HAUSDORFF MEASURES ON THE LINE

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In 1918 Hausdorff [1] defined a set of measures in metric spaces which included the Lebesgue n-dimensional measures, counting measures, as well as various non-integral-dimensional measures. These measures are the basis for various theories of generalized dimension, including Besicovitch's theory of fractionally dimensioned sets (now called Fractals). I will restrict my study to these measures on the line, and a few foundational questions.

If h is defined for $t \geq 0$, $h(t) \geq 0$, increasing and continuous on the right, it is called a Hausdorff function. Let us reserve the symbols h and g for Hausdorff functions, i.e. h and g will always denote Hausdorff functions. Given h and $E \subseteq R$ (not necessarily measurable), we form the Hausdorff outer h-measure $m^h(E)$ as follows:

$$m^h_d(E) = \inf \left\{ \sum_{i=1}^{\infty} h(b_i - a_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i),\ b_i - a_i < d \right\}$$

$$m^h(E) = \lim_{d \to 0} m^h_d(E)$$

Note that as $d \to 0$ the class of sums over which we take the infimum decreases, hence $m^h_d(E)$ increases and therefore does indeed converge (possibly to infinity). It is easy to see that m^h is a metric outer measure, that is, additive on sets separated by positive distance. Hence the field of m^h measurable sets includes the Borel sets, in particular the closed sets I construct are measurable. We shall also denote the measure m^h, and shall assume that all sets mentioned are measurable.

Since h is continuous on the right, it is clear that m^h_d can be calculated using closed intervals, in fact using any sets S_i, if we replace $b_i - a_i$ by diam(S_i). It is for this reason that h is required to be continuous on the right. The definition I have given follows Rogers [2] and is the most general definition used. Often h is required to be continuous, or satisfy h(0)=0. For example Hausdorff himself considered only concave, continuous h with h(0)=0. I shall show that, for measures in R, we may as well assume h to be continuous and subadditive, that is, satisfy $h(x+y) \leq h(x)+h(y)$.

A few questions arise immediately. It is clear that for h(t)=t, m^h is ordinary Lebesgue measure and that for h(t)=1, m^h is counting measure. Are there any other non-trivial Hausdorff measures? In his original paper, Hausdorff showed that if h is continuous, concave, and satisfies h(0)=0, then there is a set S such that $m^h(S)=1$ and proved as a specific example that $m^h(C)=1$, where C is the Cantor middle third set and $h(t)=t^{\frac{\ln 2}{\ln 3}}$. The necessary and sufficient condition on h that there exist a set with finite positive measure was a long-standing problem, solved by A. Dvoretsky [3] in 1948. The condition is only $\liminf_{t \to 0} \frac{h(t)}{t} > 0$.

A more vague question: what is the relationship between h and the measure it generates? For example, when do different functions generate the same measure? This is a difficult question which I shall answer in the case the functions are concave.

[2]

It is clear that m^h is determined by its behavior near 0. I start with:

<u>Prop. 2.1:</u> If $\limsup\limits_{t\to 0} \frac{g(t)}{h(t)} = b$, then for all E, $m^g(E) \leq bm^h(E)$.

Proof: If $b=\infty$, the inequality is trivial. Suppose b is finite, $E \subseteq R$. Given $e>0$ choose d_0 such that

$$t < d_0 \Rightarrow \frac{g(t)}{h(t)} \leq b+e \quad .$$

If $d<d_0$ and $\bigcup_{i=1}^{\infty} (\bar{a}_i, b_i) \supseteq E$, $b_i - a_i < d$, then

$$m_d^g(E) \leq \sum_{i=1}^{\infty} g(b_i - a_i) \leq (b+e) \sum_{i=1}^{\infty} h(b_i - a_i)$$

Since this holds for all d-coverings of E where $d<d_0$,

$$m_d^g(E) \leq (b+e) m_d^h(E) \qquad (d<d_0)$$

$$\therefore \; m^g(E) \leq (b+e) m^h(E)$$

As e was arbitrary, $m^g(E) \leq bm^h(E)$

Corollary 2.1: If $\liminf_{t\to 0} \frac{g(t)}{h(t)} = a$, and $\limsup_{t\to 0} \frac{g(t)}{h(t)} = b$, then for all E

$$am^h(E) \leq m^g(E) \leq bm^h(E)$$

I will show later that these bounds are the best possible, when h and g are concave.

