Method of Centers For Minimizing Generalized Eigenvalues

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February 10, 1993

[1]Research supported in part by AFOSR under F49620-92-J-0013 and DARPA AF F49620-90-C-0014.

Abstract

We consider the problem of minimizing the largest generalized eigenvalue of a pair of symmetric matrices, each of which depends affinely on the decision variables. Although this problem may appear specialized, it is in fact quite general, and includes for example all linear, quadratic, and linear fractional programs. Many problems arising in control theory can be cast in this form.

The problem is nondifferentiable but quasiconvex, so methods such as Kelley's cutting-plane algorithm or the ellipsoid algorithm of Shor, Nemirovksy, and Yudin are guaranteed to minimize it. In this paper we describe relevant background material and a simple interior point method that solves such problems more efficiently. The algorithm is a variation on Huard's method of centers, using a self-concordant barrier for matrix inequalities developed by Nesterov and Nemirovsky. (Nesterov and Nemirovsky have also extended their potential reduction methods to handle the same problem [NN91b].)

Since the problem is quasiconvex but not convex, devising a non-heuristic stopping criterion (*i.e.*, one that guarantees a given accuracy) is more difficult than in the convex case. We describe several non-heuristic stopping criteria that are based on the dual of a related convex problem and a new ellipsoidal approximation that is slightly sharper, in some cases, than a more general result due to Nesterov and Nemirovsky.

The algorithm is demonstrated on an example: determining the quadratic Lyapunov function that optimizes a decay rate estimate for a differential inclusion.

Key words: quasiconvex nondifferentiable optimization, generalized eigenvalue, linear fractional programming, analytic center, method of centers, interior point method, logarithmic barrier, Newton algorithm, path-following method, ellipsoidal approximations.

To appear in *Linear Algebra and Applications,* **special issue on Linear Algebra in Systems and Control, April 1992; revised September and December 1992.**

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1 Introduction

In this paper we consider the problem of minimizing the maximum generalized eigenvalue of a (symmetric, symmetric-positive-definite) pair of matrices that depend affinely on a variable x that is subject to some convex constraints. This problem includes linear fractional programming as a special case. Our main motivation, however, is control theory, in which generalized eigenvalue minimization arises in many contexts, *e.g.*, optimal scaling of matrices with block-structured similarity transformations, determining Lyapunov functions that optimize some objective (such as stability margin), and determining a joint Lyapunov function and state feedback that optimize some objective (see for example [BGFB93, FBB92, FBBEG92, EGBFB92, BFBG92]).

The problem is quasiconvex and so can be solved reliably by several methods, for example, the ellipsoid algorithm developed by Shor, Nemirovsky, and Yudin [Sho85, NY83, BGT81, BB91] or Kelley's cutting plane algorithm [Kel60, BB91]. In this paper we describe an interior point algorithm that solves the problem, and appears to be very efficient compared to these methods. We give a simple proof of convergence for our algorithm, but we do not give a detailed complexity analysis.

The same problem has been considered by Nesterov and Nemirovsky, who have also developed an interior point algorithm to solve it [NN91b]. Moreover, they give a complete complexity analysis of their algorithm.

Since the problem is *not* convex, the problem of developing a stopping criterion or condition is more complicated than for convex problems. (In convex problems duality theory often gives us a simple stopping condition that requires little extra computation.) We propose several stopping conditions that can be used for generalized eigenvalue minimization.

When the "denominator" matrix is constant, the problem reduces to minimizing the maximum eigenvalue of a symmetric matrix that depends affinely on x. In this case, the problem is in fact convex (but still nondifferentiable). Many researchers have considered this problem. Relevant work includes Cullum et al [CDW75], Craven and Mond [CM81], Polak and Wardi [PW82], Fletcher [Fle85], Shapiro [Sha85], Friedland et al. [FNO87], Goh and Teo [GT88], Panier [Pan89], Allwright [All89], Overton [Ove88, Ove92, OW93, OW92], Ringertz [Rin91], Fan and Nekooie [FN92], and Fan [Fan92]. In [BY89], Boyd and Yang use the cutting-plane algorithm and Shor's subgradient method [Sho85] to solve eigenvalue minimization problems that arise in control theory. They also describe a saddle point method for eigenvalue mimimization due to Pyatnitski and Skorodinsky [PS83].

Interior point methods for eigenvalue minimization have recently been developed by several researchers. The first were Nesterov and Nemirovsky [NN88, NN90b, NN90a, NN91a, NN93]; others include Alizadeh [Ali92b, Ali91, Ali92a], Jarre [Jar91a], and Vandenberghe and Boyd [VB93].

Of course, general interior point methods (and the method of centers in particular) have a long history. Early work includes the SUMT book by Fiacco and McCormick [FM68], the method of centers described by Huard et al. [LH65, Hua67], and Dikin's interior point method for linear programming [Dik67]. Interest in interior point methods, mostly for lin-

ear and quadratic programs, surged in 1979 when Khachyian used the ellipsoid method developed by Shor, Nemirovsky, and Yudin to prove that linear programs can be solved in polynomial time [Kha79, GL81]. Interest surged again in 1984 when Karmarkar [Kar84] gave his interior point method for solving linear programs, which appears to have very good practical performance as well as a good worst-case complexity bound.

Since the publication of Karmarkar's paper, many researchers have studied interior point methods for linear and quadratic programming. These methods are often described in such a way that extensions to more general (convex) constraints and objectives are not clear. However, Nesterov and Nemirovsky have developed a theory of interior point methods that applies to more general convex programming problems, and in particular, problems involving eigenvalue minimization and matrix inequality constraints (see the book [NN93]). Other recent articles that consider interior point methods for more general convex programming include Sonnevend [Son88], Jarre [Jar91b], Kortanek et al. [KPY91], and the survey by Wright [Wri92].

1.1 Outline

In the remainder of section 1 we describe the notation used throughout this paper, the problem we consider (along with the assumptions), and some duality results and optimality conditions for our problem. In section 2 we show how many convex constraints can be cast in the form of an affine matrix inequality, and similarly, how many quasiconvex objectives can be expressed as maximum generalized eigenvalues of a pair of matrices that depend affinely on a variable. This justifies our claim that the problem is much more general than it might first appear.

In section 3 we discuss the idea of the analytic center of an affine matrix inequality, and in section 4 we describe the method of centers and give a simple proof of convergence. In the two following sections we discuss some important "details" of the method of centers: nonheuristic stopping criteria and some issues that arise in implementation.

In section 7 we present an example: finding a quadratic Lyapunov function for a differential inclusion that optimizes a decay rate estimate. Numerical results are given for an instance of this problem, and compared to the performance of the ellipsoid algorithm.

1.2 Notation

Throughout this paper we use the following notation. $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}^m$ the set of real (column) vectors with m components, and $\mathbf{R}^{p\times q}$ denotes the set of real $p \times q$ matrices. I will denote the identity matrix, with size determined from context. X^T is the transpose of the matrix or vector X; for an invertible matrix we abbreviate $(X^{-1})^T = (X^T)^{-1}$ as X^{-T}. $\mathcal{N}(X)$ denotes the nullspace of X. $\mathbf{Tr}X$ is the trace of a matrix $X \in \mathbf{R}^{n\times n}$, *i.e.*, $\mathbf{Tr}X \triangleq X_{11} + \cdots + X_{nn}$. Since we will often encounter expressions of the form $\mathbf{Tr}(XY)$ with X and Y symmetric matrices ($\mathbf{Tr}(XY)$ is the natural inner product), we will write

it as $\mathbf{Tr}XY$. In other words, matrix multiplication has higher precedence than the trace operator.

For symmetric matrices $X = X^T$, $Y = Y^T \in \mathbf{R}^{n\times n}$, $X < Y$ refers to the partial ordering of symmetric matrices with respect to the cone of positive definite matrices, *i.e.*, $z^TXz < z^TYz$ for all nonzero $z \in \mathbf{R}^n$. For matrices X and Y, $X \oplus Y$ will denote the block diagonal matrix formed from X and Y, *i.e.*,

$$X \oplus Y \triangleq \left[\begin{array}{cc} X & 0 \\ 0 & Y \end{array}\right].$$

The largest eigenvalue of a symmetric matrix $X = X^T \in \mathbf{R}^{n\times n}$ will be denoted $\lambda_{\max}(X)$. For a matrix (or vector) X, $\|X\|$ will denote the spectral norm or largest singular value of X, *i.e.*, $\|X\| \triangleq \left(\lambda_{\max}(X^TX)\right)^{1/2}$. (If X is a vector, $\|X\|$ reduces to the Euclidean norm, $\|X\| = \left(X^TX\right)^{1/2}$.) $\|X\|_F$ denotes the Frobenius norm of a matrix, $\|X\|_F \triangleq \left(\mathbf{Tr}X^TX\right)^{1/2}$. For a matrix $X = X^T \geq 0$, $X^{1/2}$ will denote the symmetric square-root.

In describing algorithms, a superscript of the form (k), as in $x^{(k)}$, will denote the value of a variable at the kth iteration. The symbol := will denote assignment.

1.3 Maximum generalized eigenvalue

The generalized eigenvalues of the pair $X = X^T$, $Y = Y^T > 0$, are the roots of $\det(\lambda Y - X)$, or equivalently, the eigenvalues of $Y^{-1/2}XY^{-1/2}$ (which of course are real). Throughout this paper we only encounter generalized eigenvalues of pairs of matrices X, Y with $X = X^T$ and $Y = Y^T > 0$.

The maximum generalized eigenvalue of the pair X, Y, denoted $\lambda_{\max}(X,Y)$, can be characterized in several ways:

$$\begin{aligned}
\lambda_{\max}(X,Y) &\triangleq \max\{\, \lambda \in \mathbf{R} \mid \det(\lambda Y - X) = 0 \,\} && (1)\\
&= \lambda_{\max}\left(Y^{-1/2}XY^{-1/2}\right) && (2)\\
&= \inf\{\, \lambda \in \mathbf{R} \mid \lambda Y - X > 0 \,\} && (3)\\
&= \sup\left\{\, v^TXv \;\middle|\; v^TYv = 1 \,\right\} && (4)\\
&= \sup\left\{\, \frac{\mathbf{Tr}XU}{\mathbf{Tr}YU} \;\middle|\; U = U^T \geq 0,\ U \neq 0 \,\right\}. && (5)
\end{aligned}$$

The maximum generalized eigenspace of the pair X, Y refers to

$$\mathcal{V}_{\max}(X,Y) \triangleq \mathcal{N}\left(\lambda_{\max}(X,Y)Y - X\right).$$

Excluding 0, these are precisely the vectors that achieve the supremum in (4), when scaled so that $v^TYv = 1$. Similarly, the matrices that achieve the supremum in (5) can be described in

terms of $\mathcal{V}_{\max}(X,Y)$ as follows. Let $u_1,\ldots,u_p$, with $p \geq 1$, be nonzero vectors in $\mathcal{V}_{\max}(X,Y)$, and $\sigma_1,\ldots,\sigma_p > 0$. Then

$$U = \sigma_1 u_1 u_1^T + \cdots + \sigma_p u_p u_p^T \tag{6}$$

satisfies $U = U^T \geq 0$, $U \neq 0$, $\mathbf{Tr}XU/\mathbf{Tr}YU = \lambda_{\max}(X,Y)$. Conversely, any such U can be expressed as (6) for suitable choice of u_i and σ_i. (Indeed, we can choose these vectors to be orthonormal, but we won't need this fact.) Thus, the cone of matrices that achieve the supremum in (5) is generated by the dyads uu^T formed from $u \in \mathcal{V}_{\max}(X,Y)$.

On any region in which $Y > 0$, $\lambda_{\max}(X,Y)$ is a quasiconvex function of the matrices $X = X^T$ and $Y = Y^T$, which means that for each $\lambda \in \mathbf{R}$, the sublevel set

$$\{\ (X,Y) \mid X = X^T,\ Y = Y^T > 0,\ \lambda_{\max}(X,Y) < \lambda\ \}$$

is convex, since it can be expressed as

$$\{\ (X,Y) \mid X = X^T,\ Y = Y^T > 0,\ \lambda Y - X > 0\ \}.$$

Quasiconvexity can also be characterized as follows. For any symmetric X, $\tilde{X}$, $Y > 0$, $\tilde{Y} > 0$, and $0 \leq \theta \leq 1$,

$$\lambda_{\max}(\theta X + (1-\theta)\tilde{X}, \theta Y + (1-\theta)\tilde{Y}) \leq \max\{\ \lambda_{\max}(X,Y), \lambda_{\max}(\tilde{X},\ \tilde{Y})\ \}.$$

For fixed $Y > 0$, $\lambda_{\max}(X,Y)$ is a convex function of X, but in general it is not a convex function of X and Y.

