# A Class of Lyapunov Functionals for Analyzing Hybrid Dynamical Systems

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**Abstract**—In this paper, we introduce a new class of Lyapunov functionals for analyzing hybrid dynamical systems. This class can be thought of as a generalization of the Lyapunov functional introduced by Yakubovich for systems with hysteresis nonlinearities which incorporates path integrals that account for the energy loss or gain every time a hysteresis loop is traversed. Hence, these Lyapunov functionals capture the path-dependence of the "stored energy" in hybrid dynamical systems and are therefore less conservative over previously published approaches in analyzing such systems. More importantly, we show that searching over the proposed class of Lyapunov functionals to prove some specification (*e.g.*, stability) can be cast as a semidefinite program (SDP), which can then be efficiently solved (globally) using widely available software. Examples are presented to show the effectiveness of this class of Lyapunov functionals in analyzing hybrid dynamical systems.

## 1 Introduction

The development and analysis of hybrid systems is an active area of research, both in computer science and in the control community. Roughly speaking, hybrid dynamical systems are systems that incorporate both discrete and continuous dynamics, with the discrete dynamics governed by finite automata and the continuous dynamics represented by ordinary differential equations. The two interact at "event times" determined by the continuous state hitting certain event sets in the continuous state-space. Hybrid systems can model a vast array of important practical systems including piecewise-linear systems, systems with hysteresis or backlash nonlinearities, systems with sliding mode controllers, multi-modal systems, systems with logic, timing circuits, and computer disk drives. For formal definitions of hybrid systems in the control and dynamical systems literature refer to [1, 2].

Hybrid systems can have very complex behavior, even those with very simple continuous dynamics (*e.g.*, a second order linear system with hysteresis feedback can exhibit chaotic behavior [3]). Naturally, it is not surprising that, as in robust control, many problems in hybrid systems are known to be computationally intractable (NP-hard) or even undecidable [4]. Therefore, we do not expect to formulate such problems *exactly* as computationally efficient (polynomial-time) optimization problems. We do expect, however, to develop some *semi-heuristic* methods that are very effective on certain types of problems. By semi-heuristic we mean a method that guarantees its results when it works, but is not guaranteed to work for all input data. Such a method results, for example, when we search over a fixed, finite-dimensional class of Lyapunov functions that guarantee some specification for a given hybrid system—it may not be possible to find such a function, but if one is found, the result is unambiguous. This research is an effort in this direction.

In this paper, we introduce a new class of Lyapunov functionals for analyzing hybrid dynamical systems. This class can be thought of as a generalization of the Lyapunov functional introduced by Yakubovich [5] for systems with hysteresis nonlinearities, which incorporates path integrals that account for the energy loss or gain every time a hysteresis loop is traversed. Moreover, we will show that searching over the proposed class of Lyapunov functionals to prove some specification (*e.g.*, stability) can be cast as a *semidefinite program* (SDP), which can then be efficiently solved (globally) using widely available software (see, *e.g.*, [6, 7, 8]). Previous work in hybrid systems in the context of control theory includes [9, 10, 11, 12].

## 2 Linear hybrid dynamical systems

In what follows, linear hybrid dynamical systems (LHDS) are systems that incorporate both discrete and continuous dynamics, with the discrete dynamics governed by finite automata and the continuous dynamics represented by a set of ordinary linear differential equations at each discrete state $s \in \{1, \dots, N\}$. The two interact at "event times" determined by the continuous state $x \in \mathbf{R}^n$ hitting certain event sets, which are assumed to be hyperplanes in $\mathbf{R}^n$.

