Tradeoffs in Frequency-Weighted $\mathbf{H}_\infty$-Control

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Abstract

An important problem in $\mathbf{H}_\infty$-control is the design of a controller that minimizes the $\mathbf{H}_\infty$-norm of a closed-loop transfer matrix, multiplied by a suitable weighting function which reflects different performance requirements over different frequency bands. Often, these are competing requirements, and in this paper, we show how we may efficiently compute tradeoffs between them using a simple application of tangential Hermite-Fejér interpolation theory.

Keywords: Frequency weighted $\mathbf{H}_\infty$; tradeoffs; tangential Hermite-Fejér interpolation theory.

1 Introduction

Consider the feedback system shown in figure 1. P is a linear time-invariant (LTI) plant. K is an

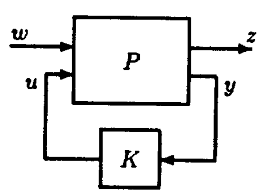

Figure 1: $\mathbf{H}_\infty$-optimal controller design framework

LTI controller to be designed so that the closed-loop system is internally stable, with the (stable) closed-loop transfer matrix H from w to z satisfying some performance requirements. (In the sequel, we will use the term stable transfer matrices to refer to those that are analytic and bounded in the closed right half complex plane $\mathbf{C}_+$.) One common design requirement leads to the frequency-weighted

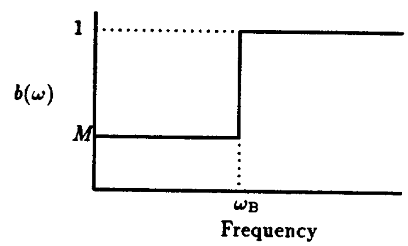

Figure 2: A typical frequency-weighting function

$\mathbf{H}_\infty$-control problem:

$$\text{Design } K \text{ so that} \quad \sigma_{\max}(H(j\omega)) \le \gamma b(\omega) \ \text{ for all } \omega \in \mathbf{R}, \tag{1}$$

where $\sigma_{\max}(Q)$ denotes the maximum singular value of the matrix Q, $b(w) : \mathbf{R} \to \mathbf{R}_+$, with $b(-\omega) = b(\omega)$ for all ω, is the "frequency-weighting" function, and $\gamma > 0$. A common choice of b is shown in figure 2; this is the frequency-weighting function that we will consider throughout the sequel.

We will call the quantity $M(\le 1)$ the ***in-band to out-band rejection ratio***, since it is the ratio of the in-band to out-band levels of the weighting function b. We will refer to ω_B as the ***bandwidth***, and γ as the ***out-band level***. A small M is usually desirable, reflecting the requirement that the closed-loop system reject low-frequency noise. The bandwidth ω_B is desired to be large, so that the system rejects noise over as wide a frequency spectrum as possible. γ is desired to be small so as to reject high-frequency disturbances from w to z.

It should be intuitively clear that the requirements above, that is, "small M", "small γ" and "large ω_B" are competing requirements: For instance, if problem (1) is feasible for $M = M_0$ and $\gamma = \gamma_0$, then it is feasible for all $M \ge M_0$ and $\gamma = \gamma_0$. In this paper, we will study the tradeoffs between M, γ and ω_B, by a simple application of tangential Hermite-Fejér interpolation theory.

A number of researchers have studied this and related problems. Zames and Francis [2, 3] have studied the effect of right-half plane transmission zeros of the plant on the weighted sensitivity transfer function for single-input single-output transfer functions. Freudenberg and Looze [4] study the Bode integral for such systems. O'Young and Francis [5, 6] consider the same problem that we consider in this paper for the special case when $H = (I + P_{yu}K)^{-1}$, the sensitivity transfer function matrix (P_{yu} is the plant transfer matrix from u to y). Their solution uses the *matrix* Nevanlinna-Pick algorithm. The approach in this paper is using the *tangential* Hermite-Fejér interpolation algorithm, and is more general.