Corollary 2.2: If $\lim_{t\to 0} \frac{g(t)}{h(t)} = a$, then for all E

$$m^g(E)=am^h(E)$$

In particular, if $\lim_{t\to 0} \frac{g(t)}{h(t)} = 1$, g and h generate the same measure. The converses of the above are quite difficult and their consideration will be postponed. I turn now to show that h may be assumed to be continuous and subadditive.

[3]

Lemma 3.1: If h(0)=0, h generates the same measure as

$$\bar{h}(t)=\inf\left\{\sum_{i=1}^{\infty} h(c_i t) \;\middle|\; \sum_{i=1}^{\infty} c_i=1,\ 0\leq c_i\leq 1\right\}$$

Proof: Let $E \subseteq R$; $h(0)=0$. $\bar{h}(t)\leq h(t)$ (just let $c_1=1$, $c_j=0$, for $j\geq 1$). Hence $\limsup_{t\to 0} \frac{\bar{h}(t)}{h(t)} \leq 1$, so by Prop.

2.1, $m^{\tilde{h}}(E) \leq m^{h}(E)$. I'll now establish the opposite inequality. If $m^{\tilde{h}}(E)=\infty$, the inequality is trivial, so assume now $m^{\tilde{h}}(E)$ is finite. Given d, e >0, choose a cover $\bigcup_{i=1}^{\infty}(a_i,b_i) \supseteq E$, $b_i - a_i < d$, such that

$$\sum_{i=1}^{\infty} \tilde{h}(b_i - a_i) - m_d^{\tilde{h}}(E) < \frac{e}{2}$$

We can do this by the definition of $m_d^{\tilde{h}}(E)$ Choose c_{ik}, i,k=1,2,... such that $0 \leq c_{ik} \leq 1$, $\sum_{k=1}^{\infty} c_{ik} = 1$, and

$$\sum_{k=1}^{\infty} h(c_{ik}(b_i - a_i)) - \tilde{h}(b_i - a_i) < e2^{-i-1}$$

We can do this by the definition of $\tilde{h}$. Consider the closed intervals $[a_{ik}, b_{ik}]$, i,k=1,2,... given by

$$a_{ik} = a_i + \{\sum_{j=1}^{k-1} c_{ij}\}(b_i - a_i)$$

$$b_{ik} = a_i + \{\sum_{j=1}^{k} c_{ij}\}(b_i - a_i)$$

These are just a subdivision of $[a_i, b_i)$. Thus

$$\bigcup_{i,k=1}^{\infty} [a_{ik}, b_{ik}] \supseteq \bigcup_{i=1}^{\infty} [a_i, b_i) \supseteq E \quad \text{and}$$

$$b_{ik} - a_{ik} = c_{ik}(b_i - a_i) \leq b_i - a_i < d$$

Hence

$$m_d^h(E) \leq \sum_{i,k=1}^{\infty} h(b_{ik}-a_{ik}) \leq \sum_{i=1}^{\infty} \{\tilde{h}(b_i-a_i) + e2^{-i-1}\} =$$

$$= \sum_{i=1}^{\infty} \tilde{h}(b_i-a_i) + \frac{e}{2} \leq m_d^{\tilde{h}}(E) + e$$

As e was arbitrary, we conclude $m_d^h(E) \leq m_d^{\tilde{h}}(E)$. Consequently $m^h(E) \leq m^{\tilde{h}}(E)$, therefore $m^h(E) = m^{\tilde{h}}(E)$.

Lemma 3.2: If $\sum_{i=1}^{\infty} c_i = 1$, $0 \leq c_i \leq 1$, then $\sum_{i=1}^{\infty} \tilde{h}(c_i t) \geq \tilde{h}(t)$ ($\tilde{h}$ as in lemma 3.1).

Proof: Since $\tilde{h}(c_i t)$ is finite, given e >0, choose e_{ik}, $0 \leq e_{ik} \leq 1$, such that

$$\sum_{k=1}^{\infty} e_{ik} = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} h(e_{ik} c_i t) - \tilde{h}(c_i t) < e2^{-i}$$

Then we note $\sum_{i,k=1}^{\infty} e_{ik} c_i = \sum_{i=1}^{\infty} c_i \sum_{k=1}^{\infty} e_{ik} = 1$, and $0 \leq e_{ik} c_i \leq 1$, so by the definition of $\tilde{h}(t)$,

$$\tilde{h}(t) \leq \sum_{i,k=1}^{\infty} h(e_{ik} c_i t) \leq \sum_{i=1}^{\infty} (\tilde{h}(c_i t) + e2^{-i})$$

$$= \sum_{i=1}^{\infty} \tilde{h}(c_i t) + e$$

As e was arbitrary, lemma 3.2 is established.

