to find an optimal choice of feedback gains. We refer to the problem of determining optimal values of the feedback gains, for a given set of specifications on overall amplifier performance, as the *local feedback allocation problem*.

In our work [4], we have shown that the local feedback allocation problem can be cast as a *geometric program* (GP), which is a special type of optimization problem. Even complicated geometric programs can be solved very efficiently, and globally, by recently developed interior-point methods (see [5]–[7]). Therefore we are able to give a complete, global, and efficient solution to the local feedback allocation problem.

In §II, we give a detailed description of the amplifier stage models used to analyze the performance of the amplifier. Though simple, the models capture the basic qualitative behavior of a source-degenerated differential pair. §III provides a brief overview of geometric programming, and an example of derived amplifier characteristics. A design example is given in §IV.

II Amplifier stage models

In this section we describe the different models of an amplifier stage used in our analysis.

A Linearized static model

The simplest model we use is the linear static model shown in figure 2. The stage is characterized by $y_i = \alpha_i e_i$, where α_i is the gain of the ith stage, which we assume to be positive. We use this simple model for determining the overall gain of the amplifier, determining the maximum signal swing, and the sensitivity of the amplifier gain to each stage gain.

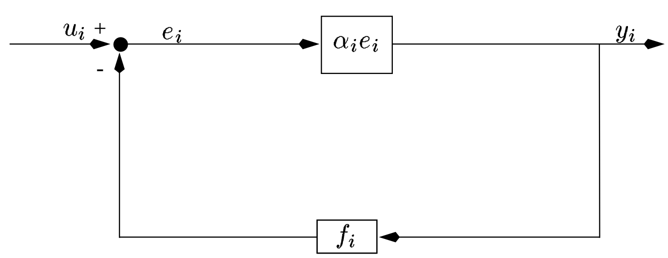

Fig. 2. Linearized static model of amplifier stage.

B Static nonlinear model

To quantify nonlinear distortion effects, we use a static nonlinear model of the amplifier stage as shown in figure 3. We assume a nonlinearity of the form

$$y_i = a_i(e) = \alpha_i e - \beta_i e^3 + o(e^3). \quad (1)$$

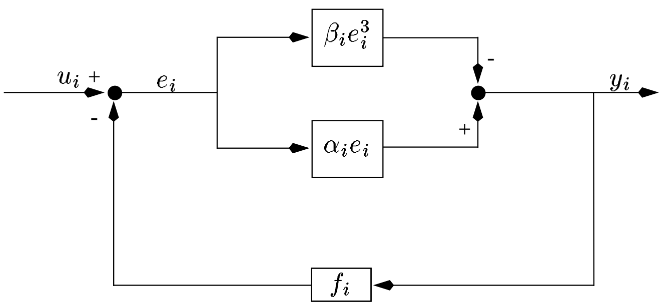

Fig. 3. Nonlinear static model of amplifier stage.

This form is inspired by the transfer characteristic of a source-coupled pair [8], and is a general model for third-order nonlinearity in a stage with an odd transfer characteristic. The function $a_i(\cdot)$ is called the *transfer characteristic* of the ith stage, and β_i is called the *third-order coefficient* of the amplifier stage. Note that the gain and third-order coefficient are related to the transfer characteristic by

$$\alpha_i = a_i'(0), \quad \beta_i = -\frac{a_i'''(0)}{6}. \tag{2}$$

We assume that $\beta_i \geq 0$, which means the third-order term is *compressive*: as the signal level increases from zero, the nonlinear term tends to decrease the output amplitude when compared to the linear model.

C Linearized dynamic model

To characterize the bandwidth, delay, and rise-time of the overall amplifier, we use the linearized dynamic model shown in figure 4. Here the stage is represented by a simple one-pole transfer function with time constant τ_i (which we assume to be positive).

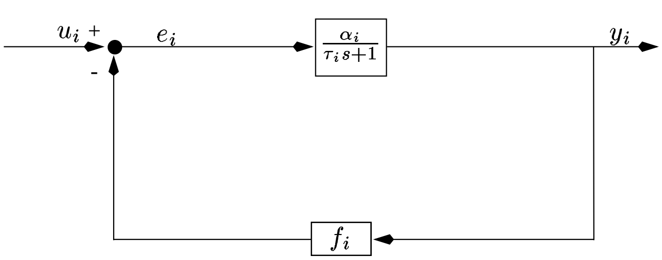

Fig. 4. Linear dynamic model of amplifier stage.

D Static noise model

Last, we have the static noise model shown in figure 5, which includes a simple output-referred noise v_i[1]. Our noise model is characterized by the RMS value of the noise source, which we denote $\overline{v_i}$. We assume that noise sources associated with different stages are uncorrelated.