2 Frequency-weighted $\mathbf{H}_\infty$-control problem as an interpolation problem

It is well-known [7, 8, 9, 10] that the set $\mathcal{H}$ of all achievable stable closed-loop transfer matrices from w to z, *i.e.*, the set of all transfer matrices in figure 1 achievable over all stabilizing controllers K may be parametrized affinely via the Youla parameter Q as

$$\mathcal{H} = \{T_1 - T_2QT_3 \mid Q \text{ stable}\},$$

where T_1, T_2 and T_3 are stable transfer matrices, of sizes $n_z \times n_w$, $n_z \times n_u$ and $n_y \times n_w$ (n_w is the number of components of the (vector) signal w of figure 1, etc.). We will refer to $H \in \mathcal{H}$ as a "closed-loop map".

We will make a number of assumptions about the system:

1. We assume that $n_y \geq n_w$ and $n_u \geq n_z$, and that $T_2(s)$ and $T_3(s)$ are full rank matrices for almost all s in $\mathbf{C}_+$. These assumptions, roughly speaking, mean that we have in effect more sensors than exogenous inputs w and more actuators than controlled variables z.

2. We assume that $T_2(s)$ and $T_3(s)$ are of full rank as $s \to \infty$. Thus T_2 and T_3 may not have transmission zeros at infinity.

3. We assume that T_2 and T_3 share no zeros in $\mathbf{C}_+$.

Let $\alpha_1, \ldots, \alpha_p$ be the transmission zeros of T_2 in $\mathbf{C}_+$, with geometric multiplicities $\mu_1, \ldots, \mu_p$ respectively. The α_is are not necessarily distinct. Then, there exist vectors $u_{i,l} \in \mathbf{C}^{n_z}$, $i = 1, \ldots, p$, $l = 1, \ldots, \mu_i$ such that

$$\sum_{k=1}^{l} u_{i,k}^* \tfrac{1}{(l-k)!} T_2^{(l-k)}(\alpha_i) = 0,$$
$$l = 1, \ldots, \mu_i,\ i = 1, \ldots, p,$$

where we have used $T_2^{(l)}$ to denote the lth derivative of T_2. The set of vectors

$$\{u_{i,l},\ l = 1, \ldots, \mu_i\}$$

is referred to as a left-null chain of H at α_i [11].

Let $\beta_1, \ldots, \beta_q$ be the transmission zeros of T_3 in $\mathbf{C}_+$, with geometric multiplicities $\nu_1, \ldots, \nu_q$ respectively. The β_is are not necessarily distinct, but they are distinct from the α_i as assumed, that is, $\alpha_i \neq \beta_l$, $i = 1, \ldots, p$, $l = 1, \ldots, q$. Then, there exist vectors $x_{i,l} \in \mathbf{C}^{n_w}$, $i = 1, \ldots, q$, $l = 1, \ldots, \nu_i$ such that

$$\sum_{k=1}^{l} \tfrac{1}{(l-k)!} T_3^{(l-k)}(\beta_i) x_{i,k} = 0,$$
$$l = 1, \ldots, \nu_i,\ i = 1, \ldots, q.$$

The set of vectors

$$\{x_{i,l},\ l = 1, \ldots, \nu_i\}$$

is referred to as a right-null chain of H at β_i [11].