Lemma 3.3: $\tilde{h}$ is increasing ($\tilde{h}$ as in lemma 3.1).

Proof: Suppose $x<y$, but $\bar{h}(y) < \bar{h}(x)$. Choose c_i such that $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$, and $\sum_{i=1}^{\infty} h(c_i y) < \bar{h}(x)$. By monotonicity of h and $x<y$,

$$\sum_{i=1}^{\infty} h(c_i x) \leq \sum_{i=1}^{\infty} h(c_i y) < \bar{h}(x)$$

contradicting the definition of $\bar{h}$. Therefore $\bar{h}$ is increasing.

Lemma 3.4: $\bar{h}$ is continuous ($\bar{h}$ as in Lemma 3.1). Proof: If $x<y$, $\bar{h}(x) \leq \bar{h}(y) \leq \bar{h}(x) + \bar{h}(y-x)$, hence

$$|\bar{h}(y) - \bar{h}(x)| \leq \bar{h}(y-x)$$

which goes to 0 as x goes to y, establishing the continuity of $\bar{h}$.

Theorem 3.1: Every Hausdorff measure in $\mathbb{R}$ is generated by a continuous, subadditive h.

Proof: Given m^h, if $h(0) > 0$ then $m^h = m^{h(0)}$, and $h(0)$ is certainly continuous and subadditive. If $h(0) = 0$, by Lemmas 3.1 through 3.4, $m^h = m^{\bar{h}}$ and $\bar{h}$ is continuous and subadditive.

Subadditive is weaker than concave, for if h is concave, $\frac{t}{h(t)}$ is increasing, hence

$$h(x+y) \leq \frac{h(y)}{y}(x+y) = h(y) + \frac{x}{y}h(y)$$

By symmetry we may assume x<y, hence

$$h(y) \leq h(x)\frac{y}{x} \qquad h(x+y) \leq h(x) + h(y)$$

Thus h is subadditive. I am not sure whether every Hausdorff measure in R is generated by a concave continuous function, but I suspect that this is not the case.

[4]

The converse of Corollary 2.1:

Theorem 4.1: If h is concave, continuous, $\liminf_{t\to 0} \frac{g(t)}{h(t)} = a$, and e>0, then there is a set $S \subseteq R$ such that

$$0 < m^h(S) < \infty$$

$$am^h(S) \leq m^g(S) \leq (1+e)am^h(S)$$

Theorem 4.1 does not appear in the literature, though it may be known. The proof is a combination of A. Dvoretsky [3] and a generalization of Hausdorff [1], though more involved than either. I have chosen the notation to agree with these sources, so that their contributions are clear.

Proof of theorem 4.1: We assume first that a is finite, and h and g satisfy the hypotheses. If h(0)>0, then m^h is counting measure, so we let S be any finite set. It is easy to check that the conclusion of theorem 4.1 is then satisfied. If $\lim_{t\to 0} \frac{h(t)}{t} < \infty$, then m^h is Lebesgue measure and a simple argument shows, so is m^g; in this case we can take $S=[0,1]$. So assume now that $\lim_{t\to 0} \frac{h(t)}{t} = \infty$, and $h(0)=0$.

I first choose two sequences which are close to each other and have nice properties with respect to h and g; from one of these sequences I construct the desired set S.

Claim 1: We can choose two sequences $\{x_j\}$, $\{x_j^*\}$ such that:

(i) $x_0 = x_0^* = 1$ (ii) $x_{j+1} \leq x_{j+1}^* \leq x_j$

(iii) $\frac{x_{j+1}^*}{h(x_{j+1}^*)} < \frac{x_j}{h(x_j)}$ (iv) $\frac{h(x_j^*)}{h(x_{j+1}^*)} > \frac{2^j}{\ln(1+e)} + 1$

(v) $\frac{g(x_{j+1}^*)}{h(x_{j+1}^*)} - a < \frac{1}{j+1}$ (vi) $h(x_{j+1}) = \frac{h(x_j)}{K_{j+1}}$

where $K_{j+1} = \left[\frac{h(x_j^*)}{h(x_{j+1}^*)}\right] + 1$ ($[\ ]$ denotes integer part)