Whenever the dimension of $\mathcal{V}_{\max}(X,Y)$ exceeds one, $\lambda_{\max}(X,Y)$ is not a differentiable function of X and Y.

1.4 The problem

We consider the optimization problem with variables $x \in \mathbf{R}^m$ and $\lambda \in \mathbf{R}$ given by:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \lambda B(x) - A(x) > 0 & \\ B(x) > 0 & \\ C(x) > 0 & \end{array} \tag{7}$$

or equivalently,

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(A(x), B(x)). \\ B(x) > 0 & \\ C(x) > 0 & \end{array} \tag{8}$$

Here, A, B, and C are symmetric matrices that depend affinely on $x \in \mathbf{R}^m$:

$$A(x) \triangleq A_0 + \sum_{i=1}^m x_i A_i, \qquad B(x) \triangleq B_0 + \sum_{i=1}^m x_i B_i, \qquad C(x) \triangleq C_0 + \sum_{i=1}^m x_i C_i, \tag{9}$$

where $A_i = A_i^T$, $B_i = B_i^T \in \mathbf{R}^{r\times r}$, and $C_i = C_i^T \in \mathbf{R}^{s\times s}$.

The form of the constraint, *i.e.*, $C(x) > 0$, may seem quite specialized, but we will see in section 2 that a large variety of constraints on x including, *e.g.*, linear and convex quadratic inequalities, can be expressed in this form with suitable C.

The optimum value of (8) will be denoted $\lambda^{\rm opt}$:

$$\lambda^{\rm opt} \triangleq \inf \{ \lambda_{\max}(A(x), B(x)) \mid B(x) > 0,\ C(x) > 0 \}. \tag{10}$$

Complex Hermitian matrices are readily handled by representing them in the standard way as real symmetric matrices which are twice as big. For example, if A and B in (8) are complex Hermitian, we form the real symmetric matrices

$$\tilde{A}(x) \triangleq \begin{bmatrix} \Re A(x) & -\Im A(x) \\ \Im A(x) & \Re A(x) \end{bmatrix}, \qquad \tilde{B}(x) \triangleq \begin{bmatrix} \Re B(x) & -\Im B(x) \\ \Im B(x) & \Re B(x) \end{bmatrix}$$

and solve (8) with $\tilde{A}$, $\tilde{B}$, and C.

1.5 Assumptions

We make the following assumptions about the data in problem (8):

1. The problem is feasible and we are given an initial feasible point, *i.e.*, we know $\lambda^{(0)}$ and $x^{(0)}$ with $\lambda^{(0)}B(x^{(0)}) - A(x^{(0)}) > 0$, $B(x^{(0)}) > 0$, and $C(x^{(0)}) > 0$.

2. B is bounded away from singularity on the feasible set, *i.e.*, we know $b_{\min} > 0$ such that $C(x) > 0 \Longrightarrow B(x) \geq b_{\min} I$.

3. The feasible set is bounded, *i.e.*, there is some R such that $C(x) > 0 \Longrightarrow \|x\| \leq R$.

Let us briefly discuss these assumptions. We can find appropriate $\lambda^{(0)}$ and $x^{(0)}$, or verify that the problem is infeasible, by solving an unconstrained ("phase I") problem, *i.e.*, by minimizing the maximum eigenvalue of $-(C(x) \oplus B(x))$ (using the algorithm described in this paper, or the more efficient methods for minimizing ordinary eigenvalues mentioned in section 1). Similarly, we can find an appropriate $b_{\min}$, or determine that assumption (2) does not hold, by minimizing the maximum eigenvalue of $-B(x)$ subject to $C(x) > 0$.

Assumption (2) implies that the constraint $B(x) > 0$ appearing in (8) is redundant. We can *enforce* the assumption (2) by augmenting the original constraint $C(x) > 0$ with $B(x) \geq b_{\min} I$, *i.e.*, replacing $C(x)$ with $C(x) \oplus (B(x) - b_{\min} I)$ (adding this constraint may, of course, change the problem).

Assumption (3) implies that B is bounded on the feasible set, *i.e.*, there is a $b_{\max}$ such that $C(x) > 0 \Longrightarrow B(x) \leq b_{\max} I$. Assumptions (1) and (3) imply that the matrices $C_1, \ldots, C_m$ are linearly independent (if not, $\{x | C(x) > 0\}$ contains a line passing through $x^{(0)}$).

Of course the assumptions (1-3) imply that $\lambda^{\rm opt}$ is finite.

We make one last comment about the assumptions. A simple transformation allows us to relax assumptions (2) and (3). With assumption (1) in force, we can replace the constraint $C(x) > 0$ with $C(x) \oplus (\lambda^{(0)}B(x) - A(x)) > 0$ without affecting the problem. (The additional constraint $\lambda^{(0)}B(x) - A(x) > 0$ is equivalent to limiting the objective $\lambda_{\max}(A(x), B(x))$ to be smaller than $\lambda^{(0)}$, which does nothing since $x^{(0)}$ is a feasible point with objective less than $\lambda^{(0)}$.) For this transformed problem, assumption (2) becomes

$$\lambda^{(0)}B(x) - A(x) > 0 \text{ and } C(x) > 0 \Longrightarrow B(x) \geq b_{\min}I.$$

This same comment holds for assumption (3) as well. This trick allows us to consider some problems that were, in original form, unconstrained.

1.6 Duality and optimality conditions

Consider first a general symmetric matrix function that depends affinely on x, $F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i$. Recall that

$$\begin{array}{rcll} \{\ x \mid F(x) > 0\ \} = \emptyset & \Longleftrightarrow & \exists U = U^T \geq 0,\ U \neq 0, & (11) \\ & & \mathbf{Tr}UF_i = 0,\ i = 1, \ldots, m, & \\ & & \mathbf{Tr}UF_0 \leq 0. & \end{array}$$

This can be seen as follows. $\{x|F(x) > 0\}$ is empty if and only if the affine set $\{F(x)|x \in \mathbf{R}^m\}$ does not intersect the cone of positive definite matrices. From convex analysis, this is equivalent to the existence of a linear functional that is positive on the positive definite cone and nonpositive on the affine set of matrices. The equivalence (11) follows from the fact that the linear functionals that are positive on the positive definite cone are exactly of the form $\psi(X) = \mathbf{Tr}UX$ where U is positive semidefinite and nonzero.

Applying (11) to $F(x) = (\lambda B(x) - A(x)) \oplus C(x)$ we have:

$$\begin{array}{rcll} \lambda \leq \lambda^{\rm opt} & \Longleftrightarrow & \{\ x \mid \lambda B(x) - A(x) > 0,\ C(x) > 0\ \} = \emptyset & (12) \\ & \Longleftrightarrow & \exists U = U^T \geq 0,\ V = V^T \geq 0,\ \ U \oplus V \neq 0, & (13) \\ & & \mathbf{Tr}U(\lambda B_i - A_i) + \mathbf{Tr}VC_i = 0,\ \ i = 1, \ldots, m, & \\ & & \mathbf{Tr}U(\lambda B_0 - A_0) + \mathbf{Tr}VC_0 \leq 0. & \end{array}$$

We will use this result in section 5 to develop appropriate stopping criteria for our algorithm.

Note that we can consider the problem

$$\begin{array}{cc} \text{maximize} & \lambda \\ U = U^T \geq 0,\ V = V^T \geq 0 & \\ \mathbf{Tr}U + \mathbf{Tr}V = 1 & \\ \mathbf{Tr}U(\lambda B_i - A_i) + \mathbf{Tr}VC_i = 0,\ \ i = 1, \ldots, m & \\ \mathbf{Tr}U(\lambda B_0 - A_0) + \mathbf{Tr}VC_0 \leq 0. & \end{array} \quad (14)$$

as a sort of dual problem to the quasiconvex problem (8). The problem (14), however, has no nice convexity or quasiconvexity properties (except that for fixed λ, the set of U and V that satisfy the constraints is convex).

Let $x^{\rm opt}$ be any optimal point, *i.e.*, a limit of feasible points with maximum generalized eigenvalue converging to $\lambda^{\rm opt}$. Such a point $x^{\rm opt}$ satisfies:

$$\begin{aligned}
\lambda_{\max}(A(x^{\rm opt}), B(x^{\rm opt})) &= \lambda^{\rm opt} && (15)\\
\lambda^{\rm opt}B(x^{\rm opt}) - A(x^{\rm opt}) &\geq 0 && (16)\\
B(x^{\rm opt}) &\geq b_{\min}I && (17)\\
C(x^{\rm opt}) &\geq 0 && (18)
\end{aligned}$$

(We note, however, that conditions (15)-(18) can also be satisfied by points that are not limits of feasible points and hence not optimal.)

Now let $U^{\rm opt}$ and $V^{\rm opt}$ be a pair of matrices that satisfy the conditions in (13) for $\lambda = \lambda^{\rm opt}$. Then for all z,

$$\mathbf{Tr}U^{\rm opt}(\lambda^{\rm opt}B(z) - A(z)) + \mathbf{Tr}V^{\rm opt}C(z) = \beta, \tag{19}$$

where β does not depend on z, and $\beta \leq 0$. In particular for $z = x^{\rm opt}$, where $x^{\rm opt}$ is any optimal point, we conclude that

$$\mathbf{Tr}U^{\rm opt}(\lambda^{\rm opt}B(x^{\rm opt}) - A(x^{\rm opt})) + \mathbf{Tr}V^{\rm opt}C(x^{\rm opt}) = \beta. \tag{20}$$

Each term on the left-hand side of this equation is the trace of the product of two nonnegative definite matrices, and so must be nonnegative. So we conclude that $\beta = 0$ and moreover, both of the terms are zero:

$$\begin{aligned}
\mathbf{Tr}U^{\rm opt}(\lambda^{\rm opt}B(x^{\rm opt}) - A(x^{\rm opt})) &= 0, && (21)\\
\mathbf{Tr}V^{\rm opt}C(x^{\rm opt}) &= 0. && (22)
\end{aligned}$$

From (13) we know that at least one of $U^{\rm opt}$ and $V^{\rm opt}$ is nonzero. In fact, our assumptions imply that $U^{\rm opt} \neq 0$. If $U^{\rm opt} = 0$, then $V^{\rm opt}$ satisfies $\mathbf{Tr}V^{\rm opt}C_i = 0$, $i = 0, \ldots, m$, which by (11) implies that the constraint $C(x) > 0$ is infeasible.

We can describe the matrices $U^{\rm opt}$ and $V^{\rm opt}$ in terms of generalized eigenvectors of A, B, and C at $x^{\rm opt}$, as follows. From (21), $U^{\rm opt}$ is one of the matrices that achieves the supremum in the characterization (5), *i.e.*, $U^{\rm opt} \geq 0$, $U^{\rm opt} \neq 0$, and $\mathbf{Tr}U^{\rm opt}A(x^{\rm opt})/\mathbf{Tr}U^{\rm opt}B(x^{\rm opt}) = \lambda^{\rm opt}$. Therefore, we can express $U^{\rm opt}$ as

$$U^{\rm opt} = \sum_{i=1}^{p} \sigma_i u_i u_i^T,$$

where $u_i \in \mathcal{V}_{\max}(A(x^{\rm opt}), B(x^{\rm opt}))$, $u_i \neq 0$, and $\sigma_i > 0$. Similarly, from (22) we have

$$V^{\rm opt} = \sum_{i=1}^{q} \tau_i v_i v_i^T$$

where $v_i \in \mathcal{N}(C(x^{\rm opt}))$ and $\tau_i > 0$. (Here, however, it is possible that $q = 0$, *i.e.*, $V^{\rm opt} = 0$.)

2 Convex constraints as affine matrix inequalities

In this section we discuss ways of representing convex constraints on the variable x in the form of an affine matrix inequality $C(x) > 0$. The idea that affine matrix inequalities can be used to represent a wide variety of convex constraints can be found in Nesterov and Nemirovsky [NN90b, NN90a, NN93] (who formalize the idea of a "positive definite representable" function) and Alizadeh [Ali92b, Ali91].

2.1 Multiple constraints

We first note that multiple constraints on x, expressed as the affine matrix inequalities $C_i(x) > 0$, $i = 1, \ldots, l$, are equivalent to the single affine matrix inequality $C_1(x) \oplus \cdots \oplus C_l(x) > 0$.

2.2 Linear constraints

The constraint $a^T x < b$, where $a \in \mathbf{R}^m$ and $b \in \mathbf{R}$, is represented by $C(x) > 0$, where $C(x) = b - a^T x$. (Here $C(x) \in \mathbf{R}^{1\times 1}$.)