More formally, a linear hybrid dynamical system $\mathcal{H}$ is a tuple

$$\mathcal{H} = \Big(\mathbf{R}_+, \{1, \dots, N\}, \mathbf{R}^n, (R_i)_{i=1}^M, \mathcal{I}, (A_i)_{i=1}^N, (b_i)_{i=1}^N\Big) \quad (1)$$

where $\mathbf{R}_+$ is time, $\{1, \dots, N\}$ is the discrete state-space, $\mathbf{R}^n$ is the continuous state-space, $(R_1, \dots, R_M)$ are the invariant regions, $\mathcal{I} : \{1, \dots, N\} \to \{1, \dots, M\}$ is the invariant region function, and $(A_1, \dots, A_N)$ and $(b_1, \dots, b_N)$ are the system matrices. Below is a brief description of each of these and other related concepts used in this paper.

- *Time.* This is the set $\mathbf{R}_+$. Time is denoted by $t$.
- *Discrete state, continuous state, state-space.* The discrete and continuous states at time $t$ are denoted by $s(t)$ and $x(t)$ respectively. For all $t$, $s(t) \in \{1, \dots, N\}$ and $x(t) \in \mathbf{R}^n$. The state-space of $\mathcal{H}$ is the set $\{1, \dots, N\} \times \mathbf{R}^n$.
- *System matrices.* $A_i \in \mathbf{R}^{n\times n}$ and $b_i \in \mathbf{R}^n$ for $i = 1, \dots, N$. These matrices determine the dynamics of $x$ at each discrete state $s = i$ (more below).
- *Invariant regions.* It is assumed that each invariant region $R_i$ is polytopic, *i.e.*, for $i = 1, \dots, M$

  $$R_i = \left\{x \in \mathbf{R}^n \mid h_{ij}^T x < g_{ij},\ j = 1, \dots, J_i\right\}.$$

  Moreover the sets $R_i$ partition $\mathbf{R}^n$ such that

  $$\textstyle\bigcup_{i=1}^M \bar{R}_i = \mathbf{R}^n, \quad R_i \bigcap R_j = \emptyset,\ i \neq j.$$

  ($\bar{R}_i$ denotes the *closure* of $R_i$.)
- *Invariant region for a discrete state.* The function $\mathcal{I} : \{1, \dots, N\} \to \{1, \dots, M\}$ assigns an invariant region to each discrete state (we assume $M \geq N$[1]). In other

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[1]For $M = N$ the LHDS becomes a *piecewise-linear* system.

words, $R_{\mathcal{I}(i)}$ is the invariant region for discrete state $i$. The invariant region sets a condition under which $s(t)$ remains unchanged—if $s(t_0) = i$, as long as $x(t) \in R_{\mathcal{I}(i)}$ for $t \geq t_0$, we have $s(t) = i$ (more below).

- *State dynamics.* The evolution of the continuous state $x$ over time is given by the integral equation

$$x(t) = \int_0^t (A_{s(\tau)}x(\tau) + b_{s(\tau)})\, d\tau + x(0).$$

In other words, at discrete state $s$, the continuous dynamics are governed by the *affine* dynamical equation $\dot{x} = A_s x + b_s$.

The evolution of the discrete state $s$ is as follows. Assuming $s(t) = i$ and $\dot{x}(t) = A_i x(t) + b_i$

$$s(t^+) = \begin{cases} i & \text{if } x(t^+) \in R_{\mathcal{I}(i)}, \\ j & \text{if } x(t^+) \in \bar{R}_{\mathcal{I}(i)} \bigcap \bar{R}_{\mathcal{I}(j)}. \end{cases}$$

Hence, the discrete state $s(t)$ remains unchanged until $x(t)$ hits the boundary of the invariant region for the current discrete state $i$ and another discrete state $j$. At that point $s(t)$ changes to $j$.