Proof by induction. Suppose we've picked x_j, x_j^* for $j=0,1,\ldots n$ satisfying (i)-(vi). I will show we may choose x_{n+1}^*, x_{n+1} satisfying (i)-(vi). Since $h(t)\to 0$ as $t\to 0$, for sufficiently small x_{n+1}^*, (iv) will be satisfied. Since $\frac{t}{h(t)}\to 0$ as $t\to 0$, for sufficiently small x_{n+1}^* (iii) will be satisfied. The second half of (ii) is clearly satisfied for sufficiently small x_{n+1}^*. Since (v) is satisfied for arbitrarily small x^*_{n+1}, we may choose x^*_{n+1} satisfying (i)-(vi) simultaneously. Having chosen x^*_{n+1} we choose x_{n+1} so that (vi) is satisfied. This we may do because h is continuous. Now

$$h(x^*_{n+1}) > \frac{h(x^*_n)}{K_{n+1}} > \frac{h(x_n)}{K_{n+1}} = h(x_{n+1})$$

by the definition of K_{n+1}, choice of x_{n+1}, and the inductive hypothesis. Since h is monotone, we conclude $x_{n+1} < x^*_{n+1}$, so (i) through (vi) are satisfied, proving claim 1.

Claim 2:

$$\text{(i) } K_{j+1} \geq 2 \qquad \text{(ii) } K_{j+1} x_{j+1} < x_j$$

$$\text{(iii) } \frac{g(x_{j+1})}{h(x_{j+1})} \leq (1+e)\left(a + \frac{1}{j+1}\right)$$

Proof of Claim 2:

(i) is immediate from (iv) of claim 1. Since $\frac{t}{h(t)}$ increases (as h is concave), by claim 1 (ii),(iii),

$$\frac{x_{j+1}}{h(x_{j+1})} \leq \frac{x^*_{j+1}}{h(x^*_{j+1})} < \frac{x_j}{h(x_j)}$$

$$\frac{h(x_j)}{h(x_{j+1})} x_{j+1} = K_{j+1} x_{j+1} < x_j$$

establishing (ii). Since

$$\frac{h(x_j^*)}{h(x_{j+1}^*)} < K_{j+1} \leq \frac{h(x_j^*)}{h(x_{j+1}^*)} + 1$$

$$\text{and} \qquad K_{j+1} = \frac{h(x_j)}{h(x_{j+1})},$$

$$\frac{h(x_j^*)}{h(x_{j+1}^*)}\{1 + \frac{h(x_{j+1}^*)}{h(x_j^*)}\} \geq \frac{h(x_j)}{h(x_{n+1})}$$

$$\therefore \frac{h(x_{j+1}^*)}{h(x_{j+1})} \leq \{1 + \frac{1}{K_{j+1}-1}\}\frac{h(x_j^*)}{h(x_j)} \leq \ldots$$

$$\leq \prod_{i=1}^{j+1}\{1 + \frac{1}{K_j - 1}\}$$

But we've arranged $K_i > \frac{2^{i+1}}{\ln(1+e)} + 1$. Since $\frac{1}{K_i - 1} < 1$, it is easy to verify that

$$\ln(1 + \frac{1}{K_i - 1}) \leq \frac{2}{K_i - 1} \leq \ln(1+e)2^{-i}$$

$$\ln \prod_{i=1}^{j+1}(1 + \frac{1}{K_j - 1}) = \sum_{i=1}^{j+1} \ln(1 + \frac{1}{K_j - 1}) \leq$$

$$\leq \sum_{i=1}^{j+1} \ln(1+e)2^{-i} \leq \ln(1+e)$$

Thus $\frac{h(x^*_{j+1})}{h(x_{j+1})} \leq 1+e$. Since g is increasing and $x_{j+1} < x^*_{j+1}$,

$$\frac{g(x_{j+1})}{h(x_{j+1})} \leq (1+e)\frac{g(x^*_{j+1})}{h(x^*_{j+1})} \leq (1+e)(1 + \frac{1}{j+1})$$

establishing claim 2.