2.3 Convex quadratic constraints

The constraint $\|Z(x)\| < 1$, where Z is an affine function from $\mathbf{R}^m$ into $\mathbf{R}^p$, is represented as

$$C(x) = \begin{bmatrix} I & Z(x) \\ Z(x)^T & 1 \end{bmatrix} > 0.$$

The ellipsoid described by $(x - x_c)^T P^{-1}(x - x_c) < 1$, where $P = P^T > 0$, can be expressed in the alternate form

$$C(x) = \begin{bmatrix} P & x - x_c \\ (x - x_c)^T & 1 \end{bmatrix} > 0$$

(this matrix is related to the one above by a congruence).

2.4 Matrix norm constraints

More generally, a constraint on the norm of a matrix $Z(x) \in \mathbf{R}^{p\times q}$ that depends affinely on x, *i.e.*, $\|Z(x)\| < 1$, is represented as

$$C(x) = \begin{bmatrix} I & Z(x) \\ Z(x)^T & I \end{bmatrix} > 0.$$

2.5 Algebraic Riccati inequality

The (convex) Riccati constraint

$$A^T P(x) + P(x)A + P(x)BR^{-1}B^T P(x) + Q < 0,$$

where $P(x) = P(x)^T$ is an affine function of x, and A, B, $Q = Q^T$, $R = R^T > 0$, are matrices of appropriate size, can be expressed as

$$C(x) = \begin{bmatrix} -A^T P(x) - P(x)A - Q & P(x)B \\ B^T P(x) & R \end{bmatrix} > 0.$$

These inequalities arise in control theory[BGFB93].

2.6 Schur complement constraints

The constraints described above are special cases of constraints having a "Schur complement form":

$$Q(x) - S(x)R(x)^{-1}S(x)^T > 0 \quad \text{and} \quad R(x) > 0, \tag{23}$$

where $Q(x) = Q(x)^T$, $S(x)$ and $R(x) = R(x)^T$ are matrices of appropriate size that depend affinely on the vector x. The constraint (23) can be represented as

$$C(x) = \begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0.$$

2.7 Quasiconvex functions as generalized eigenvalues

Analogously, many quasiconvex functions can be represented in the form $\lambda_{\max}(A(x), B(x))$ (with some suitable constraint that ensures $B(x) > 0$). For example, the maximum of two functions expressed in this form can be expressed in this form by forming block diagonal matrices.

The sum of two quasiconvex functions expressed in the form $\lambda_{\max}(A(x), B(x))$ need not be quasiconvex, and therefore cannot in general be expressed in the same form. However, the sum of the (convex) objectives $\lambda_{\max}(A_1(x)) + \lambda_{\max}(A_2(x))$ is readily handled. The problem

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(A_1(x)) + \lambda_{\max}(A_2(x)) \\ x \in \mathbf{R}^m & \\ C(x) > 0 & \end{array}$$

is equivalent in the obvious way to the problem with $m + 2$ variables

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(z_1 + z_2) \\ x \in \mathbf{R}^m,\ z \in \mathbf{R}^2 & \\ C(x) > 0 & \\ z_1 I - A_1(x) > 0 & \\ z_2 I - A_2(x) > 0 & \end{array}$$

which is of the form (8).

Several common convex objectives can be expressed as ordinary maximum eigenvalues, *i.e.*, in the form $\lambda_{\max}(A(x))$. The objective $\|Z(x)\|^2$, where $Z(x) \in \mathbf{R}^{p\times q}$ is an affine function of x, is given by

$$\lambda_{\max}\left(\begin{bmatrix} 0 & Z(x) \\ Z(x)^T & 0 \end{bmatrix}\right).$$

This includes all quadratic ($q = 1$) and (squared) matrix norm objectives.

The usual linear fractional objective is given by

$$\frac{a^Tx + b}{c^Tx + d} = \lambda_{\max}(a^Tx + b, c^Tx + d)$$

(where $c^Tx + d > 0$). So the problem (7) includes all linear, linear fractional, and quadratic programs.

The linear fractional objective can be generalized to a quasiconvex "norm of matrix fractional" objective as follows. Given affine functions $N(x) = N(x)^T$, $D(x) = D(x)^T$, we have

$$\left\|N(x)^{1/2}D(x)^{-1/2}\right\|^2 = \lambda_{\max}\left(N(x), D(x)\right)$$

(for $N(x)$, $D(x) > 0$).

Using Schur complements we can express several interesting convex and quasiconvex functions as maximum eigenvalues or maximum generalized eigenvalues. As an example consider the convex function

$$\mathbf{Tr}N(x)^TD(x)^{-1}N(x) = \left\|N(x)^TD(x)^{-1/2}\right\|_F^2 \tag{24}$$

where $N(x) \in \mathbf{R}^{p\times q}$ and $D(x) = D(x)^T$ are affine functions of x (and $D(x) > 0$). The objective (24) can be minimized by introducing a "slack matrix" $Y \in \mathbf{R}^{q\times q}$:

$$\begin{array}{c} \text{minimize} \qquad\qquad \lambda. \\ x \in \mathbf{R}^m,\ Y \in \mathbf{R}^{q\times q},\ \lambda \in \mathbf{R} \\ \mathbf{Tr}Y < \lambda \\ \begin{bmatrix} Y & N(x)^T \\ N(x) & D(x) \end{bmatrix} > 0 \end{array}$$

The function (24) includes as a special case the quadratic-over-linear objective $\|Ax + b\|/(c^Tx + d)$. Note also that by substituting $Y < \lambda I$ for $\mathbf{Tr}Y < \lambda$ we can minimize the convex function

$$\lambda_{\max}\left(N(x)^TD(x)^{-1}N(x)\right) = \left\|N(x)^TD(x)^{-1/2}\right\|^2.$$

As a final example we consider the condition number of a positive definite matrix A that depends affinely on x, which is readily minimized as follows:

$$\begin{array}{c} \text{minimize} \qquad\qquad \lambda_{\max}(A(x), \mu I). \\ A(x) - \mu I > 0,\ \mu > 0 \end{array}$$

3 Analytic center of an affine matrix inequality

Throughout this section, which is completely independent of the others, we consider a general affine matrix inequality $F(x) > 0$, where

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i$$

and $F_i = F_i^T \in \mathbf{R}^{n\times n}$. We will assume that the matrices $F_1, \ldots, F_m$ are linearly independent. We denote the feasible set by $\mathbf{X}$:

$$\mathbf{X} \triangleq \{\ x \in \mathbf{R}^m \ \mid F(x) > 0\ \}.$$

3.1 A barrier function for X

The function

$$\phi(x) \triangleq \begin{cases} \log\det F(x)^{-1} & x \in \mathbf{X} \\ \infty & x \notin \mathbf{X} \end{cases} \tag{25}$$

is finite if and only if $x \in \mathbf{X}$, and becomes infinite as x approaches the boundary of $\mathbf{X}$, *i.e.*, it is a *barrier function* for $\mathbf{X}$. There are many other barrier functions for $\mathbf{X}$ (for example, trace can be substituted for determinant in (25)), but this one enjoys many special properties. In particular, when $x \in \mathbf{X}$, it is analytic and strictly convex.

We first give formulas for the gradient $g(x)$ and Hessian $H(x)$ of ϕ at $x \in \mathbf{X}$. It is readily shown (see appendix A) that

$$\begin{aligned} g_i(x) &= -\mathbf{Tr} F(x)^{-1} F_i & (26)\\ &= -\mathbf{Tr} F(x)^{-1/2} F_i F(x)^{-1/2} & (27) \end{aligned}$$

for $i = 1, \ldots, m$. Similarly,

$$\begin{aligned} H_{ij}(x) &= \mathbf{Tr} F(x)^{-1} F_i F(x)^{-1} F_j & (28)\\ &= \mathbf{Tr}\left(F(x)^{-1/2} F_i F(x)^{-1/2}\right)\left(F(x)^{-1/2} F_j F(x)^{-1/2}\right) & (29) \end{aligned}$$

for $i, j = 1, \ldots, m$.

From (29) we can verify that ϕ is strictly convex for $x \in \mathbf{X}$. For $x \in \mathbf{X}$ and $y \in \mathbf{R}^m$,

$$\begin{aligned} y^T H(x) y &= \sum_{i,j=1}^{m} y_i y_j \mathbf{Tr}\left(F(x)^{-1/2} F_i F(x)^{-1/2}\right)\left(F(x)^{-1/2} F_j F(x)^{-1/2}\right) & (30)\\ &= \mathbf{Tr}\left(F(x)^{-1/2}\left(\sum_{i=1}^{m} y_i F_i\right) F(x)^{-1/2}\right)^2 & (31)\\ &= \left\| F(x)^{-1/2}\left(\sum_{i=1}^{m} y_i F_i\right) F(x)^{-1/2}\right\|_F^2 \geq 0 & (32) \end{aligned}$$

which establishes that ϕ is convex in x. From (32) we see that $y^T H(x)y = 0$ if and only if $\sum_{i=1}^m y_i F_i = 0$. By independence of $F_1, \ldots, F_m$, we conclude that $H(x) > 0$, *i.e.*, ϕ is strictly convex.

For future use we note a few more important formulas. From (27) we see that for $x \in \mathbf{X}$ and $z \in \mathbf{R}^m$ we have

$$\begin{aligned} \mathbf{Tr} F(x)^{-1/2} F(z) F(x)^{-1/2} &= \mathbf{Tr} F(x)^{-1/2} \left(F(x) + \sum_{i=1}^m (z_i - x_i) F_i \right) F(x)^{-1/2} \quad (33) \\ &= n - g(x)^T (z - x). \quad (34) \end{aligned}$$

From (29) we have:

$$\begin{aligned} \left\| F(x)^{-1/2} F(z) F(x)^{-1/2} \right\|_F^2 &= \mathbf{Tr} \left(F(x)^{-1/2} \left(F(x) + \sum_{i=1}^m (z_i - x_i) F_i \right) F(x)^{-1/2} \right)^2 \quad (35) \\ &= (z - x)^T H(x)(z - x) - 2g(x)^T (z - x) + n. \quad (36) \end{aligned}$$

The barrier function ϕ is bounded below if and only if $\mathbf{X}$ is bounded. The "if" part is clear. To see the "only if" part, suppose that $\mathbf{X}$ is unbounded. Since it is convex it must contain a ray, say $\{x_0 + \alpha v | \alpha \geq 0\}$, where $v \neq 0$. Since $F(x) > 0$ along this ray we conclude that

$$\tilde{F} \triangleq \sum_{i=1}^m v_i F_i \geq 0.$$

By independence of $F_1, \ldots, F_m$, $\tilde{F}$ is nonzero. It follows that $\det F(x_0 + \alpha v)$, which is a polynomial in α with degree equal to the rank of $\tilde{F}$, grows at least linearly with α. Therefore, ϕ is unbounded below on the ray.

3.2 Analytic center

We suppose now that $\mathbf{X}$ is nonempty and bounded. From the discussion above we conclude that ϕ has a unique minimizer, which we denote x^*:

$$x^* \triangleq \operatorname*{argmin}_x \phi(x). \quad (37)$$

We refer to x^* as the *analytic center* of the affine matrix inequality $F(x) > 0$. Equivalently,

$$x^* = \operatorname*{argmax}_{x \in \mathbf{X}} \det F(x), \quad (38)$$

that is, $F(x^*)$ has maximum determinant, among all positive definite matrices of the form $F(x)$. Note that the analytic center is invariant with respect to congruence transformations, *i.e.*, the analytic center of $F(x) > 0$ is the same as the analytic center of $Z^T F(x) Z > 0$ for any nonsingular matrix Z.

From (27) we see that x^* is characterized by

$$\mathbf{Tr} F(x^*)^{-1} F_i = 0, \quad i = 1, \ldots, m, \tag{39}$$

or equivalently

$$\mathbf{Tr} F(x^*)^{-1} F(x) = n, \quad \text{for all } x \tag{40}$$

(since the left-hand side is independent of x, and the right-hand side is its value at x^*). Thus, $F(x^*)^{-1}$ is orthogonal to the span of $F_1, \ldots, F_m$.

The definition (37) of the analytic center of an affine matrix inequality follows Nesterov and Nemirovsky [NN93] (see also Sonnevend [Son91]). It agrees with the usual definition of the analytic center of a set of linear inequalities (see *e.g.*, Sonnevend [Son86]), $a_i^T x < b_i$, $i = 1, \ldots, n$ (which can be represented as an affine matrix inequality with diagonal matrices). In this case, x^* maximizes among feasible points $\prod_{i=1}^n (b_i - a_i^T x)$, or equivalently, the product of the distances to the constraint planes $a_i^T x = b_i$.