Since Q is stable, $H \triangleq T_1 - T_2QT_3$ must satisfy, for $l = 1, \ldots, \mu_i$, $i = 1, \ldots, p$,

$$\begin{aligned} \sum_{k=1}^{l} u_{i,k}^* \tfrac{1}{(l-k)!} H^{(l-k)}(\alpha_i) &= v_{i,l}^* \\ \sum_{k=1}^{l} \tfrac{1}{(l-k)!} H^{(l-k)}(\beta_i) x_{i,k} &= y_{i,l} \end{aligned} \tag{2}$$

where

$$\begin{aligned} v_{i,l}^* &\triangleq \sum_{k=1}^{l} u_{i,k}^* \tfrac{1}{(l-k)!} T_1^{(l-k)}(\alpha_i), \\ y_{i,l} &\triangleq \sum_{k=1}^{l} \tfrac{1}{(l-k)!} T_1^{(l-k)}(\beta_i) x_{i,k}. \end{aligned}$$

Conversely, it can be shown that if H satisfies (2), then there exists a stable transfer matrix Q of size $n_y \times n_u$ such that $H = T_1 - T_2QT_3$. Thus (2) provides an interpolation characterization of the set $\mathcal{H}$. Conditions (2) are referred to as "tangential interpolation conditions" in the literature, to contrast them from matrix interpolation conditions [11].

In view of the interpolation characterization (2) for $\mathcal{H}$, the frequency weighted $\mathbf{H}_\infty$-control problem (1) becomes

$$\begin{array}{l} \textit{Find } H \textit{ that satisfies (2) with} \\ \quad \sigma_{\max}(H(j\omega)) \leq \gamma b(\omega) \textit{ for all } \omega \in \mathbf{R}. \end{array} \tag{3}$$

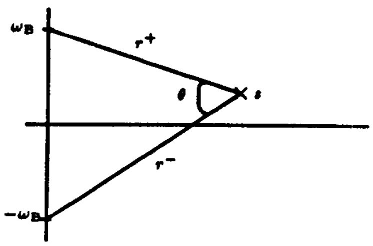

Figure 3: Definition of $\theta(\omega_B, s)$, $r^+(\omega_B, s)$ and $r^-(\omega_B, s)$

We may solve this problem using the classical Hermite-Fejér interpolation theory. As a first step, we construct an analytic function W with W and W^{-1} stable, satisfying $|W(s)| \longrightarrow b(\omega)$ as $s \to j\omega$ for almost all $\omega \in \mathbf{R}$: $W(s)$ is uniquely given [12] by $e^{G(s)}$, where

$$G(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \log b(j\omega) \frac{1}{1+\omega^2} \frac{j\omega s - 1}{j\omega - s} \, d\omega. \quad (4)$$

It may be verified that equation (4) yields the following formula for G: Given $s \in \mathbf{C}_+$, let $\theta(\omega_B, s)$ be the angle subtended at s by the line segment $[-j\omega_B, j\omega_B]$ in the complex plane, and let $r^+(\omega_B, s)$ and $r^-(\omega_B, s)$ be the distance between s and the points $j\omega_B$ and $-j\omega_B$ respectively (see figure 3). Then, it may be verified that

$$\Re G(s) = \frac{\theta(\omega_B, s)}{\pi} \log M, \quad (5)$$

and

$$\Im G(s) = \frac{1}{\pi} \log\left(\frac{r^+(\omega_B, s)}{r^-(\omega_B, s)}\right) \log M. \quad (6)$$

Then, $e^{G(s)} = M^{\phi(\omega_B, s)}$, where

$$\phi(\omega_B, s) = \frac{\theta(\omega_B, s)}{\pi} + \frac{j}{\pi} \log\left(\frac{r^+(\omega_B, s)}{r^-(\omega_B, s)}\right). \quad (7)$$

Returning to problem (3), suppose that there exists $\tilde{H}$ that satisfies the modified interpolation conditions: For $l = 1, \ldots, \mu_i$, $i = 1, \ldots, p$,

$$\begin{aligned} \textstyle\sum_{k=1}^{l} u_{i,k}^* \frac{1}{(l-k)!} \tilde{H}^{(l-k)}(\alpha_i) &= \tilde{v}_{i,l} \triangleq e^{-G(\alpha_i)} v_{i,l}^*, \\ \textstyle\sum_{k=1}^{l} \frac{1}{(l-k)!} \tilde{H}^{(l-k)}(\beta_i) x_{i,k} &= \tilde{y}_{i,l} \triangleq e^{-G(\beta_i)} y_{i,l} \end{aligned} \quad (8)$$