[1] More complicated noise models can also be handled by our method.

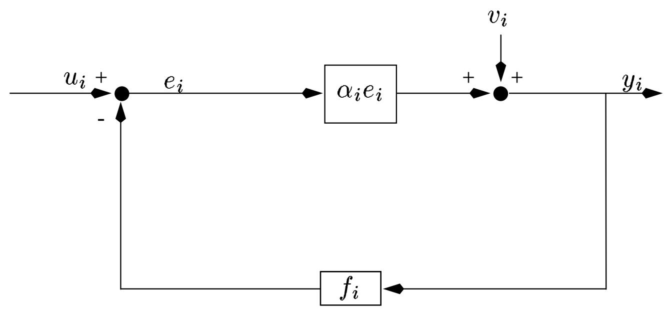

Fig. 5. Static noise model of amplifier stage.

III Amplifier Analysis

Having established the models of §II, there are many figures of merit that are straightforward to derive. For our purposes we require that such derivations result in *posynomials*, thereby enabling the use of geometric programming. To this end, it is useful to adopt *return differences*, $l_i = 1 + f_i\alpha_i$, as our design variables.

This section provides a basic description of geometric programming, followed by an example of a derived amplifier characteristic. A more complete treatment of geometric programming can be found in [9], [7], [5]; an extensive series of derivations can be found in [4].

A Geometric programming

Let f be a real-valued function of n real, positive variables $x_1, x_2, \ldots, x_n$. It is called a *posynomial* function if it has the form

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{t} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}} \tag{3}$$

where $c_j \geq 0$ and $\alpha_{ij} \in \mathbf{R}$. When $t = 1$, f is called a *monomial* function. Thus, for example, $0.7 + 2x_1/x_3^2 + x_2^{0.3}$ is posynomial and $2.3(x_1/x_2)^{1.5}$ is a monomial. Posynomials are closed under sums, products, and nonnegative scaling.

A *geometric program* (GP) has the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, 2, \ldots, m, \\ & g_i(x) = 1, \quad i = 1, 2, \ldots, p, \\ & x_i > 0, \quad i = 1, 2, \ldots, n, \end{array} \tag{4}$$

where f_i are posynomial functions and g_i are monomial functions. Geometric programs were introduced by Duffin, Peterson, and Zener in the 1960s [9].

The most important property of geometric programs for us is that they can be solved, with great efficiency, and globally, using recently developed interior-point methods [7], [5]. Geometric programming has recently been used to optimally design electronic circuits including CMOS op-amps [10], [11], and planar spiral inductors [12].

B Example Derivation

Here we examine the static nonlinearity of a cascade of stages. This derivation would be useful for detemining the spurious-free dynamic range, or for evaluating intermodulation distortion products.

We begin by deriving the closed-loop third-order coefficient of a single feedback amplifier stage, using the static nonlinear model of §II. The output y is related to the input u through the relation

$$y = a(u - fy). \tag{5}$$

Differentiating both sides with respect to u leads to the familiar result from elementary feedback theory:

$$y'(0) = \frac{a'(0)}{1 + fa'(0)} = \frac{\alpha}{l} = \tilde{\alpha}. \tag{6}$$

Differentiating again yields

$$y''(0) = \frac{a''(0)}{l^3} = 0, \tag{7}$$

and, once more,

$$y'''(0) = \frac{a'''(0)l - 3fa''(0)^2}{l^5} = -\frac{6\beta}{l^4}, \tag{8}$$

using $a'''(0) = -6\beta$ and $a''(0) = 0$ from the previous equation. This equation shows that the third-order coefficient of the closed-loop transfer characteristic is given by

$$\tilde{\beta} = \frac{y'''(0)}{6} = \frac{\beta}{l^4}. \tag{9}$$

This is the well-known result showing the linearizing effect of (linear) feedback on an amplifier stage.

More generally, the third-order coefficient of a cascade of n stages can be expressed as [4]

$$\tilde{\beta} = \sum_{i=1}^{n} \left[\left(\prod_{k=1}^{i-1} \tilde{\alpha}_k^3 \right) \tilde{\beta}_i \left(\prod_{j=i+1}^{n} \tilde{\alpha}_j \right) \right]. \tag{10}$$

This formula gives the relation between the local return differences and the third-order coefficient of the overall amplifier.

IV Design Example

We find that complicated problems of feedback allocation can be solved, globally and efficiently, using geometric programming. We can take as an objective any posynomial performance measure, and apply any combination of posynomial constraints. We can also compute optimal trade-off curves by varying one of the specifications or constraints over a range, computing the optimal value of the objective for each value of the specification.