3.3 Ellipsoidal approximations

The level curves of the barrier function ϕ give a smooth approximation of the shape of the boundary of $\mathbf{X}$, which of course need not be smooth. Near x^* the shape of these level curves is determined by $H(x^*)$, so it seems plausible that the ellipsoids centered at x^* and with shape determined by $H(x^*)$ should give a good quadratic approximation of the shape of $\mathbf{X}$. Alternatively, it seems that $\mathbf{X}$ should be reasonably well conditioned in the coordinates given by $\bar{x} = H(x^*)^{-1/2} x$.

This intuition is correct. The following inner and outer ellipsoidal approximations hold for $\mathbf{X}$:

$$\mathcal{E}_{\text{in}} \subseteq \mathbf{X} \subseteq \mathcal{E}_{\text{out}},$$

where the ellipsoids $\mathcal{E}_{\text{in}}$ and $\mathcal{E}_{\text{out}}$ are given by

$$\mathcal{E}_{\text{in}} \triangleq \left\{ x \in \mathbf{R}^m \;\middle|\; (x - x^*)^T H(x^*)(x - x^*) < 1 \right\}, \tag{41}$$

$$\mathcal{E}_{\text{out}} \triangleq \left\{ x \in \mathbf{R}^m \;\middle|\; (x - x^*)^T H(x^*)(x - x^*) \leq n(n-1) \right\}. \tag{42}$$

A proof is given in appendix B. The inner ellipsoidal approximation holds for a general class of barrier functions (called *self-concordant*), which includes our barrier function (25), and is given in [NN88, NN93]. The outer approximation (42) is similar to an outer approximation given by Nesterov and Nemirovsky, which holds for these more general (self-concordant) barriers.

3.4 Nesterov and Nemirovsky's Newton algorithm

Newton's method, with appropriate step length selection, can be used to efficiently compute x^*, given an initial point in $\mathbf{X}$. We consider the algorithm:

$$x^{(k+1)} := x^{(k)} - \alpha^{(k)} H(x^{(k)})^{-1} g(x^{(k)}), \tag{43}$$

where $\alpha^{(k)}$ is the damping factor of the kth iteration.

In [NN93], Nesterov and Nemirovsky give a simple step length rule appropriate for the general class of self-concordant barrier functions mentioned earlier, along with a complete convergence analysis and sharp bounds on the number of iterations required to compute the analytic center to within a given accuracy, starting from a given initial feasible point. We refer the reader to [NN93] for details of this generalization.

Their damping factor depends on a quantity which they call the *Newton decrement* of ϕ at x:

$$\delta(x) \triangleq \left\| H(x)^{-1/2} g(x) \right\|.$$

(The name comes from the observation that $\delta(x)^2/2$ is the difference between $\phi(x)$ and the minimum value of the quadratic approximation of ϕ at x. Alternatively, $\delta(x)$ is the length of the Newton step $-H(x)^{-1}g(x)$ measured in the norm induced by the Hessian $H(x)$.) The Nesterov-Nemirovsky damping factor is:

$$\alpha^{(k)} := \begin{cases} 1 & \text{if } \delta(x^{(k)}) \leq 1/4 \\ 1/(1+\delta(x^{(k)})) & \text{if } \delta(x^{(k)}) > 1/4 \end{cases} \tag{44}$$

Nesterov and Nemirovsky show that this step length always results in $x^{(k+1)} \in \mathbf{X}$ (the inner ellipsoidal approximation in appendix (B) shows that $x^{(k+1)} \in \mathbf{X}$ provided $\alpha^{(k)} < 1/\delta(x^{(k)})$). Moreover, for $\delta(x^{(k)}) < 1/4$, we have $\delta(x^{(k+1)}) \leq 2\delta(x^{(k)})^2$, *i.e.*, the algorithm converges quadratically once we start taking undamped Newton steps. They show that whenever $\delta(x^{(k)}) > 1/4$, $\phi(x^{(k)}) - \phi(x^{(k+1)}) > c$, where c is some absolute constant. Using this fact they bound the number of iterations required to reach the region of quadratic convergence.

Their analysis holds for step length given by exact line search, *i.e.*,

$$\alpha^{(k)} := \operatorname*{argmin}_{\alpha} \ \phi\left(x^{(k)} - \alpha H(x^{(k)})^{-1} g(x^{(k)})\right),$$

since the reduction of ϕ while $\delta > 1/4$ must exceed the absolute constant c guaranteed using the step length rule (44). (See section 6.5 for a discussion of exact line search.)

3.5 A least-squares interpretation

The undamped Newton step $-H(x)^{-1}g(x)$ can be interpreted as the solution of an appropriate weighted least-squares problem:

$$-H(x)^{-1}g(x) = \operatorname*{argmin}_{v \in \mathbf{R}^m} \left\| F(x)^{-1/2} F(x-v) F(x)^{-1/2} \right\|_F \tag{45}$$

$$= \operatorname*{argmin}_{v \in \mathbf{R}^m} \left\| I - \sum_{i=1}^{m} v_i F(x)^{-1/2} F_i F(x)^{-1/2} \right\|_F \tag{46}$$

Thus, the Newton step at x is given by projecting I onto the span of the normalized matrices $F(x)^{-1/2}F_iF(x)^{-1/2}$.

We can give a rough interpretation of this result. We are trying to make $F(z)$ "large" (as measured by the determinant). To do this, we first normalize the problem by a congruence transformation (multiply each F_i on the left and right by $F(x)^{-1/2}$), so that, in effect, we have $F(x) = I$. Now we find the "smallest" $F(z)$, as measured by the Frobenius norm. Let us call the minimizer $x_{\rm small}$. The Newton step is then given by the *opposite* of the step from x to $x_{\rm small}$. (Roughly speaking, if stepping from x to $x_{\rm small}$ makes F "smaller", then stepping in the opposite direction should make F "larger".)

The result can also be seen as follows. Suppose the problem has been normalized by a congruence transformation so that $F(x) = I$. Now consider the two functions $\phi(x) = \log\det F(x)^{-1}$ and $\psi(x) = \frac{1}{2}\|F(x)\|_F^2$. From the formulas for the gradient and Hessian of ϕ (with $F(x) = I$) we see that the gradients of ϕ and ψ at x are the same, except for a change of sign, and the Hessians are identical. Therefore the Newton step for ϕ is the negative of the Newton step for ψ. Since ψ is quadratic, its Newton step is the difference between x and its minimizer.

The Newton decrement at x is related to the distance between I and the span of the normalized matrices:

$$n - \delta(x)^2 = \min_{v\,\in\,\mathbf{R}^m} \left\| I - \sum_{i=1}^m v_i F(x)^{-1/2}F_iF(x)^{-1/2} \right\|_F^2. \tag{47}$$

Equivalently,

$$\delta(x) = \left\| \sum_{i=1}^m v_i F(x)^{-1/2}F_iF(x)^{-1/2} \right\|_F \tag{48}$$

where $v = -H(x)^{-1}g(x)$ is the Newton step. These results follow from the formula (36) noted in section 3.2.

This least-squares interpretation of the Newton step generalizes a well known fact for the linear inequalities $a_i^Tx < b_i,\ \ i = 1,\ldots,n$. In this case the Newton step is given by the diagonally weighted least-squares problem

$$-H(x)^{-1}g(x) = \operatorname*{argmin}_{v\,\in\,\mathbf{R}^m} \sum_{i=1}^n \left(\frac{a_i^Tv}{b_i - a_i^Tx} - 1 \right)^2.$$

4 The method of centers

We now consider again the problem (7):

$$\begin{array}{ll} \text{minimize} & \lambda. \\ \lambda B(x) - A(x) > 0 & \\ C(x) > 0 & \end{array}$$

Let $n \triangleq r + s$, so that $(\lambda B - A) \oplus C \in \mathbf{R}^{n\times n}$ (recall that $A,\ B \in \mathbf{R}^{r\times r}$ and $C \in \mathbf{R}^{s\times s}$).

4.1 Path of centers

The assumptions of section (1.5) imply that for $\lambda > \lambda^{\rm opt}$, the set $\{x|(\lambda B(x)-A(x))\oplus C(x) > 0\}$ is nonempty and bounded; therefore the analytic center of the inequality $(\lambda B(x)-A(x))\oplus C(x) > 0$ is well defined. We will denote this analytic center by $x^*(\lambda)$ when we need to emphasize its dependence on the parameter λ. To simplify notation we will write $x^*(\lambda)$ as x^* when λ is understood.

From (39) we see that x^* is characterized by

$$\mathbf{Tr}(\lambda B(x^*) - A(x^*))^{-1}(\lambda B_i - A_i) + \mathbf{Tr}C(x^*)^{-1}C_i = 0, \quad i = 1, \ldots, m. \tag{49}$$

The curve given by $x^*(\lambda)$ for $\lambda > \lambda^{\rm opt}$ is called the *path of centers*. It can be shown that it is analytic and has a limit as $\lambda \to \lambda^{\rm opt}$, which we denote $x^{\rm opt}$ (see *e.g.*, [FM68]). $x^{\rm opt}$ is optimal, since for all $\lambda > \lambda^{\rm opt}$, $x^*(\lambda)$ is feasible and

$$\lambda^{\rm opt} \leq \lambda_{\max}(A(x^*(\lambda)), B(x^*(\lambda))) \leq \lambda.$$

(There may be other optimal points too.) Since $x^{\rm opt}$ is optimal it satisfies the conditions given in (15)–(18).

4.2 A dual bound on the path of centers

Let us fix $\lambda > \lambda^{\rm opt}$. Let A^* denote $A(x^*(\lambda))$, and similarly for B^* and C^*.

We will show that

$$\lambda - \lambda^{\rm opt} \leq \eta\,(\lambda - \lambda_{\max}(A^*, B^*)) \tag{50}$$

where $\eta \stackrel{\Delta}{=} nb_{\max}/b_{\min}$ (recall that $n = r + s$ is the size of $(\lambda B - A) \oplus C$, and $b_{\min}$ and $b_{\max}$ are defined in section 1.5). We can put (50) in the form:

$$\lambda_{\max}(A^*, B^*) - \lambda^{\rm opt} \leq \left(1 - \frac{1}{\eta}\right)(\lambda - \lambda^{\rm opt}). \tag{51}$$

This equation shows that the maximum generalized eigenvalue at the analytic center of $(\lambda B - A) \oplus C > 0$ is guaranteed to be a fixed fraction closer to $\lambda^{\rm opt}$ than λ.

Define

$$U \stackrel{\Delta}{=} (\lambda B^* - A^*)^{-1}, \qquad V \stackrel{\Delta}{=} C^{*-1}.$$

From (49) we see that

$$\mathbf{Tr}U(\lambda B(z) - A(z)) + \mathbf{Tr}VC(z) = n \tag{52}$$

for all z (*cf.* (19)), so in particular

$$\mathbf{Tr}U(\lambda B(x^{\rm opt}) - A(x^{\rm opt})) + \mathbf{Tr}VC(x^{\rm opt}) = n. \tag{53}$$

Since $V > 0$ and $C(x^{\rm opt}) \geq 0$, $\mathbf{Tr}VC(x^{\rm opt}) \geq 0$, so we have

$$\lambda \mathbf{Tr}UB(x^{\rm opt}) - n \leq \mathbf{Tr}UA(x^{\rm opt}). \tag{54}$$

Since $U > 0$ and $B(x^{\rm opt}) > b_{\min} I$, $\mathbf{Tr} U B(x^{\rm opt}) > 0$ and therefore

$$\lambda - \frac{n}{\mathbf{Tr} U B(x^{\rm opt})} \leq \frac{\mathbf{Tr} U A(x^{\rm opt})}{\mathbf{Tr} U B(x^{\rm opt})} \leq \lambda_{\max}(A(x^{\rm opt}), B(x^{\rm opt})) = \lambda^{\rm opt} \tag{55}$$

(the second inequality uses (5)). Thus we have:

$$\lambda - \lambda^{\rm opt} \leq \frac{n}{\mathbf{Tr} U B(x^{\rm opt})} \leq \frac{n}{b_{\min} \mathbf{Tr} U} \tag{56}$$

(The second inequality uses $\mathbf{Tr} U B(x^{\rm opt}) \geq b_{\min} \mathbf{Tr} U$, which follows from $B(x^{\rm opt}) \geq b_{\min} I$).