with $\sigma_{\max}\left(\tilde{H}(j\omega)\right) \leq \gamma$ for all $\omega \in \mathbf{R}$. Then $H(s) = e^{G(s)} \tilde{H}(s)$ solves problem (3). Conversely, if H solves problem (3), then $\tilde{H}(j\omega)$ must satisfy (8) and the condition that $\sigma_{\max}\left(\tilde{H}(j\omega)\right) \leq \gamma$ for all $\omega \in \mathbf{R}$.

Thus, we have reduced the frequency-weighted $\mathbf{H}_\infty$-optimal control problem to the standard two-sided Hermite-Fejér interpolation problem [11]:

$$\textit{Find } \tilde{H} \textit{ that satisfies (8) with } \sigma_{\max}\left(\tilde{H}(j\omega)\right) \leq \gamma \textit{ for all } \omega \in \mathbf{R}. \quad (9)$$

3 Computing tradeoffs efficiently

We first state the condition for the existence of a solution to problem (9) in a form that will be most useful to us: There exists a solution to problem (9) if and only the solution N to the Lyapunov equation

$$\Lambda^* N + N\Lambda = \begin{bmatrix} \tilde{V}^*\tilde{V} - \gamma^2 U^* U & \gamma\left(\tilde{V}^* X - U^* \tilde{Y}\right) \\ \gamma\left(X^* \tilde{V} - \tilde{Y}^* U\right) & \gamma^2 X^* X - \tilde{Y}^* \tilde{Y} \end{bmatrix}$$

satisfies $N \geq 0$, where $\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2)$,

$$\Lambda_1 = \operatorname{diag}\left(-J_{\overline{\alpha_1}, \mu_1}, \ldots, -J_{\overline{\alpha_p}, \mu_p}\right), \quad \Lambda_2 = \operatorname{diag}\left(J_{\beta_1, \nu_1}, \ldots, J_{\beta_q, \nu_q}\right),$$

$$\tilde{V} = \left[\tilde{V}_1 \ \cdots \ \tilde{V}_p\right], \ \tilde{Y} = \left[\tilde{Y}_1 \ \cdots \ \tilde{Y}_p\right],$$

$$\tilde{V}_i = [\tilde{v}_{i1} \ \cdots \ \tilde{v}_{i\mu_i}], \ i = 1, \ldots, p,$$

$$\tilde{Y}_i = [\tilde{y}_{i1} \ \cdots \ \tilde{y}_{i\nu_i}], \ i = 1, \ldots, q.$$

(We have used $J_{\lambda,m}$ to denote a Jordan block of size m and eigenvalue λ.) N is a "generalized" Pick matrix. Though the above condition for the existence of a solution to problem (9) does not appear anywhere in the literature, it is a straightforward extension of the results in [11].

With

$$V = [V_1 \ \cdots \ V_p], \ Y = [Y_1 \ \cdots \ Y_p],$$

$$V_i = [v_{i1} \ \cdots \ v_{i\mu_i}], \ i = 1, \ldots, p,$$

$$Y_i = [y_{i1} \ \cdots \ y_{i\nu_i}], \ i = 1, \ldots, q,$$

and

$$D_1 = \operatorname{diag}\left(e^{-G(\overline{\alpha_1})} I_{\mu_1}, \ldots, e^{-G(\overline{\alpha_p})} I_{\mu_p}\right), \quad D_2 = \operatorname{diag}\left(e^{-G(\beta_1)} I_{\nu_1}, \ldots, e^{-G(\beta_q)} I_{\nu_q}\right),$$