- *Trajectory.* Roughly speaking, a trajectory of $\mathcal{H}$ is any function $(s, x) : \mathbf{R}_+ \to \{1, \ldots, N\} \times \mathbf{R}^n$ where $s$ and $x$ satisfy the discrete and continuous state dynamics of $\mathcal{H}$ respectively.
- *Transition diagram.* It is possible to associate to $\mathcal{H}$ a directed graph $\mathcal{G}(\mathcal{H})$ called a transition diagram. In this graph the nodes correspond to the discrete states and the arcs between nodes correspond to an existing boundary between the invariant regions of those nodes or discrete states. The nodes are labeled with the discrete state number and the arcs are labeled with the hyperplane equation of the boundary between invariant regions of those nodes (see example below). Any trajectory of the discrete state corresponds to the sequence of nodes along some forward path in $\mathcal{G}(\mathcal{H})$.
- *Set of transition vectors.* This is the set $\mathcal{T}$ defined as

$$\mathcal{T} = \left\{ c_i \;\middle|\; \begin{array}{l} \text{there is an arc labeled} \\ c_i^T x = d_i \text{ in } \mathcal{G}(\mathcal{H}) \end{array} \right\}, \quad |\mathcal{T}| = a. \quad (2)$$

Note that it is sometimes preferred to label the arcs as $c_i^T x \leq d_i$ (with inequality instead of equality) to show that $c_i^T x > d_i$ before the transition.

The linear system with piecewise-linear hysteresis nonlinearity of Figure 1 is a simple example of an LHDS. The transition diagram is given in Figure 2. It is easy to verify that

$$R_1 = \{x \mid c^T x < 1\},\ R_2 = \{x \mid 1 < c^T x < 2\},$$
$$R_3 = \{x \mid c^T x > 2\},\ \mathcal{I} = \{(1,3), (2,2), (3,1), (4,2)\},$$
$$A_1 = A_2 = A_3 = A_4 = A,\ b_1 = b_4 = b,\ b_2 = b_3 = 0.$$

Other examples include sliding mode controllers [13] (with hysteresis switching to reduce "chattering") and multi-modal systems such as computer disk drives (cf. §6).

Note that in our definition of LHDS we have skipped some details and omitted many subtleties (*e.g.*, uniqueness and existence of trajectories, motion on discontinuity surfaces or sliding modes, etc.). Also it should be mentioned that there is no standard definition for LHDS. For example, in the computer science literature a *linear* hybrid dynamical system is usually referred to a hybrid dynamical system in which, at each discrete state, the continuous *trajectory* is *linear* (or has constant slope). Here, on the other hand, the definition is more general and by *linear* we mean that, at each discrete state, the continuous *dynamics* is *linear.* This definition of LHDS encompasses many systems of practical interest.

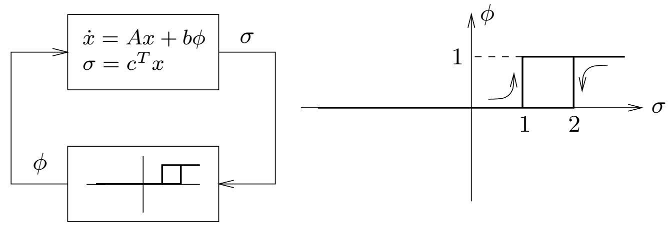


**Figure 1:** Linear system with hysteresis element.

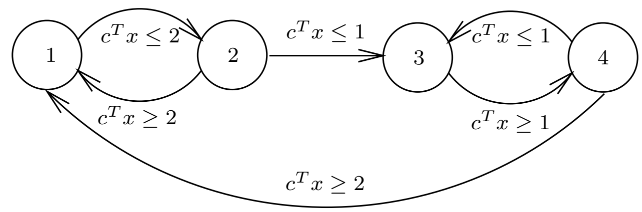


**Figure 2:** Transition diagram $\mathcal{G}(\mathcal{H})$ of LHDS $\mathcal{H}$ of Figure 1.

## 3 Motivation

As seen in the previous section, the feedback interconnection of a linear system with a hysteresis nonlinearity is a hybrid dynamical system. Yakubovich [5] introduced a class of Lyapunov functionals for analyzing such systems when the hysteresis nonlinearity is sector-bounded and passive (roughly meaning that the hysteresis loop is clockwise as in Figure 1). Specifically, for the system

$$\dot{x}(t) = Ax(t) + b\phi(c^T x),$$
$$\sigma(t)(\phi(\sigma) - \sigma(t)) \leq 0, \qquad \int_\Gamma \phi(\sigma)\, d\sigma \geq -\beta \;\text{ for all } \sigma,$$