Now we note that

$$\mathbf{Tr} U = \mathbf{Tr}(\lambda B^* - A^*)^{-1} \geq \frac{1}{(\lambda - \lambda_{\max}(A^*, B^*))\|B^*\|}$$

which we prove in appendix C. Finally, noting that $\|B^*\| \leq b_{\max}$, we have

$$\lambda - \lambda^{\rm opt} \leq (n b_{\max}/b_{\min})(\lambda - \lambda_{\max}(A^*, B^*)) \tag{57}$$

which is the desired result.

This is the simplest dual bound for the objective that can be obtained; in section 5 we derive more complicated, but better, bounds.

4.3 Basic algorithm

Perhaps the simplest optimization algorithm based on the notion of analytic center is the *method of centers* due to Lieu and Huard [LH65, Hua67]. We describe here a simple variation on the method of centers.

The algorithm is initialized with $\lambda^{(0)}$ and $x^{(0)}$ with $\lambda^{(0)} B(x^{(0)}) - A(x^{(0)}) > 0$ and $C(x^{(0)}) > 0$, and proceeds as follows:

$$\lambda^{(k+1)} := (1-\theta)\lambda_{\max}(A(x^{(k)}), B(x^{(k)})) + \theta\lambda^{(k)} \tag{58}$$

$$x^{(k+1)} := x^*(\lambda^{(k+1)}) \tag{59}$$

where θ is a parameter with $0 < \theta < 1$.

The classic method of centers is obtained with $\theta = 0$. In this case, however, $x^{(k)}$ does not (quite) satisfy the new inequality $\lambda^{(k+1)} B(x) - A(x) \oplus C(x) > 0$. With $\theta > 0$, however, the current iterate $x^{(k)}$ is feasible for the tightened inequality $\left(\lambda^{(k+1)} B(x) - A(x)\right) \oplus C(x) > 0$, and therefore can be used as the initial point in computing the next iterate $x^*(\lambda^{(k+1)})$.

We now give a simple proof of convergence. From (51), we have

$$\lambda_{\max}(A(x^{(k)}), B(x^{(k)})) - \lambda^{\rm opt} \leq \left(1 - \frac{1}{\eta}\right)\left(\lambda^{(k)} - \lambda^{\rm opt}\right). \tag{60}$$

Subtracting $\lambda^{\rm opt}$ from both sides of (58) yields

$$\lambda^{(k+1)} - \lambda^{\rm opt} = (1-\theta)(\lambda_{\max}(A(x^{(k)}), B(x^{(k)})) - \lambda^{\rm opt}) + \theta(\lambda^{(k)} - \lambda^{\rm opt}). \tag{61}$$

Substituting (60) into this, we have

$$\lambda^{(k+1)} - \lambda^{\rm opt} \leq \left(1 - \frac{1-\theta}{\eta}\right)(\lambda^{(k)} - \lambda^{\rm opt}),$$

so that

$$\lambda^{(k)} - \lambda^{\rm opt} \leq \left(1 - \frac{1-\theta}{\eta}\right)^k (\lambda^{(0)} - \lambda^{\rm opt}).$$

Thus, $\lambda^{(k)}$ converges to $\lambda^{\rm opt}$ at least geometrically.

5 Stopping criteria

5.1 Objective duality gap

From (57), we see that the stopping criterion

$$\lambda^{(k)} - \lambda_{\max}(A(x^{(k)}), B(x^{(k)})) \leq \frac{\epsilon b_{\min}}{n b_{\max}}$$

guarantees that on exit, $\lambda^{(k)} - \lambda^{\rm opt} \leq \epsilon$, and therefore

$$\lambda_{\max}(A(x^{(k)}), B(x^{(k)})) - \lambda^{\rm opt} \leq \epsilon.$$

This simple stopping criterion has essentially no computational cost, since $\lambda_{\max}(A(x^{(k)}), B(x^{(k)}))$ must be computed to find $\lambda^{(k+1)}$ anyway. In this section we investigate better lower bounds on $\lambda^{\rm opt}$ that can be obtained with a little more computation.

Using the notation of section 4.2, we derive from (53) the inequality

$$\lambda - \lambda^{\rm opt} \leq \frac{n - \mathbf{Tr} VC(x^{\rm opt})}{\mathbf{Tr} UB(x^{\rm opt})}. \tag{62}$$

The idea is to derive some computable upper bounds on the right-hand side.

Let us list some information we have, once we have computed $x^*(\lambda)$:

$$C(x^{\rm opt}) \geq 0, \tag{63}$$
$$B(x^{\rm opt}) \geq b_{\min} I, \tag{64}$$
$$x^{\rm opt} \in \mathcal{E}_{\rm out} \tag{65}$$

where $\mathcal{E}_{\rm out}$ is the outer ellipsoid given in (42).

Using (63) and (64), we derive the bound

$$\lambda - \lambda^{\rm opt} \leq \frac{n}{b_{\rm min}\mathbf{Tr}U} \tag{66}$$

(which is always better than the simple bound (50)).

Another bound can be derived using (65) and the inequality (62):

$$\lambda - \lambda^{\rm opt} \leq \max_{z \in \mathcal{E}_{\rm out}} \frac{n - \mathbf{Tr}VC(z)}{\mathbf{Tr}UB(z)}. \tag{67}$$

Note that $(n - \mathbf{Tr}VC(z))/\mathbf{Tr}UB(z)$ is a linear fractional form in z. Therefore the right-hand side of (67) is readily computed—there is a "closed form" expression for the maximum of a linear fractional form over an ellipsoid, which is derived in appendix D. The bound (67) has one major drawback, however: it can be worse than the simple bound (50). It can even happen that the hyperplane $\{z|\mathbf{Tr}UB(z) = 0\}$ intersects the ellipsoid $\mathcal{E}_{\rm out}$, in which case the right-hand side of (67) is infinite.

This problem can be circumvented. From (64) we know that $\mathbf{Tr}UB(x^{\rm opt}) \geq b_{\rm min}\mathbf{Tr}U$. Hence $x^{\rm opt}$ lies in the halfspace $\{z|\mathbf{Tr}UB(z) \geq b_{\rm min}\mathbf{Tr}U\}$. Similarly, from (63) we know that $\mathbf{Tr}VC(x^{\rm opt}) \geq 0$, *i.e.*, $x^{\rm opt}$ lies in the halfspace $\{z|\mathbf{Tr}VC(z) \geq 0\}$. Therefore we can localize $x^{\rm opt}$ to the intersection of $\mathcal{E}_{\rm out}$ and these two halfspaces.

A bound that uses this information is:

$$\lambda - \lambda^{\rm opt} \leq \max_{\substack{z \in \mathcal{E}_{\rm out} \\ n - \mathbf{Tr}VC(z) \leq n \\ \mathbf{Tr}UB(z) \geq b_{\rm min}\mathbf{Tr}U}} \frac{n - \mathbf{Tr}VC(z)}{\mathbf{Tr}UB(z)}. \tag{68}$$

This bound is always better than all the bounds described so far. To compute it requires the solution of the following problem: maximize a linear fractional form over an ellipsoid, subject to an upper bound on the numerator and a (positive) lower bound on the denominator. This problem also has a "closed form" solution, given in appendix E. This solution is harder to describe than in the unconstrained case. Computing it, however, requires essentially no additional effort compared to the unconstrained problem.

5.2 Constraint duality gap

We continue to use the notation of section 4.2. In this section we show that

$$\lambda \leq \min_{C(x) > n\lambda_{\rm min}(C^*)} \lambda_{\rm max}(A(x), B(x)). \tag{69}$$

This means that λ is a lower bound on the minimum value of the maximum generalized eigenvalue of the pair (A, B), subject to the tightened constraint $C(x) > n\lambda_{\rm min}(C^*)$. If the

constraint $C(x) > 0$ is active at $x^{\rm opt}$ for the original problem, then $\lambda_{\min}(C^*)$ converges to zero as λ approaches $\lambda^{\rm opt}$. The result (69) shows that in this case, the stopping criterion

$$\lambda_{\min}(C(x^{(k)})) < \epsilon/n$$

guarantees that on exit, $x^{(k)}$ is ϵ-optimal for the "tightened" problem

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(A(x), B(x)). \\ C(x) > \epsilon I & \end{array}$$

To show (69), recall that

$$\mathbf{Tr}U(\lambda B_i - A_i) + \mathbf{Tr}VC_i = 0, \quad i = 1, \ldots, m \tag{70}$$

and

$$\mathbf{Tr}U(\lambda B_0 - A_0) + \mathbf{Tr}VC_0 = \mathbf{Tr}U(\lambda B^* - A^*) + \mathbf{Tr}VC^* = n.$$

Let $\mu \triangleq \lambda_{\min}(C^*)$. Then

$$\mathbf{Tr}U(\lambda B_0 - A_0) + \mathbf{Tr}V(C_0 - \mu I) = n - \mu \mathbf{Tr}V.$$

From

$$\mathbf{Tr}V = \mathbf{Tr}C^{*-1} \geq \frac{n}{\lambda_{\min}(C^*)}$$

we conclude

$$\mathbf{Tr}U(\lambda B_0 - A_0) + \mathbf{Tr}V(C_0 - \mu I) \leq 0. \tag{71}$$

Now note that (70) and (71) establish that the affine matrix inequality

$$(\lambda B(x) - A(x)) \oplus (C(x) - \mu I) > 0$$

is infeasible (by the duality result (11)). This establishes (69).

6 Some notes on implementation

In this section we briefly mention some of the issues that arise in implementing the method of centers.

6.1 Problem structure

In many problems, the matrices A, B, and C, and hence $F = (\lambda B - A) \oplus C$ have a block diagonal structure, say,

$$F_i \in \bigoplus_{j=1}^{K} \mathbf{R}^{n_j \times n_j},$$

where $n_1 + \cdots + n_K = n$. Moreover, each of these K blocks may have one of the special structures mentioned in section 2, *e.g.*, the special structure that corresponds to a quadratic constraint.

The choice of method used to compute the Newton step depends on how much of the problem structure we choose to exploit. As far as we know, there is not a simple description of a "best" method that exploits all of the structure.

For future reference we note an inequality relating m and $\sum_{j=1}^K n_j^2$. Since the dimension of the set of symmetric matrices in $\bigoplus_{j=1}^K \mathbf{R}^{n_j \times n_j}$ is $\sum_{j=1}^K n_j(n_j+1)/2$ and the matrices F_i are independent (otherwise the feasible set contains a line, violating the assumptions), we have

$$m \leq \sum_{j=1}^K n_j(n_j+1)/2. \tag{72}$$

6.2 Normalizing with Cholesky factors

In numerical computations based on the formulas of section 3, it is more convenient to use a triangular factor of $F(x)^{-1}$ instead of the symmetric square-root $F(x)^{-1/2}$ that appears throughout section 3. All of the formulas of that section are readily modified to use triangular factors rather than $F(x)^{-1/2}$.

Let L be the Cholesky factor of $F(x)$, *i.e.*, L is lower triangular with $LL^T = F(x)$. The gradient of ϕ is given by

$$\begin{aligned} g_i(x) &= -\mathbf{Tr} F(x)^{-1} F_i & (73)\\ &= -\mathbf{Tr} L^{-T} L^{-1} F_i & (74)\\ &= -\mathbf{Tr} L^{-1} F_i L^{-T}, & (75) \end{aligned}$$

and the Hessian is given by

$$\begin{aligned} H_{ij}(x) &= \mathbf{Tr} F(x)^{-1} F_i F(x)^{-1} F_j & (76)\\ &= \mathbf{Tr} L^{-T} L^{-1} F_i L^{-T} L^{-1} F_j & (77)\\ &= \mathbf{Tr} \left(L^{-1} F_i L^{-T}\right) \left(L^{-1} F_j L^{-T}\right). & (78) \end{aligned}$$

Similarly, the least-squares characterization of the Newton step, given by formula (46), becomes

$$-H(x)^{-1} g(x) = \begin{array}{l} \mathrm{argmin} \\ v \in \mathbf{R}^m \end{array} \left\| I - \sum_{i=1}^m v_i L^{-1} F_i L^{-T} \right\| \tag{79}$$

(this follows from the fact that there is an orthogonal matrix Q such that $F(x)^{-1/2} = QL^{-1} = L^{-T}Q^T$, so that $F(x)^{-1/2} F_i F(x)^{-1/2} = QL^{-1} F_i L^{-T} Q^T$).

In other words, the congruence $\tilde{F} := L^{-1} F L^{-T}$ normalizes the problem so that $\tilde{F}(x) = I$.

6.3 Full blocks

Suppose first that the blocks in F are "full" (or, we choose to ignore any structure the individual blocks in F may have). Of course, L^{-1} will have the same block structure as F. Let's give a rough operation count for computing the Newton step at a given x. We will ignore constant factors and keep only dominant terms.