where I_m denotes the $m \times m$ identity matrix, we observe that

$$\tilde{V} = V D_1, \ \tilde{Y} = Y D_2.$$

Note that using (7), D_1 and D_2 may be expressed in terms of M and ω_B as

$$D_1 = M^{-\Phi_1(\omega_B)} \text{ and } D_2 = M^{-\Phi_2(\omega_B)}, \quad (10)$$

where

$$\Phi_1(\omega_B) = \text{diag}\left(\phi(\omega_B, \overline{\alpha_1}) I_{\mu_1}, \ldots, \phi(\omega_B, \overline{\alpha_p}) I_{\mu_p}\right),$$
$$\Phi_2(\omega_B) = \text{diag}\left(\phi(\omega_B, \beta_1) I_{\nu_1}, \ldots, \phi(\omega_B, \beta_q) I_{\nu_q}\right). \quad (11)$$

(The notation a^B with a scalar a and a diagonal matrix B denotes a diagonal matrix with diagonal entries $a^{B_{ii}}$.)

Therefore, we conclude that a solution to problem (9) exists if and only if the solution N to the Lyapunov equation

$$\begin{aligned} \Lambda^* N + N\Lambda = &\begin{bmatrix} D_1^* V^* \\ \gamma X^* \end{bmatrix} \begin{bmatrix} D_1^* V^* \\ \gamma X^* \end{bmatrix}^* \\ &- \begin{bmatrix} \gamma U^* \\ D_2^* Y^* \end{bmatrix} \begin{bmatrix} \gamma U^* \\ D_2^* Y^* \end{bmatrix}^* \end{aligned} \quad (12)$$

satisfies $N \geq 0$.

We may further simplify the expression for N. Let us define two Gramians W_{in} and W_{out} via

$$\begin{aligned} \Lambda^* W_{\text{in}} + W_{\text{in}} \Lambda &= \begin{bmatrix} V^*V & V^*X \\ X^*V & X^*X \end{bmatrix}, \\ \Lambda^* W_{\text{out}} + W_{\text{out}} \Lambda &= \begin{bmatrix} U^*U & U^*Y \\ Y^*U & Y^*Y \end{bmatrix}. \end{aligned} \quad (13)$$

Then, it is easily verified that

$$\begin{aligned} N = &\begin{bmatrix} D_1^* & 0 \\ 0 & \gamma I \end{bmatrix} W_{\text{in}} \begin{bmatrix} D_1 & 0 \\ 0 & \gamma I \end{bmatrix} \\ &- \begin{bmatrix} \gamma I & 0 \\ 0 & D_2^* \end{bmatrix} W_{\text{out}} \begin{bmatrix} \gamma I & 0 \\ 0 & D_2 \end{bmatrix}. \end{aligned} \quad (14)$$

Thus in summary, given M, γ and ω_B, the existence of a solution to the problem

> *Find H that satisfies (2) with*
> $\sigma_{\max}(H(j\omega)) \leq \gamma b(\omega)$ *for all* $\omega \in \mathbf{R}$

may be checked with the following steps:

1. Compute W_{in} and W_{out} using equations (13).
2. Form D_1 and D_2 using equations (10) and (11).
3. Check if N, computed using equation (14), satisfies $N \geq 0$.

Thus, the main contribution of the paper is the following observation:

> *Given M, γ and ω_B, we may check if problem (1) is feasible by essentially an eigenvalue computation.*

This observation enables us to compute efficiently the tradeoffs between M, γ and ω_B. Suppose that for for fixed M and ω_B, we wish to compute the smallest value of γ for which (9) has a solution (let us denote this value of γ by γ_{opt}). We start by computing W_{in} and W_{out} by solving the Lyapunov equations (13). Then, γ_{opt} may be computed by a simple bisection scheme, every iteration of which requires:

1. The evaluation of D_1 and D_2 (using equations (10) and (11)),
2. Computing N (using equation (14)), and
3. Checking if the minimum eigenvalue of N is nonnegative.