Yakubovich proposed the candidate Lyapunov functional

$$V(x) = x(t)^T P x(t) + \int_\Gamma \phi(\sigma) d\sigma, \quad P \succ 0, \qquad (3)$$

(parameterized by $P$) where $\Gamma$ is the path traversed in the $(c^T x, \phi)$ plane. The path integral in $V$ captures the energy applied to the hysteresis element, and takes into account the fact that every time a hysteresis loop is traversed, a certain amount of energy (equal to the area of the hysteresis loop) is dissipated. This Lyapunov functional has the usual property of becoming unbounded as $\|x(t)\|$ becomes unbounded. However, the Lyapunov functional can also become unbounded if the hysteresis loop is traversed an infinite number of times, which can easily happen for *bounded* $\|x(t)\|$.

Since the Lyapunov functional $V$ proposed by Yakubovich has proven to be very effective in analyzing systems with hysteresis nonlinearities (see, *e.g.*, [5, 14, 15]), and that such systems are special cases of hybrid dynamical systems, it is natural to believe that any useful class of Lyapunov functionals for general hybrid systems should include $V$ in (3) as a special case.

Piecewise-quadratic Lyapunov functions have already been proposed for analyzing hybrid dynamical systems [11]. However, piecewise-quadratic Lyapunov functions do not possess the property of becoming unbounded for bounded $\|x(t)\|$ and therefore do not include $V$ in (3) as a special case. In fact, as shown in §6, piecewise-quadratic Lyapunov functions can be very conservative in analyzing hybrid systems, because for example, they cannot distinguish between a clockwise or counterclockwise hysteresis nonlinearity. (Piecewise-quadratic Lyapunov functions are very effective for analyzing *piecewise-linear* systems [16] however, since they can be thought of as a generalization of the "quadratic plus integral of nonlinearity" Lyapunov function for the Luré system in robust control [17].)

By using the Lyapunov functional of Yakubovich as a guideline, it is possible to come up with a *numerically computable* class of Lyapunov functionals for analyzing general LHDS effectively. The *key* observation is that going around the hysteresis loop of the system (Figure 1) is equivalent to going around the (1,2,3,4,1) loop of the corresponding transition diagram $\mathcal{G}(\mathcal{H})$ (Figure 2). Each time the loop is traversed, $V$ is increased by a net amount equal to the area of the hysteresis loop (due to the path integral term). Therefore, any Lyapunov functional that generalizes $V$ for LHDS must have this property of increasing by a net amount every time a loop (or forward cycle) in $\mathcal{G}(\mathcal{H})$ is traversed.

## 4 Lagrange stability

The focus of this paper is the analysis of *Lagrange stability* for LHDS. A dynamical system is Lagrange stable if the *continuous state* remains bounded from any initial condition. For example, if the continuous state converges to a *stationary set*, the dynamical system is Lagrange stable. Stable hybrid systems typically converge to stationary sets. For example, hysteresis nonlinearities are multi-valued, so systems with such nonlinearities have a stationary set defined by the intersection of the nonlinearity and the DC value of the linear part (see, *e.g.*, [15]). Piecewise-linear systems, which are another subclass of hybrid systems, can also easily have more than one stationary point [16]. The Lyapunov theorem for Lagrange stability is as follows [18].

**Theorem 1 (Lagrange stability)** *Suppose the functional $V$ maps the trajectories of a dynamical system into $\mathbf{R}$ such that $V(t) \to \infty$ as $\|x(t)\| \to \infty$ ($x$ is the continuous state) and $V$ is bounded below, i.e., there exists $\beta \geq 0$ such that $V \geq -\beta$. $V$ becomes a function of time $t$ along trajectories of the dynamical system. The dynamical system is stable in the Lagrange sense if along every trajectory we have $\frac{d}{dt}V(t) \leq 0$.*

## 5 Lyapunov functional for LHDS

In this subsection we introduce a class of Lyapunov functionals for analyzing LHDS and we show how they can be computed using SDP and basic network optimization.