Forming $F(x)$ given x costs $m\sum n_j^2$. We can compute L^{-1} by Cholesky factorization of each block of $F(x)$ and then inversion. The cost is $\sum n_j^3$. Normalizing the problem, *i.e.*, forming $L^{-T}F_iL^{-1}$, costs $m\sum n_j^3$. This cost dominates so far.

We suppose first that we compute the Newton direction by forming $g(x)$ and $H(x)$ and solving $H(x)v = -g(x)$. Forming $g(x)$ costs mn, and forming $H(x)$ (which is the Gram matrix of the normalized F_i) costs $m^2\sum n_j^2$. Finding v then costs m^3. The dominant term is thus $m^2\sum n_j^2$ (since by (72), $m \leq \sum n_j^2$).

Now suppose that we compute the Newton direction by solving the least-squares problem (79), which has m variables and (ignoring constant factors) $\sum n_j^2$ "equations". Using for example QR factorization, the cost is $m^2\sum n_j^2$, which is the same cost as forming the gradient and Hessian and solving for the Newton direction. (Computing the Newton step via QR factorization will have better numerical properties, however, since we don't "square up," *i.e.*, form, the Hessian.)

Therefore, the total operation count for one step of the Newton method is of order $\max\{m^2\sum n_j^2, m\sum n_j^3\}$.

6.4 Exploiting internal block structure

We can exploit additional structure that the blocks may have to reduce the computation required for the Newton step. As an example, consider a single block that arises from the quadratic constraint $\|Ax - b\| < 1$, where $A \in \mathbf{R}^{N\times m}$ and is full rank. We may assume that $N \leq m$.

The block associated with this constraint is

$$F_{\mathrm{quad}}(x) = \begin{bmatrix} I & Ax - b \\ (Ax-b)^T & 1 \end{bmatrix}$$

(we ignore for now the other blocks in $F(x)$). Suppose that we use the method described above in section 6.3, *i.e.*, normalize and then solve a least-squares problem. If we treat this block as full, it incurs a cost of $\max\{m^2N^2, mN^3\} = m^2N^2$ for one Newton step. We will see that by exploiting the special structure, this can be reduced to m^2N, or even m^2, along with some initial precomputation. For a quadratic constraint of high rank (*i.e.*, N significant compared to m), this factor of N^2 is significant.

The barrier term for the constraint $\|Ax - b\| < 1$ is

$$\log\det F_{\mathrm{quad}}(x)^{-1} = -\log(1 - y^Ty)$$

where $y \stackrel{\Delta}{=} Ax - b$. Hence from (27) and (29) (or direct calculation), the gradient and Hessian of this barrier term are

$$g(x) = \frac{2}{1 - y^T y} A^T y \tag{80}$$

$$H(x) = \frac{2}{1 - y^T y} A^T A + g(x) g(x)^T \tag{81}$$

Given x, the cost of computing y and $1/(1 - y^T y)$ is mN; forming $g(x)$ via (80) costs mN, and forming $H(x)$ via (81) costs $m^2 N$. Hence by computing $g(x)$ and $H(x)$ in this way, the cost incurred by the constraint $\|Ax - b\| < 1$ is $m^2 N$ per Newton step.

Moreover, suppose that we precompute and store the Gram matrix $A^T A$. Then the cost of forming $H(x)$ via (81) drops to m^2, so the overall cost incurred by the block is m^2. (Note that the cost of the precomputation, *i.e.*, forming $A^T A$, is $m^2 N$, but this cost is amortized over all of the Newton steps performed throughout the whole algorithm.)

This computational savings can also be understood in the context of the method described in section 6.3. It is possible to derive a simple explicit expression for the inverse Cholesky factor L^{-1}. We save computation by simply evaluating this expression rather than performing a Cholesky factorization and inversion.

More generally, by exploiting the special structure of the blocks that arise from the constraints described in section 2, we can lower the computational cost per Newton step below the "full" block cost described in section 6.3, although in many cases the savings is not as large as in this quadratic constraint example.

6.5 Line search

We noted in section 3.4 that Nesterov and Nemirovsky's analysis holds for exact line search step length selection, *i.e.*,

$$\alpha^{(k)} := \operatorname*{argmin}_{\alpha} \; \phi\left(x^{(k)} - \alpha H(x^{(k)})^{-1} g(x^{(k)})\right) \tag{82}$$

With exact line search, the number of iterations required to compute the analytic center is typically smaller than with the Nesterov-Nemirovsky step length (44), but of course each iteration involves the extra computation required to determine the step length $\alpha^{(k)}$. In many cases, there is an overall advantage in using exact line search.

We need to compute

$$\alpha^* \stackrel{\Delta}{=} \operatorname{argmax}\{\det(I + \alpha P) | I + \alpha P > 0\} \tag{83}$$

where

$$P = \sum_i v_i F(x)^{-1/2} F_i F(x)^{-1/2}$$

and $v = -H(x)^{-1}g(x)$. (Note the similarity to the standard problem of computing eigenvalues—the difference is that here, we want to compute a zero of the derivative of the characteristic polynomial instead of the characteristic polynomial itself.)

We first reduce P to tridiagonal or even diagonal form, which costs $\sum n_i^3$. Now the derivatives $g(\alpha)$ and $H(\alpha)$, and hence the Newton step, can be computed at a cost of n. (This reduction is used in the algorithm described in [NN90a].) Several methods can be used to find α^*, *e.g.*, we can use Newton's method with the Nesterov-Nemirovsky step length, or bisect until $\delta < 1/4$ and then switch to Newton's method. For this latter method, it can be shown that in the worst case we perform no more than (a constant times) $\log n$ bisections to reach the region of quadratic convergence, *i.e.*, $\delta < 1/4$ (after which we perform at most a small fixed number of iterations). Thus, in the worst case the cost of computing α^* is no more than $n \log n$, once we have reduced the pencil. So the cost of exact line search is at most $\max\{n \log n, \sum n_j^3\}$, which in many cases is small compared to the cost of computing the Newton direction.

7 An example

7.1 A Lyapunov function search problem

We consider a simple example of determining a Lyapunov function that optimizes a decay rate estimate for a linear differential inclusion. More detail on this and similar problems can be found in [BGFB93] or [BY89].

We consider the differential equation

$$\frac{dy}{dt}(t) = \left(\sum_{i=1}^{L} \theta_i(t) G_i\right) y(t) \tag{84}$$

where $G_i \in \mathbf{R}^{N\times N}$ (and do not depend on t) and the $\theta_i(t)$ satisfy $\sum \theta_i(t) = 1$, $\theta_i(t) \geq 0$, but are otherwise arbitrary.

Given any $P = P^T > 0$, let $V(z) = z^T P z$. For $y(t)$ satisfying (84), we have

$$\begin{aligned}
\frac{d}{dt}V(y(t)) &= \sum_{i=1}^{L} \theta_i(t) y(t)^T \left(G_i^T P + P G_i\right) y(t) && (85)\\
&\leq \max_i \; y(t)^T \left(G_i^T P + P G_i\right) y(t) && (86)\\
&\leq \max_i \; \lambda_{\max}\left(G_i^T P + P G_i, P\right) V(y(t)). && (87)
\end{aligned}$$

This proves that

$$V(y(t)) \leq e^{\alpha t} V(y(0)) \tag{88}$$

where

$$\alpha \triangleq \max_i \; \lambda_{\max}\left(G_i^T P + P G_i, P\right) = \lambda_{\max}\left(\bigoplus_{i=1}^{L} (G_i^T P + P G_i), \bigoplus_{i=1}^{L} P\right)$$

(and, moreover, α is the smallest number for which we can guarantee that $dV/dt \leq \alpha V$ regardless of $y(t)$ and the particular $\theta_i(t)$).

We can interpret $-\alpha/2$ in (88) as a conservative stability degree estimate or guaranteed decay rate (if $\alpha < 0$) of the differential inclusion (84), and $V(z) = z^T P z$ as a Lyapunov function that proves it. Our problem is to determine the Lyapunov function that gives the best guaranteed decay rate estimate for the system (84):

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}\left(\bigoplus_{i=1}^{L}(G_i^T P + P G_i), \bigoplus_{i=1}^{L} P\right). \\ P > 0 & \end{array}$$

Since the objective is homogeneous in P of degree zero, we can normalize P by, *e.g.*, $\mathbf{Tr}P = N$. We also impose the constraint that $P > b_{\min} I$ where $1 > b_{\min} > 0$, which essentially limits the condition number of the Lyapunov functions we are willing to consider, but in most cases is irrelevant if $b_{\min}$ is small enough (see[BY89]). This results in:

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}\left(\bigoplus_{i=1}^{L}(G_i^T P + P G_i), \bigoplus_{i=1}^{L} P\right). \\ \mathbf{Tr}P = N & \\ P - b_{\min} I > 0 & \end{array} \tag{89}$$

When this problem is put in the form (8), by eliminating the equality constraint, we find that the number of variables is $m = N(N+1)/2 - 1$, the size of the matrices $\lambda B - A$ is $r = LN$, and the size of the constraint matrix C is $s = N$. The matrix $F = (\lambda B - A) \oplus C$ has size $n = r + s = (L+1)N$, and consists of $L+1$ blocks each of size N.

As initial feasible point we can take

$$P^{(0)} = I, \tag{90}$$

$$\lambda^{(0)} = \lambda_{\max}\left(\bigoplus_{i=1}^{L}(G_i^T + G_i)\right) + 1. \tag{91}$$

Since the set $\{P | \mathbf{Tr}P = N,\ P > 0\}$ is bounded, it is clear that all the assumptions of section 1.5 are satisfied. Moreover we can take $b_{\max} = N$, which can be seen as follows. Since $P > 0$ and $\mathbf{Tr}P = N$, we have $\lambda_{\max}(P) < N$, so $\lambda_{\max}\left(\oplus_{i=1}^{L} P\right) < N$.

7.2 An instance of the problem

In the next two sections we give some numerical results for an instance of the problem (89). We consider a physical system consisting of two unit masses, which are connected to each other by a spring. In addition, one of the masses is connected to a wall (infinite mass) by another spring. The two springs can instantly change stiffness over the range of $[1, 2]$. By loosening and stiffening the springs appropriately we can pump energy into our system; our task is derive the best upper bound, based on a quadratic Lyapunov function, on the rate at which this can be done.

With y_1 and y_2 denoting the positions of the masses and y_3 and y_4 denoting their velocities, the differential inclusion describing this system is

$$\frac{dy}{dt}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1(t) - k_2(t) & k_2(t) & 0 & 0 \\ k_2(t) & -k_2(t) & 0 & 0 \end{bmatrix} y(t), \qquad 1 \leq k_i(t) \leq 2, \ i = 1, 2.$$

Thus we have $N = 4$ and $L = 4$ (representing the four extreme matrices). There are 9 variables, and the matrix $F = (\lambda B - A) \oplus C$ is 20×20, and consists of five 4×4 blocks. The Lyapunov function P is initialized as I, and we take $b_{\min} = 0.01$ (which limits the condition number of P below 400). The optimal Lyapunov function turns out to have a minimum eigenvalue of about 0.42, so the constraint $P > b_{\min} I$ is (quite) inactive. The optimum value is $\lambda^{\rm opt} = 0.6056$, which has multiplicity four. (The multiplicity is split between one active eigenvalue corresponding to the case of both springs loose, *i.e.*, $k_1 = k_2 = 1$, and one active eigenvalue corresponding to the case of both springs tight, *i.e.*, $k_1 = k_2 = 2$.)

7.3 Some numerical results: method of centers

The table below shows the progress of the method of centers with the parameter θ set at 0.001, *i.e.*, the next λ is set 99.9% of the way towards the current objective value, from the current value of λ. The first and second columns show the iteration number and objective value. The third column, labeled gap 1, shows the simple bound on the difference between the current value of the objective and the optimal value from the simple formula (50), *i.e.*, $(\lambda^{(k)} - \lambda_{\max}(A(x^{(k)}), B(x^{(k)})))nb_{\max}/b_{\min}$. The fourth column, labeled gap 2, shows the better bound obtained using (68). The next column, labeled NeNe, shows the number of Newton steps that were required to compute the analytic center (*i.e.*, the current iterate) using the Nesterov-Nemirovsky step length. The last column shows the number of Newton steps that were required to compute the analytic center using exact line search step length. (In both cases, the stopping criterion for the analytic center computation is $\delta < 0.001$, which can be shown to imply that $\phi(x) - \phi(x^*) < 0.001^2$.)