A Trade-offs among bandwidth, gain, and noise

Consider a three-stage amplifier, all stages identical, with parameters

$$\alpha_i = 10, \qquad \tau_i = 10^{-6}\text{sec}, \qquad \overline{v_{n,i}} = 4.07\mu\text{V}. \tag{11}$$

The required closed-loop gain is 23.5dB. We maximized the bandwidth, subject to the equality constraint on closed-loop gain, and a maximum allowed value of input-referred noise.

Figure 6 shows the optimal bandwidth achieved, as a function of the maximum allowed input-referred noise. As it must, the optimal bandwidth increases as we relax (increase) the input-referred noise limit. Figure 7 shows the optimal values of the feedback gains as the input-referred noise limit varies.

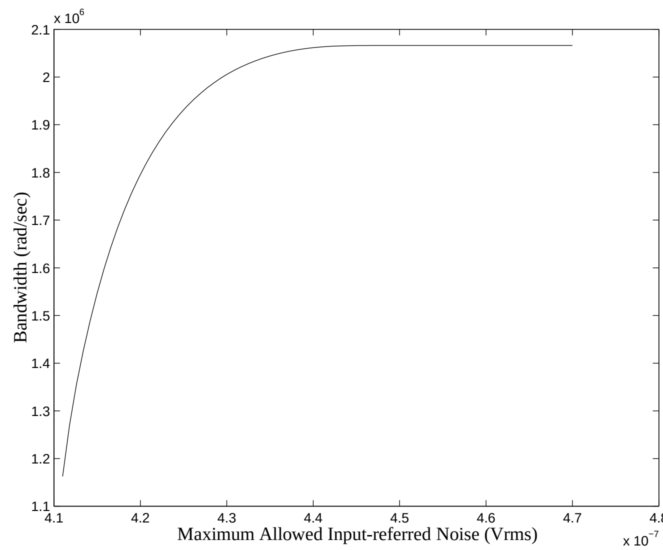

Fig. 6. Maximum bandwidth versus limit on input-referred noise.

These curves roughly identify two regions in the design space. In one, the noise constraint is so relaxed as to not be an issue. The program identifies the optimum bandwidth solution for the given gain, which is to place all of the closed loop poles in the same place. In the other, the tradeoff between bandwidth and noise is strong. The noise contribution of $\overline{v_{n,1}}$ is independent of l_1, but the noise contributions of the following stages can be diminished by making l_1 (and therefore f_1) small. It follows that f_3 is the greatest of the feedback gains, followed by f_2 and f_1.

We can also examine the optimal trade-off between bandwidth and required DC gain. Here we impose the fixed limit on input-referred noise at 4.15×10^{-7} V rms, and maximize the bandwidth subject to a required closed-loop gain.

Figures 8 and 9 show the maximum attainable bandwidth and the optimal feedback gain allocation as a function of the re-

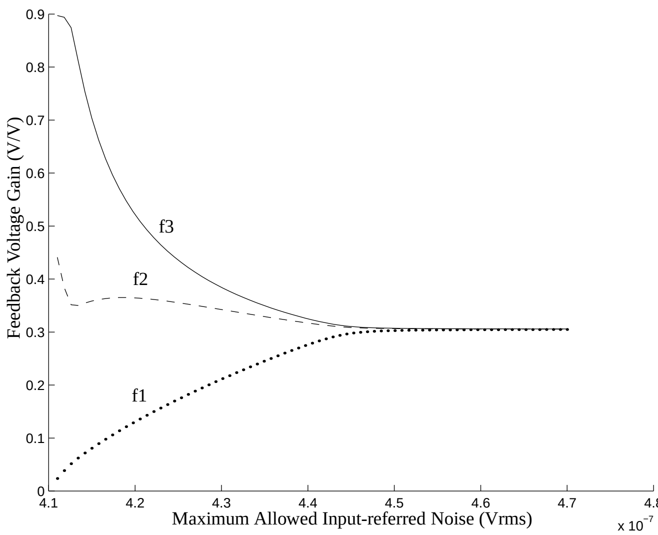

Fig. 7. Optimal feedback allocation pattern, for maximum bandwidth with limit on input-referred noise. Gain = 23.5dB.

quired closed-loop gain. Again we see two regions in the design space caused by the noise constraint.