We now construct S, using the sequence $\{x_j\}$. Let $S_0 = [0,1]$, $B[0] = (-\infty,0)$, $B(1) = (1,\infty)$

Let $S_1 = [0,x_1] \cup [x_1+y_1, 2x_1+y_1] \cup \ldots \cup [1-x_1, 1]$

consist of the closed intervals J_1^j, $j=1,2,\ldots K_1$ got by deleting the K_1-1 open intervals $B[j]$, $j=1,2,\ldots$ K_1-1 (ordered left to right) of length y_1 from S_0. By claim 2, $K_1 \geq 2$ and $K_1 x_1 \leq 1$, so that none of the B intervals is empty. Proceeding inductively we let S_{n+1} be the $\prod_{j=1}^{n+1} K_j$ closed intervals J^j_{n+1} got by deleting the open intervals $B[k_1,\ldots k_n,i]$, $i=1,2,\ldots K_{n+1}-1$ (ordered left to right) of length y_{n+1} from the J_n to the right of $B[k_1,\ldots k_n]$. As above we note that $K_{n+1} \geq 2$, and $K_{n+1} x_{n+1} < x_n$, so this slicing is really possible.

$S = \bigcap_{n=1}^{\infty} S_n$ is the desired set.

Claim 3: $m^h(S) \leq 1$

Proof: Given $d>0$ we choose n to be large enough that $x_n < d$. Consider the $\prod_{j=1}^{n} K_j$ closed intervals J_n^j which make up S_n.

They cover S, since S S_n, and they are each of length x_n.

Hence

$$m_d^h(S) \leq \sum_{i=1}^{\prod_{j=1}^{n} K_j} h(x_n) = \prod_{j=1}^{n} K_j h(x_n) = 1$$

As d was arbitrary, we conclude $m^h(S) \leq 1$.

The converse is true, but the proof is more involved. It is easy to see that the B's are disjoint and lexicographically ordered left to right. We let

$$|B[k_1,\ldots k_n]| = \sum_{j=1}^{n} k_j h(x_j)$$

and rank $B[k_1,\ldots k_n] = n$ (we assume $k_n>0$) For technical convenience, we let $|B(1)|=1$, rank $B[0]=$ rank $B(1)=0$. Say B = (uB, vB).

Claim 4: If $0<r<K_n$, then $h(rx_n+(r-1)y_n) \geq rh(x_n)$

Proof by induction on r. The inequality is clear for r=1. Suppose now $h(rx_n+(r-1)y_n) \geq rh(x_n)$ and $r+1 < K_n$. Then

$$rx_n+(r-1)y_n \leq (r+1)x_n+ry_n \leq K_n x_n+(K_n-1)y_n = x_n$$

So by convexity of h and inductive hypothesis,

$$h((r+1)x_n+y_n) \geq$$

$$\geq \frac{rh(x_n)(K_n-r-1)(x_n+y_n)+K_n h(x_n)(x_n+y_n)}{(K_n-r)(x_n+y_n)}$$

$= (r+1)h(x_n)$ establishing claim 4.

Claim 5: If $|B_2| > |B_1|$, (B_2 lies to the right of B_1), then $h(uB_2-vB_1) > |B_2|-|B_1|$

Proof by induction on the ranks of the B´s. When the ranks are 0, we must have $B_1=B[0]$ and $B_2=B(1)$, then $h(uB_2-vB_1) = 1 = |B_2|-|B_1|$. Now suppose Claim 5 holds for B´s of ranks $\leq n$, and B_2, B_1 have ranks $\leq n+1$. There are three nontrivial cases:

Case 1: $\text{rank}B_1=n+1$, $\text{rank}B_2\leq n$

Say $B_1 = B[k_1,\ldots k_n,r]$

Let $L = B[k_1,\ldots k_n]$

$$\text{and}\quad R=\begin{cases} B[k_1,\ldots k_n+1] & \text{if}\quad k_n+1<K_n \\ B[k_1,\ldots k_m+1] & \text{if}\quad k_n+1=K_n,\ldots k_{m+1}+1=K_{m+1} \\ B(1) & \text{if}\quad k_j+1=K_j\quad j=1,2,\ldots n \end{cases}$$

L and R are merely the left and right nearest neighbors of B_1 which have rank $\leq n$.