$\theta = 0.001$

iteration	$\lambda_{\max}$	gap 1	gap 2	NeNe	LS
1	$2.6511e+00$	$1.03e+05$	$6.21e+01$	5	3
2	$1.4645e+00$	$4.76e+04$	$3.36e+01$	12	6
3	$8.5597e-01$	$2.44e+04$	$6.05e-01$	18	8
4	$6.6293e-01$	$7.74e+03$	$4.32e-01$	17	8
5	$6.6093e-01$	$8.76e+01$	$5.79e-02$	9	5
6	$6.6061e-01$	$1.29e+01$	$8.74e-03$	15	7
7	$6.6057e-01$	$1.82e+00$	$1.27e-03$	14	7
8	$6.6056e-01$	$2.88e-01$	$2.02e-04$	7	4

Five iterations (30 LS Newton steps) are required to reduce the objective value to within 0.001 of the optimal value (but of course, we don't *know* this at the fifth iteration). Eight iterations (48 Newton steps) are required to reduce the better gap (gap 2) below 0.001.

To see the effect of the parameter θ, we now consider the large value $\theta = 0.5$. Note that the table does not show every iteration.

$\theta = 0.5$

iteration	$\lambda_{\max}$	gap 1	gap 2	NeNe	LS
1	$2.6511e+00$	$1.03e+05$	$6.21e+01$	5	3
5	$1.0344e+00$	$3.18e+04$	$1.41e+00$	3	3
10	$6.6751e-01$	$2.68e+03$	$1.03e+00$	2	2
15	$6.6117e-01$	$1.44e+02$	$9.36e-02$	2	2
20	$6.6060e-01$	$8.88e+00$	$6.09e-03$	2	2
24	$6.6056e-01$	$9.61e-01$	$6.61e-04$	2	2

As expected, convergence is slower—15 iterations (37 Newton steps) are required to converge to within 0.001 of the optimal value and 24 iterations (55 Newton steps) are required to reduce the better gap below 0.001. Also as expected, the number of Newton steps required to compute each analytic center is smaller than in the case $\theta = 0.001$, since the initial points for the analytic center computations are "more feasible" than in the case $\theta = 0.001$. Note that the total numbers of Newton steps required (37 and 55, respectively) are not much larger than the numbers required in the case $\theta = 0.001$ (30 and 48, respectively).

We now consider the value $\theta =$ 1e-6, which is very nearly the classical method of centers. The results are shown below:

$\theta = 1e-6$

iteration	$\lambda_{\max}$	gap 1	gap 2	NeNe	LS
1	$2.6511e+00$	$1.03e+05$	$6.21e+01$	5	3
2	$1.4726e+00$	$4.72e+04$	$3.35e+01$	12	6
3	$8.6616e-01$	$2.42e+04$	$6.21e-01$	12	6
4	$6.6201e-01$	$8.16e+03$	$2.70e-01$	11	5
5	$6.6078e-01$	$4.92e+01$	$3.32e-02$	25	11
6	$6.6059e-01$	$7.38e+00$	$5.03e-03$	10	5
7	$6.6056e-01$	$1.04e+00$	$7.28e-04$	31	11

While the convergence is essentially the same as for the case $\theta = 0.001$, the number of Newton steps required per iteration is larger, since the initial points for the analytic center computations are "less feasible" than in the $\theta = 0.001$ case.

7.4 Some numerical results: ellipsoid algorithm

For comparison we solve the same problem using the ellipsoid algorithm, which is a general algorithm that can minimize a quasiconvex function subject to a convex constraint. We give a brief but complete description of the algorithm here. More details can be found in, *e.g.*, [BB91].

The ellipsoid algorithm must be initialized with an ellipsoid that contains a minimizer. As initial ellipsoid we take

$$\mathcal{E}^{(0)} = \left\{ P \;\middle|\; \|P - I\|_F \leq \sqrt{N(N-1)},\ \mathbf{Tr}P = N \right\}$$

which (by our outer ellipsoidal bound) contains the set of positive definite matrices with trace N, and so contains the entire feasible set for our problem.

At each iteration, we produce an ellipsoid of smaller volume that is still guaranteed to contain a minimizer, as follows. First we find a cutting-plane that separates the center of the current ellipsoid from the set of minimizers, so the minimizer is now localized to the intersection of a half-space and the current ellipsoid. Then, the next ellipsoid is the minimum volume ellipsoid that contains this intersection. (There are simple formulas for this update.)

The cutting-plane is computed as follows. If the current iterate $P^{(k)}$ (which is the center of the ellipsoid $\mathcal{E}^{(k)}$) is not feasible, *i.e.*, does not satisfy $P^{(k)} > b_{\min}I$, we compute the minimum eigenvalue of $P^{(k)}$ along with a corresponding eigenvector v with $\|v\| = 1$. The cutting-plane is then given by $v^T P v = b_{\min}$, which describes a hyperplane in $\{P|\mathbf{Tr}P = N\}$. In other words, the minimizer is contained in the half-space

$$\left\{ P \;\middle|\; v^T P v \geq b_{\min},\ \mathbf{Tr}P = N \right\},$$

since P's not in this half-space are surely infeasible. ($P^{(k)}$ is not in this half-space, so intersecting $\mathcal{E}^{(k)}$ with this half-space "cuts away" more than half of $\mathcal{E}^{(k)}$. For this reason this is called a "deep-cut.")

If $P^{(k)}$ is feasible, *i.e.*, satisfies $P^{(k)} > b_{\min}I$, then we generate a cutting-plane from the objective, as follows. we compute

$$\lambda^{(k)} \triangleq \lambda_{\max}\left(\bigoplus_{i=1}^{L} (G_i^T P^{(k)} + P^{(k)} G_i), \bigoplus_{i=1}^{L} P^{(k)} \right)$$

along with a corresponding generalized eigenvector v with $\|v\| = 1$. Then any minimizer must lie in the half-space

$$\left\{ P \;\middle|\; v^T \bigoplus_{i=1}^{L} \left(\lambda^{(k)} P - G_i^T P - P G_i \right) v \geq 0,\ \mathbf{Tr}P = N \right\}$$

(since any other P will either be infeasible or have an objective value larger than $\lambda^{(k)}$). In

this case we can also compute a lower bound on the optimal objective value:

$$\lambda^{\rm opt} \geq \min_{\substack{\mathbf{Tr}P = N,\ P \in \mathcal{E}^{(k)} \\ v^T \bigoplus_{i=1}^L Pv \geq b_{\min}}} \frac{v^T \bigoplus_{i=1}^L \left(G_i^T P + P G_i\right) v}{v^T \bigoplus_{i=1}^L Pv}. \tag{92}$$

The table below shows the progress of the ellipsoid algorithm. The column labeled gap shows the difference between $\lambda^{(k)}$ and the lower bound (92). (The iterates shown in the table are all feasible.)

iteration	$\lambda_{\max}$	gap
1	4.2361e+00	6.48e+02
10	6.8871e-01	2.67e+02
100	6.7125e-01	2.92e+01
200	6.6213e-01	3.54e+00
300	6.6075e-01	1.20e+00
400	6.6059e-01	2.20e-01
500	6.6057e-01	1.30e-02
600	6.6056e-01	4.51e-03
674	6.6056e-01	9.27e-04

The ellipsoid algorithm requires 190 iterations to converge within 0.001 of $\lambda^{\rm opt}$ and 674 iterations to reduce the gap below 0.001.

7.5 Comparison

For this problem, the computation cost of an ellipsoid algorithm iteration is less than but still roughly comparable to the cost of a Newton step and line search in the method of centers.

The table below summarizes the numbers of Newton/line search steps for the method of centers, and the number of iterations for the ellipsoid algorithm, required for convergence within 0.001 of the optimal value ($\lambda^{(k)} - \lambda^{\rm opt} \leq 0.001$) and for reduction of the gap below 0.001.

criterion	θ = 1e-3	θ =5e-1	ell. alg.
$\lambda^{(k)} - \lambda^{\rm opt} \leq .001$	30	37	190
gap $\leq .001$	48	55	674

We should make several comments concerning this comparison. First, we were able to initialize the ellipsoid method with an efficient ellipsoid (indeed, the minimum volume ellipsoid that contains $\{P | P \geq 0,\ \mathbf{Tr}P = N\}$). In the general problem, no such efficient ellipsoid is available. Second, the efficiency of the method of centers, as compared to the ellipsoid method, rapidly increases with problem size.

8 Conclusions

The method of centers is a simple interior point algorithm that appears to be very efficient when compared to other algorithms for minimizing the maximum generalized eigenvalue of a pair of matrices that depend affinely on a decision variable.

We do not, however, present the algorithm as described in section 4.3 as the "fastest" measured either by typical practical performance or by bounds on worst case performance. In particular, the algorithm can be made to run faster using standard techniques, three of which we mention here:

- *First-order predictor.*

  It is possible to cheaply compute $\partial x^*/\partial\lambda$ at $\lambda^{(k)}$. This can be used to initialize the Newton algorithm for computing $x^{(k+1)}$. This reduces the number of Newton steps per iteration.

- *Weighted analytic centers.*

  Let $\nu \geq 1$ be some integer. Then we apply the method of centers to the problem with data $\tilde{A}$, $\tilde{B}$, and C where $\tilde{A} = \oplus_{i=1}^{\nu} A$ and $\tilde{B} = \oplus_{i=1}^{\nu} B$. Of course, working with ν "copies" of the inequality $\lambda B - A > 0$ does not change the optimal value or set of minimizers for the problem. In effect, we substitute the barrier function $\nu \log\det(\lambda B - A)^{-1} + \log\det C^{-1}$ for $\log\det(\lambda B - A)^{-1} + \log\det C^{-1}$.

  This results in a larger reduction of λ per iteration but more Newton steps required per iteration. In practice, this can lead to substantially faster convergence of $\lambda^{(k)}$ to $\lambda^{\rm opt}$ (measured in total Newton steps). However, the dual bounds are often worse than for $\nu = 1$.

  For ν large, the method of centers will approach an analog of Dikin's affine scaling algorithm [Dik67].

- *Switching to a quadratically convergent local method.*

  We note the possibility of combining the method of centers with a quadratically convergent local method. The method of centers identifies the active eigenvalues and eigenvectors (via the dual matrices U and V) as it proceeds (or more precisely, it identifies the branches of the eigenvalue functions that are active at the optimal point $x^{\rm opt}$). We presume that once these active eigenvalues are identified, an optimum point can be computed more rapidly by switching to a quadratically convergent method such as Overton's (see [Ove88]; the extension to the generalized eigenvalue case is considered in [Hae91]).

We have not given a complete complexity analysis (worst case operation count) of the algorithm, since we have not given any bound on the number of Newton steps required to reach (in some appropriate approximate sense) the analytic center. To do this would require modifying the algorithm to use some appropriate approximate analytic center instead of

the analytic center (which of course cannot be computed in a finite number of steps for $n \geq 6$) and in addition restricting θ to be close enough to one to get a suitable bound on the number of Newton iterations required to compute the approximate center. We remind the reader that in [NN91b], Nesterov and Nemirovsky describe a potential reduction algorithm for generalized eigenvalue minimization and give a complete worst case complexity analysis.

In any case, the material of sections 2, 3, and 6 only concern the notion of the analytic center of an affine matrix inequality, and is independent of the method of centers.

9 Acknowledgments

This paper was written over a long period, and we have benefited from useful discussion and interaction with many colleagues, especially Florian Jarre, Farid Alizadeh, Arkadii Nemirovsky, and Yuri Nesterov. We are especially grateful to Arkadii Nemirovsky and Yuri Nesterov for providing us with preliminary versions of their book and many papers, advice, discussion, and software. We acknowledge useful comments and suggestions from Michael Overton, Jean-Pierre Haeberly, Jeffrey Coleman, Venkataramanan Balakrishnan, Eric Feron, Michael Grant, and Craig Barratt.

A Derivative of $\log\det F$

The derivatives given in section 3.1 are readily derived once the reader knows that

$$\left.\frac{d}{dt}\right|_{t=0} \log\det(F_0 + tF_1) = \mathbf{Tr} F_0^{-1} F_1 \tag{93}$$

(assuming of course that $\det F_0 \neq 0$). This is shown as follows.

$$\begin{aligned}
\log\det(F_0 + tF_1) &= \log\det\left(F_0\left(I + tF_0^{-1}F_1\right)\right) && (94)\\
&= \log\det F_0 + \log\det\left(I + tF_0^{-1}F_1\right) && (95)\\
&= \log\det F_0 + \log\left(1 + t\mathbf{Tr}F_0^{-1}F_1 + o(t)\right) && (96)
\end{aligned}$$

from which (93) follows.