Consider the candidate Lyapunov functional $V$ such that along any trajectory of $\mathcal{H}$

$$V(t) = x(t)^T P_{s(t)} x(t) + 2q_{s(t)}^T x(t) + r_{s(t)} + 2\sum_{i=1}^{a} \int_0^t \left(\alpha_i(\tau) c_i^T x(\tau) + \beta_i(\tau)\right) c_i^T \dot{x}(\tau)\, d\tau. \quad (4)$$

($c_i$'s and $a$ defined in (2)). Hence, $V$ consists of a piecewise-quadratic term (depending on the discrete state $s$) parameterized by $P_s$, $q_s$, $r_s$ for $s = 1, \dots, N$, and an integral term parameterized by the functions $\alpha_i, \beta_i : \mathbf{R}_+ \to \mathbf{R}$ for $i = 1, \dots, a$ as explained below.

Let

$$S_i = \left\{ (k, m, \sigma) \,\middle|\, \begin{array}{l} \text{node } m \text{ of } \mathcal{G}(\mathcal{H}) \text{ has an incoming} \\ \text{arc labeled } c_i^T x = \sigma \text{ from node } k. \end{array} \right\}$$

and define $\mathcal{P}_1 S_i = \{(k,m) \mid (k,m,\sigma) \in S_i\}$ ($\mathcal{P}_1$ is a projection). $\alpha_i$ and $\beta_i$ are piecewise-constant functions of time satisfying

$$\alpha_i(t^+) = \begin{cases} \alpha_{ikm} & \big(s(t), s(t^+)\big) = (k,m) \in \mathcal{P}_1 S_i, \\ \alpha_i(t) & \text{otherwise}, \end{cases}$$

$$\beta_i(t^+) = \begin{cases} \beta_{ikm}, & \big(s(t), s(t^+)\big) = (k,m) \in \mathcal{P}_1 S_i, \\ \beta_i(t) & \text{otherwise}, \end{cases}$$

where $\alpha_{ikm}, \beta_{ikm} \in \mathbf{R}$. In other words, $\alpha_i$ and $\beta_i$ only change when the discrete state changes by the continuous state hitting a hyperplane $c_i^T x = \text{constant}$. $\alpha_i$ and $\beta_i$ are fully determined once the values for the parameters $\alpha_{ikm}$ and $\beta_{ikm}$ are given for all $(k,m) \in \mathcal{P}_1 S_i$. (Note that $\alpha_i(0)$ and $\beta_i(0)$ can always be consistently chosen to be equal to one of the $\alpha_{ikm}$'s and $\beta_{ikm}$'s, respectively, depending on the value for $s(0)$.)

In what follows, we give conditions on the (design) parameters $P_s$, $q_s$, $r_s$, $\alpha_{ikm}$, $\beta_{ikm}$ for $V$ to be a valid Lyapunov functional. In particular, we show that $\alpha_{ikm}$ and $\beta_{ikm}$ can be chosen such that $V$ is increased by a net amount around loops in $\mathcal{G}(\mathcal{H})$ (as discussed in §3).

### 5.1 Continuity of $V$

For $V$ to be continuous we must have $x^T P_i x + 2q_i^T x + r_i = x^T P_j x + 2q_j^T x + r_j$ for $x \in \bar{R}_{\mathcal{I}(i)} \bigcap \bar{R}_{\mathcal{I}(j)}$ (the integral term in $V$ is already continuous). Since the invariant regions are polytopic, their intersections are hyperplanes, and we can assume that these hyperplanes are parameterized by matrices $F_{\mathcal{I}(i)\mathcal{I}(j)}$ and $l_{\mathcal{I}(i)\mathcal{I}(j)}$ so that $\bar{R}_{\mathcal{I}(i)} \bigcap \bar{R}_{\mathcal{I}(j)} = \{F_{\mathcal{I}(i)\mathcal{I}(j)} z + l_{\mathcal{I}(i)\mathcal{I}(j)} \mid z \in \mathbf{R}^{n-1}\}$. Then $V$ is continuous if and only if