Thus each bisection iteration requires essentially an eigenvalue computation. By computing γ_{opt} for various values of M, we may compute the tradeoff between M and γ for fixed ω_B. The above remarks hold for the computation of tradeoffs between other quantities as well.

We note that instead of a bisection scheme, we may also use more sophisticated methods such as the Newton-Raphson method to compute γ_{opt}. We will not discuss the details here.

4 A simple example

We demonstrate the results of the previous section on a simple example. We consider an example where the set of achievable stable closed-loop transfer matrices for system in figure 1 are given by

$$\mathcal{H} = \{T_1 - T_2 Q T_3 \mid Q \text{ stable}\},$$

where

$$T_1 = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+2} \\ \dfrac{1}{s+2} & \dfrac{1}{s+3} \end{bmatrix},$$

$$T_2 = \begin{bmatrix} \dfrac{(s-1)^2+1}{(s+1)^2} & 0 \\ \dfrac{1}{s+2} & 1 \end{bmatrix}$$

and

$$T_3 = \begin{bmatrix} 1 & \frac{1}{s+1} \\ 0 & \frac{(s-1)^2}{(s+1)^2} \end{bmatrix}.$$

A stable transfer matrix H belongs to $\mathcal{H}$ if and only if it satisfies the following interpolation conditions:

$$\begin{aligned} u_{1,1}^* H(1+j) &= v_{1,1}^* \\ u_{2,1}^* H(1-j) &= v_{2,1}^* \\ H(1)x_{1,1} &= y_{1,1} \\ H^{(1)}(1)x_{1,1} + H(1)x_{1,2} &= y_{1,2}, \end{aligned} \qquad (15)$$

where

$$[u_{1,1} \;\; u_{2,1}] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$[v_{1,1} \;\; v_{2,1}] = \begin{bmatrix} \frac{1}{(2-j)} & \frac{1}{(2+j)} \\ \frac{1}{(3-j)} & \frac{1}{(3+j)} \end{bmatrix},$$

$$[x_{1,1} \;\; x_{1,2}] = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{bmatrix},$$

$$[y_{1,1} \;\; y_{1,2}] = \begin{bmatrix} \frac{1}{12} & \frac{5}{36} \\ \frac{1}{12} & \frac{11}{144} \end{bmatrix}.$$

We will now impose the frequency-weighted $\mathbf{H}_\infty$ condition

$$\sigma_{\max}(H(j\omega)) \le \gamma b(\omega) \;\text{ for all } \omega \in \mathbf{R}. \qquad (16)$$

on $H \in \mathcal{H}$, and study the tradeoffs between M, γ and ω_B.

4.1 Tradeoff between M and γ

For various values of ω_B, the smallest achievable γ (i.e., γ_{opt}), as a function of M are shown in figure 4.

We first note that the case $\omega_B = 0$ corresponds to the situation where there is no in-band specification; in this case, irrespective of M, the condition (16) is merely the "$\mathbf{H}_\infty$-norm condition"

$$\sigma_{\max}(H(j\omega)) \le \gamma \;\text{ for all } \omega \in \mathbf{R}. \qquad (17)$$

We see that γ_{opt} in this case, which we will denote γ_0, is about 0.83.

For nonzero values of ω_B, decreasing the in-band to out-band rejection ratio M corresponds to requiring increasingly stringent conditions on $\sigma_{\max}(H(j\omega))$ over $\omega \in [-\omega_B, \omega_B]$. This can be achieved only at the cost of increased out-band levels of $\sigma_{\max}(H(j\omega))$ (recall that γ is precisely this quantity). Moreover, for nonzero ω_B, $\gamma_{opt} \to \infty$ as $M \to 0$. Similarly, increasing ω_B corresponds to requiring the in-band to out-band rejection ratio to hold over larger bandwidths, and the cost of requiring this is reflected in larger values of γ_{opt} as well.