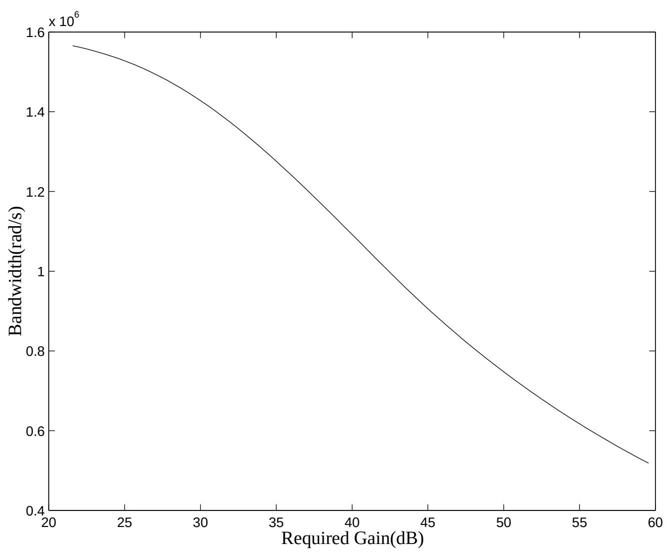

Fig. 8. Maximum bandwidth versus required closed-loop gain. Maximum input-referred noise = 4.15e-7 Vrms.

V Conclusion

In our work we have demonstrated that the local feedback allocation problem is globally solvable by the use of geometric programming. We emphasize the advantages of this method over most general methods of nonlinear optimization: there is no danger of getting "trapped" in a local extremum; there is no need for a user-supplied starting point; infeasibility can be unambiguously detected.

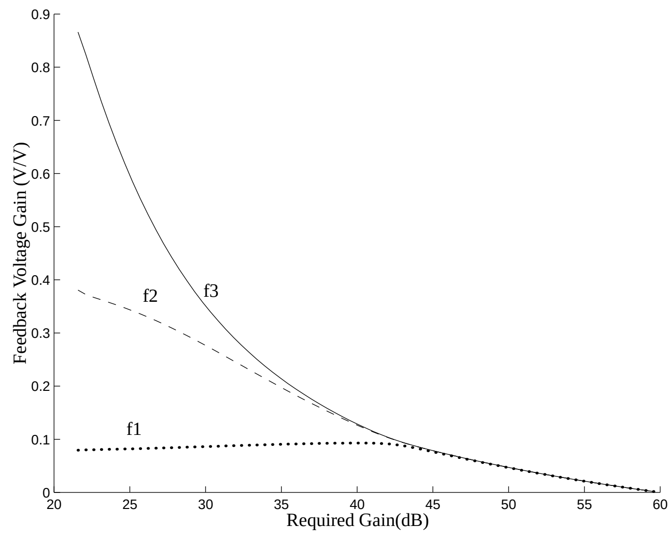

Fig. 9. Optimal feedback allocation pattern for maximum bandwidth versus required closed-loop gain. Maximum Input-referred noise = 4.15e-7 Vrms.

References

[1] H. S. Black. Stabilized feedback amplifiers. *Bell System Technical Journal*, 13:1–18, 1934.

[2] H. W. Bode. *Network Analysis and Feedback Amplifier Design*. Van Nostrand, New York, 1945.

[3] J. K. Roberge. *Operational Amplifiers: Theory and Practice*. John Wiley and Sons, Inc., New York, New York, 1975.

[4] J. L. Dawson, S. P. Boyd, M. Hershenson, and T. H. Lee. Optimal allocation of local feedback in multistage amplifiers via geometric programming. *IEEE Transactions on Circuits and Systems I*, 2000. In press.

[5] S. Boyd and L. Vandenberghe. Introduction to convex optimization with engineering applications. Course Notes, 1997. http://www.stanford.edu/class/ee364/.

[6] J. Ecker. Geometric programming: methods, computations and applications. *SIAM Rev.*, 22(3):338–362, 1980.

[7] Y. Nesterov and A. Nemirovsky. *Interior-point polynomial methods in convex programming*, volume 13 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.

[8] P. R. Gray and R. G. Meyer. *Analysis and Design of Analog Integrated Circuits*. John Wiley and Sons, New York, NY, 1993.

[9] R. J. Duffin, E. L. Peterson, and C. Zener. *Geometric Programming — Theory and Applications*. Wiley, 1967.

[10] M. Hershenson, S. Boyd, and T. H. Lee. GPCAD: A tool for CMOS op-amp synthesis. In *IEEE/ACM International Conference on Computer Aided Design*, pages 296–303, San Jose, CA, 1998.

[11] M. Hershenson, S. P. Boyd, and T. H. Lee. Automated design of folded-cascode op-amps with sensitivity analysis. In *International Conference on Electronics, Circuits and Systems*, volume 1, pages 1–1, September 1998.

[12] M. Hershenson, S. Mohan, S. Boyd, and T. H. Lee. Optimization of inductor circuits via geometric programming. In 36^{th} *Design Automation Conference*, pages 994–998, 1999.