B Proof of ellipsoidal bounds

Suppose that x satisfies $F(x) = F_0 + \sum_{i=1}^m x_i F_i > 0$. We assume without loss of generality that $x = 0$ and $F_0 = I$ (the latter by multiplying the original matrices on the left and right by $F_0^{-1/2}$). From the formulas for the gradient and Hessian of the barrier, we have $g_i(0) = -\mathbf{Tr}F_i$ and $H_{ij}(0) = \mathbf{Tr}F_iF_j$.

We first establish the inner ellipsoidal bound. Suppose that $z^T H(0) z < 1$. Since

$$z^T H(0) z = \sum_{i,j=1}^{m} z_i z_j \mathbf{Tr} F_i F_j = \|F(z) - I\|_F^2 \geq \|F(z) - I\|^2$$

we conclude that $\|F(z) - I\| < 1$, and hence $F(z) > 0$. This proves the inner ellipsoidal bound (41). Note that we did not use here the fact that 0 is the analytic center of the matrix inequality. Therefore the inner ellipsoidal approximation holds for any feasible point x:

$$F(x) > 0 \text{ and } (z-x)^T H(x)(z-x) < 1 \implies F(z) > 0$$

(see Nesterov and Nemirovsky [NN93] for a generalization to any self-concordant barrier).

Now we prove the outer ellipsoidal bound, assuming that 0 is the analytic center of $F(x) > 0$, so that $\mathbf{Tr} F_i = 0$. Then for any $z \in \mathbf{R}^m$, we have $\mathbf{Tr} F(z) = \mathbf{Tr} F(0) = n$. Similarly,

$$\begin{aligned} \mathbf{Tr} F(z)^2 &= \sum_{i,j=1}^{m} z_i z_j \mathbf{Tr} F_i F_j + 2 \sum_{i=1}^{m} z_i \mathbf{Tr} F_i + \mathbf{Tr} I \qquad (97) \\ &= z^T H(0) z + n. \qquad (98) \end{aligned}$$

Now suppose that z satisfies $F(z) > 0$. For any $X = X^T \in \mathbf{R}^{n \times n}$ with $X > 0$, $\mathbf{Tr} X^2 \leq (\mathbf{Tr} X)^2$ (this can be seen by diagonalizing X). Therefore we have

$$z^T H(0) z + n = \mathbf{Tr} F(z)^2 \leq (\mathbf{Tr} F(z))^2 = n^2 \qquad (99)$$

so

$$F(z) > 0 \implies z^T H(0) z \leq n(n-1)$$

which is the outer ellipsoidal bound.

The same type of argument can be used to derive an outer ellipsoidal bound centered at any point x with $\delta(x) < 1$ (again, see Nesterov and Nemirovsky [NN93] for a generalization to any self-concordant barrier). We proceed as follows. Suppose now that $x = 0$ is not necessarily the analytic center. Then (99) becomes:

$$z^T H(0) z - 2 g(0)^T z + n = \mathbf{Tr} F(z)^2 \leq (\mathbf{Tr} F(z))^2 = (n - g(0)^T z)^2.$$

This implies that

$$z^T \left(H(0) - g(0) g(0)^T \right) z + 2(n-1) g(0)^T z \leq n(n-1),$$

which can be put in the form

$$(z - x_c)^T \left(H(0) - g(0) g(0)^T \right) (z - x_c) \leq \frac{(n-1)(n - \delta(0)^2)}{1 - \delta(0)^2}, \qquad (100)$$

where

$$x_c \triangleq -\frac{n-1}{1 - \delta(0)^2} H(0)^{-1} g(0).$$

The inequality (100) defines an ellipsoid if and only if $H(0) - g(0) g(0)^T > 0$, which is the same as $\delta(0)^2 = g(0)^T H(0)^{-1} g(0) < 1$. The center of the ellipsoid, x_c, is displaced along the Newton direction from the point $x = 0$.

C A generalized resolvent inequality

Suppose that $Y > 0$ and $\lambda Y - X > 0$, and let $\lambda_{\max} \triangleq \lambda_{\max}(X, Y)$. We will show that

$$\mathbf{Tr}(\lambda Y - X)^{-1} \geq \frac{1}{(\lambda - \lambda_{\max})\|Y\|}.$$

Find $v \neq 0$ such that $Xv = \lambda_{\max} Y v$. Then we have

$$(\lambda Y - X)^{-1} Y v = \frac{1}{\lambda - \lambda_{\max}} v,$$

so that

$$\left\|(\lambda Y - X)^{-1} Y\right\| \geq \frac{1}{\lambda - \lambda_{\max}}.$$

Therefore

$$\left\|(\lambda Y - X)^{-1}\right\| \geq \frac{1}{(\lambda - \lambda_{\max})\|Y\|)}.$$

Our conclusion follows from

$$\mathbf{Tr}(\lambda Y - X)^{-1} \geq \left\|(\lambda Y - X)^{-1}\right\|.$$

D Maximum of a linear fractional form on an ellipsoid

In section (5.1) we derive an upper bound for $\lambda^{(k)} - \lambda^{\text{opt}}$ that is given by the maximum of a linear fractional form over an ellipsoid. We use a similar bound for the ellipsoid algorithm in (92). Here we show how this can be computed.

By a suitable change of coordinates we may assume that the problem is to determine

$$\delta \triangleq \max_{x^T x \leq 1} \frac{a^T x}{c^T x + 1}.$$

We will assume that $a \neq 0$ (if $a = 0$ then obviously $\delta = 0$).

If $\|c\| > 1$, then $\delta = \infty$, since the linear fractional form is unbounded above near $x = -c/\|c\|$.

Assume now that $\|c\| < 1$. Then,

$$\delta \leq \gamma \iff a^T x/(c^T x + 1) \leq \gamma \text{ for all } x \text{ with } \|x\| \leq 1 \tag{101}$$

$$\iff (a - \gamma c)^T x \leq \gamma \text{ for all } x \text{ with } \|x\| \leq 1 \tag{102}$$

$$\iff \|a - \gamma c\| \leq \gamma. \tag{103}$$

Therefore δ is equal to the larger root of the quadratic equation $\|a - \gamma c\|^2 = \gamma^2$.

In the case $\|c\| = 1$, the solution depends on the sign of $a^T c$. (Although this case is irrelevant in any numerical computation, we include it below for completeness.) In summary, the solution is:

$$\delta = \begin{cases} \left(\sqrt{(a^T c)^2 + a^T a(1 - c^T c)} - a^T c\right) \big/ (1 - c^T c) & \|c\| < 1, \\ a^T a/(2a^T c) & \|c\| = 1,\ a^T c > 0, \\ \infty & \text{otherwise.} \end{cases} \tag{104}$$

E Solution of the constrained problem

Here we describe the solution of the following problem: maximize a linear fractional form over an ellipsoid, subject to an upper bound on the numerator and a (positive) lower bound on the denominator. As in the previous section we change coordinates so the ellipsoid becomes the unit ball. The problem we must solve assumes the following form: determine

$$\delta = \max_{\substack{x^T x \le 1 \\ a^T x \le \alpha \\ c^T x + 1 \ge \beta}} \frac{a^T x}{c^T x + 1} \tag{105}$$

In our problem, α and β are positive, and 0 is feasible, *i.e.*, $1 > \beta$.

We can always reduce the problem (105) to a two dimensional problem (which is not surprising since the two affine functions $a^T x$ and $c^T x + 1$ do not vary along directions lying in a subspace of dimension $n - 2$). Using a Lagrange multiplier or direct argument, it can be shown that a maximizer of (105) always lies in the span of a and c. (This is clear in the case where the Lagrange multiplier corresponding to the constraint $x^T x \le 1$ is nonzero, in which case there is exactly one maximizer. When this Lagrange multiplier is zero, the problem (105) can have multiple maximizers, one of which, however, lies in the span of a and c.)

We proceed under the assumption that a and c are linearly independent. (If they are not, the problem reduces to a trivial one dimensional problem.) We define the new optimization variable $w \in \mathbf{R}^2$ given by

$$x = \begin{bmatrix} a & c \end{bmatrix} G^{-1} \left(w - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

where G is the Gram matrix

$$G \triangleq \begin{bmatrix} a^T a & a^T c \\ c^T a & c^T c \end{bmatrix}.$$

Therefore,

$$\begin{aligned} a^T x &= w_1, \\ c^T x + 1 &= w_2, \end{aligned}$$

and the constraint $x^T x \leq 1$ becomes $w \in \mathcal{E}$ where

$$\mathcal{E} \triangleq \left\{ w \;\middle|\; \left(w - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^T G^{-1} \left(w - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \leq 1 \right\}.$$

Our problem (105) becomes: determine

$$\delta \;=\; \max_{\substack{w \in \mathcal{E} \\ w_1 \leq \alpha \\ w_2 \geq \beta}} \frac{w_1}{w_2}. \tag{106}$$

The solution to this problem is readily obtained by solving a few quadratic equations, and so has essentially no computational cost (*e.g.*, when compared to the computational cost of reducing the original problem to the five parameters appearing in (106)). Its solution is cumbersome to describe, however.

We first note that the solution must lie in the first quadrant ($w_1 > 0$, $w_2 > 0$), and on the boundary of the feasible set $\mathcal{E} \cap \{w | w_1 \leq \alpha,\ w_2 \geq \beta\}$. We distinguish several cases:

Case I: $[\alpha\ \beta]^T \in \mathcal{E}$.
In this case the maximizer is the point $w^I = [\alpha\ \beta]^T$, and we have $\delta = \alpha/\beta$. Henceforth we assume that $[\alpha\ \beta]^T \notin \mathcal{E}$.

Case II: $[\alpha\ \beta]^T \notin \mathcal{E}$, $0 \in \mathcal{E}$.
In this case the maximizer lies on the line segment $\{w | w_2 = \beta\} \cap \mathcal{E}$ (which is readily found by solving a quadratic equation; the assumptions ensure that the line segment is nonempty). The maximizer is

$$w^{(\beta)} = \begin{bmatrix} \left((\beta - 1)a^T c + \sqrt{(c^T c - (\beta-1)^2) \det G}\,\right) \Big/ c^T c \\ \beta \end{bmatrix}.$$

Henceforth we assume that $0 \notin \mathcal{E}$, *i.e.*, $a^T a > \det G$.

We now compute $\tilde{w}$, the local maximum of w_1/w_2 on $\partial\mathcal{E}$ which satisfies $w_1 > 0$, $w_2 \neq 0$. By solving a quadratic equation we find:

$$\tilde{w} = \begin{bmatrix} \sqrt{a^T a - \det G} \\ 1 + \left(a^T c \sqrt{a^T a - \det G} - \det G\right) \Big/ a^T a \end{bmatrix}.$$

We distinguish three more cases depending on $\tilde{w}$.

Case III: $[\alpha\ \beta]^T \notin \mathcal{E}$, $0 \notin \mathcal{E}$, $\tilde{w}_1 \leq \alpha$, $\tilde{w}_2 \geq \beta$.
The condition is simply that $\tilde{w}$ is feasible. In this case, $\tilde{w}$ is the maximizer.

Case IV: $[\alpha\ \beta]^T \notin \mathcal{E}$, $0 \notin \mathcal{E}$, $\tilde{w}_1 > \alpha$, $\tilde{w}_2 \geq \beta$.
The condition is that $\tilde{w}$ violates violates the first linear constraint. In this case the maximizer

lies on the line segment $\{w|w_1 = \alpha\} \cap \mathcal{E}$. By solving a quadratic equation we find the maximizer as:

$$w^{(\alpha)} = \begin{bmatrix} \alpha \\ 1 + \left(\alpha a^T c - \sqrt{(a^T a - \alpha^2)\det G}\right) \Big/ a^T a \end{bmatrix}.$$

Case V: $[\alpha\ \beta]^T \notin \mathcal{E}$, $0 \notin \mathcal{E}$, $\tilde{w}_1 \leq \alpha$, $\tilde{w}_2 < \beta$.
If $\tilde{w}$ violates the second linear constraint, then the maximizer lies on the line segment $\{w|w_2 = \beta\} \cap \mathcal{E}$. In this case the maximizer is $w^{(\beta)}$.

In summary, the solution is given by

$$\delta = \begin{cases} \alpha/\beta & \text{Case I} \\ \left((\beta-1)a^T c + \sqrt{(c^T c - (\beta-1)^2)\det G}\ \right) \Big/ (\beta c^T c) & \text{Case II or V} \\ (\alpha a^T a) \Big/ \left(a^T a + \alpha a^T c - \sqrt{(a^T a - \alpha^2)\det G}\ \right) & \text{Case IV} \\ (a^T a) \Big/ \left(a^T c + \sqrt{a^T a - \det G}\ \right) & \text{Case III} \end{cases}$$

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