Then $|L|<|B_1|<|R|\leq|B_2|$. If $R=B_2=B(1)$,

$$h(uB_2-vB_1) = h((K_{n+1}-r)x_{n+1} + (K_{n+1}-r-1)y_{n+1})$$

$$\geq(K_{n+1}-r)h(x_{n+1}) = |B_2| - |B_1|$$

using claim 4. If $|R|<|B(1)|$ we note

$$uB_2 - vR \leq uB_2 - vB_1 \leq uB_2 - vL$$

Therefore by inductive hypothesis and convexity of h,

$$h(uB_2 - vB_1) \geq \frac{|B_2|-|R|)(vB_1 - vL) + (|B_2|-|L|)(vR - vB_1)}{vR - vL}$$

Now $|L|=|B_1|-rh(x_{n+1})$, and $|R|=|B_1|+(K_{n+1}-r)h(x_{n+1})$,

$$vB_1 = vL + rx_{n+1} + ry_{n+1}$$

$$vR = vL+x_n+y_n = vL+K_{n+1}x_{n+1}+(K_{n+1}-1)y_{n+1}+y_n$$

Hence $h(uB_2-vB_1) \geq$

$$= |B_2| - |B_1| + \frac{rh(x_{n+1})(y_n - y_{n+1})}{x_n + y_n} \geq |B_2| - |B_1|$$

since $y_n \geq y_{n+1}$

Case 2: $\text{rank} B_1 \leq n$, $\text{rank} B_2 = n+1$

Again, let L, R be the the rank $<n+1$ left and right nearest neighbors of B_2. Then $|B_1|\leq|L|<|B_2|<|R|$. Say $B_2 = B[k_1,\ldots k_n,r]$. If $B_1=L=B[0]$,

$$h(uB_2-vB_1) = h(rx_{n+1}+(r-1)y_{n+1}) \geq rh(x_{n+1})$$

$=|B_2|-|B_1|$ using claim 4. If $0<|L|$, we note

$$uL - vB_1 \leq uB_2 - vB_1 \leq uR - vB_1$$

Hence by inductive hypothesis and convexity of h,

$$h(uB_2-vB_1) \geq \frac{(|L|-|B_1|)(uR-uB_2) + (|R|-|B_1|)(uB_2-uL)}{uR - uL}$$

Again, $|L|=|B_2|-rh(x_{n+1})$, $|R|=|B_2|+(K_{n+1}-r)h(x_{n+1})$,

$$uR = uB_2+(K_{n+1}-r)(x_{n+1}+y_{n+1})$$

$$uL = uB_2 - y_n - rx_{n+1} - (r-1)y_{n+1}$$

$$h(uB_2-vB_1) \geq |B_2|-|B_1|+ \frac{h(x_{n+1})(K_{n+1}-r)(y_n-y_{n+1})}{x_n + y_n}$$

$\geq|B_2|-|B_1|$ since $r\leq K_{n+1}-1$.

Case 3: rank B_1= rank B_2=n+1.

Let L, R be the rank <n+1 left and right nearest neighbors of B_2. Then $0<|B_1|<|L|<|B_2|<|R|<1$, and

$$uL - vB_1 \leq uB_2 - vB_1 \leq uR - vB_1$$

Now using case 1 and convexity,

$$h(uB_2-vB_1)\geq \frac{(|L|-|B_1|)(uR-uB_2)) + (|R|-|B_1|)(uB_2-uL)}{uR - uL}$$

$$= |B_2| - |B_1| + \frac{h(x_{n+1})(K_{n+1}-r)(y_n - y_{n+1})}{x_n + y_n} \geq |B_2| - |B_1|$$

proving Claim 5.

Claim 6: $m^h(S)=1$.

Proof: Suppose $S \subseteq \bigcup_{i=1}^{\infty} I_i$, I_i open intervals. Since S is compact, S is covered by a finite number of the I_i which intersect S, say

$$S \subseteq \bigcup_{i=1}^{N} (a_i, b_i)$$

where (a_i, b_i) are some of the I_i's which intersect S and

$$a_1 < 0 < b_1 < a_2 \ldots a_N < 1 < b_N$$

I claim $\sum_{i=1}^{N} h(b_i - a_i) \geq 1$. $b_1 \notin S$, say $b_1 \in B_1$. Now $vB_1 \in S$, $a_2 < vB_1$, $\therefore a_2 \in B_1$. Continuing, we get $B_2, \ldots B_{N-1}$ with $a_{j+1}, b_j \in B_j$, $a_{j+1} < vB_j$, $b_j > uB_j$. Let $B_0 = B[0]$, $B_N = B(1)$. Then $b_j - a_j \geq uB_j - vB_{j-1}$, therefore by claim 5, $h(b_j - a_j) \geq |B_j| - |B_{j-1}|$,

$$\therefore \sum_{i=1}^{N} h(b_j - a_j) \geq 1$$

But clearly $\sum_{i=1}^{\infty} h(\text{diam} I_i) \geq \sum_{i=1}^{N} h(b_i - a_i) \geq 1$. Thus $m_d^h(S) \geq 1$ for $d>0$, hence $m^h(S) \geq 1$, so by Claim 3, $m^h(S)=1$, establishing Claim 6.