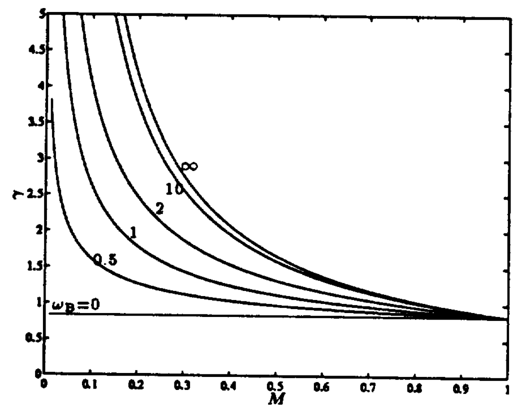

Figure 4: Tradeoff between M and γ for various values of ω_B.

Finally, when $\omega_B \to \infty$, there is no out-band. As with the case $\omega_B = 0$, the condition (16) reduces to the $\mathbf{H}_\infty$-norm condition

$$\sigma_{\max}(H(j\omega)) \le \gamma M \;\text{ for all } \omega \in \mathbf{R}. \qquad (18)$$

Since the smallest achievable $\mathbf{H}_\infty$-norm is γ_0, the tradeoff curve between M and γ is given by $\gamma_{opt} M = \gamma_0$.

4.2 Tradeoff between ω_B and γ

The tradeoff between ω_B and γ for various fixed values of M is shown in figure 5.

We start with the case $M = 1$, which translates to the simple $\mathbf{H}_\infty$-norm condition (17). Therefore, the smallest achievable γ (i.e., γ_{opt}), irrespective of ω_B, is γ_0.

For $M < 1$, increasing ω_B requires an in-band to out-band rejection ratio of M over larger bandwidths, the cost of which is reflected in larger values of γ_{opt}. Similarly, decreasing M leads to an increase in γ_{opt}, for all ω_B. As $\omega_B \to \infty$, the condition (16) reduces to (18), and therefore $\gamma_{opt} \to \gamma_0/M$.

Finally, we note that we may study the tradeoff between ω_B and M as well.

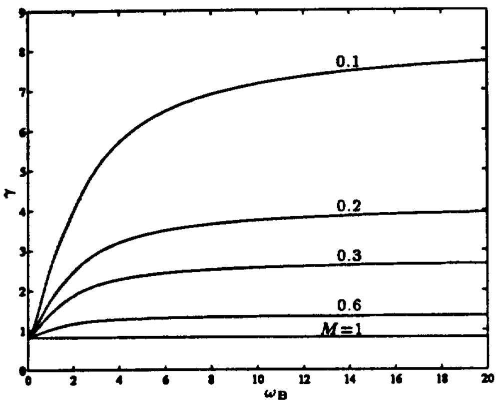

Figure 5: Tradeoff between ω_B and γ for various values of M.

5 Conclusions

We have shown how we may very efficiently plot tradeoff curves for the frequency-weighted $\mathbf{H}_\infty$-control problem, by reducing the question of whether a point lies above or below a tradeoff curve to an eigenvalue calculation. Each tradeoff curve in figures 4 and 5, comprising 100 data points each, took only about 20 seconds to compute, using a Newton search, on a lightly-loaded SUN Sparc 2 workstation. The implication is that least for moderate-size problems, we may interactively study the tradeoffs between the various parameters that comprise the frequency-weighting function. We also note that the results presented here can be extended to more complicated frequency weighting functions in a straightforward manner.

In this paper, we have only concentrated on *achievable performance*; we have not concerned ourselves with *designing controllers* that achieve a given frequency-weighted $\mathbf{H}_\infty$-norm specification. However, from standard results in interpolation theory, an explicit parametrization of all interpolants $\tilde{H}$ that solve the Hermite-Fejér problem (9) is readily available; thus, we may immediately write down an explicit parametrization of all stabilizing controllers that achieve a given frequency-weighted $\mathbf{H}_\infty$-norm specification. We refer the reader, once again, to [11] for details.

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