Claim 7: $a \le m^g(S) \le (1+e)a$

Proof: As in Claim 3, given $d>0$ choose n large enough that $x_{n+1} < d$. Consider the $\prod_{j=1}^{n+1} K_j$ closed intervals which make up S_{n+1}. They cover S and have length $<d$, hence

$$m_d^g(S) \le \prod_{j=1}^{n+1} K_j\, g(x_{n+1}) =$$

$$= \prod_{j=1}^{n+1} K_j\, h(x_{n+1}) \frac{g(x_{n+1})}{h(x_{n+1})} = \frac{g(x_{n+1})}{h(x_{n+1})}$$

$\le (1+e)\left(a + \frac{1}{j+1} \right)$ by Claim 2 (iii). Thus $m^g(S) \le (1+e)a$. By Corollary 2.1, $m^g(S) \ge am^h(S) = a$, hence

$$a \le m^g(S) \le (1+e)a$$

Thus by Claims 5 and 7,

$$m^h(S) = 1$$

$$am^h(S) \le m^g(S) \le a(1+e)m^h(S)$$

establishing Theorem 4.1 in the case $a<\infty$. If $a=\infty$, Claim 5 (setting $g=h$, say) yields a set S with $m^h(S)=1$. By Corollary 2.1, though, $m^g(S)=\infty$, establishing Theorem 4.1 when $a=\infty$.

I don't know whether Theorem 4.1 is true when h is not concave.

Corollary 4.1: If h is concave and continuous, there is a set S R with $0<m^h(S)<\infty$.

This is Hausdorff's result, slightly weaker than Dvoretsky's result, which assumes only $\liminf_{t\to 0}\frac{h(t)}{t}>0$.

Corollary 4.2: If h and g are concave and $\liminf_{t\to 0}\frac{g(t)}{h(t)}=a$, $\limsup_{t\to 0}\frac{g(t)}{h(t)}=b$, e>0, then there are sets S_1, $S_2 \subseteq R$ such that

$$o < m^h(S_i) < \infty$$

$$am^h(S_1) \leq m^g(S_1) \leq (1+e)am^h(S_1)$$

$$b(1-e)m^h(S_2) \leq m^g(S_2) \leq bm^h(S_2)$$

This and corollary 3.2 are the relationship between h and m^h referred to in section 2.

Corollary 4.3: Concave functions h and g generate the same measures in R if and only if $\lim_{t\to 0}\frac{h(t)}{g(t)}=1$.

Corollary 4.3 answers another query of section 2, and is by no means obvious.

We have shown that the Hausdorff measures in R generated by concave functions are in one to one correspondence with the equivalence classes of concave continuous functions whose ratio tends to one as t goes to 0. I close by remarking that this set C has a very complicated structure. It is not linearly ordered by any of the natural partial orders, for example $h \precsim g \Leftrightarrow \limsup_{t\to 0}\frac{g(t)}{h(t)}\leq 1$ or $h \lhd g \Leftrightarrow \lim_{t\to 0}\frac{g(t)}{h(t)}=0$. Nor do either of these orders have a countable basis in C. Hausdorff constructed the "Logarithmic Scale"

$$h[a_1,\ldots a_k](t) = t^{a_1}|\ln t|^{a_2}\ldots(\ln\ln\ldots|\ln t|\ )^{a_k}$$

(first nonzero a_j is >0)

which is a countably based linear chain in C, but by the preceeding remarks is only a (very) small part of C.

Cambridge, Mass. 1980

Footnotes

[1] Hausdorff, F.: 1919, Dimension und ausseres mass, Math. Ann. 79, 157-179

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[3] Dvoretsky, A.: 1948, A note on Hausdorff dimension functions, Proc. Camb. Phil. Soc. 44, 13-16

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