$$\begin{aligned} &F_{\mathcal{I}(i)\mathcal{I}(j)}^T (P_i - P_j) F_{\mathcal{I}(i)\mathcal{I}(j)} = 0, \\ &F_{\mathcal{I}(i)\mathcal{I}(j)}^T (P_i - P_j) l_{\mathcal{I}(i)\mathcal{I}(j)} + F_{\mathcal{I}(i)\mathcal{I}(j)}^T (q_i - q_j) = 0, \\ &l_{\mathcal{I}(i)\mathcal{I}(j)}^T (P_i - P_j) l_{\mathcal{I}(i)\mathcal{I}(j)} + 2(q_i - q_j)^T l_{\mathcal{I}(i)\mathcal{I}(j)} + (r_i - r_j) = 0, \end{aligned} \quad (5)$$

for all $i, j$ such that $\bar{R}_{\mathcal{I}(i)} \bigcap \bar{R}_{\mathcal{I}(j)} \neq \emptyset$. (5) is a set of linear equality constraints in the design parameters $P_s$, $q_s$, and $r_s$.

### 5.2 Positivity of $V$

First we require the quadratic part of $V$ to be positive. This will also guarantee that $V(t) \to \infty$ as $\|x(t)\| \to \infty$. Using the $\mathcal{S}$-procedure, a sufficient condition for this is the existence of $Z_i$ for $i = 1, \dots, N$ such that

$$\begin{bmatrix} P_i - H_{\mathcal{I}(i)}^T Z_i H_{\mathcal{I}(i)} & q_i + H_{\mathcal{I}(i)}^T Z_i g_{\mathcal{I}(i)} \\ q_i^T + g_{\mathcal{I}(i)}^T Z_i H_{\mathcal{I}(i)} & r_i - g_{\mathcal{I}(i)}^T Z_i g_{\mathcal{I}(i)} \end{bmatrix} \succ 0, \quad Z_i = Z_i^T,$$

$$(Z_i)_{k_1 k_2} = 0, \quad (Z_i)_{k_1 k_2} \geq 0, \quad k_1, k_2 = 1, \dots, J_{\mathcal{I}(i)} \quad (6)$$

where

$$H_{\mathcal{I}(i)} = \begin{bmatrix} 0 \\ h_{\mathcal{I}(i)1}^T \\ \vdots \\ h_{\mathcal{I}(i)J_{\mathcal{I}(i)}}^T \end{bmatrix}, \quad g_{\mathcal{I}(i)} = \begin{bmatrix} -1 \\ g_{\mathcal{I}(i)1} \\ \vdots \\ g_{\mathcal{I}(i)J_{\mathcal{I}(i)}} \end{bmatrix}.$$

(6) is a linear matrix inequality (LMI) constraint in the design parameters $P_s$, $q_s$, and $r_s$.

Second we require the integral terms of $V$ to be bounded below. Equivalently, 1) the net amount of the integral terms around any loop in the transition diagram $\mathcal{G}(\mathcal{H})$ should be non-negative, and 2) the integral terms should remain bounded below in any node of $\mathcal{G}(\mathcal{H})$ (since any trajectory of $\mathcal{H}$ corresponds to a simple path and possibly infinite number of loops in $\mathcal{G}(\mathcal{H})$).

Suppose that $\ell_j$ denotes the sequence of nodes in the $j$th loop of $\mathcal{G}(\mathcal{H})$ for $j = 1, \dots, L$. Given $S_i$ and $\ell_j$ it is easy to construct the sequence of nodes $\ell_{ij} = (k_{ij}^{(1)}, m_{ij}^{(1)}, \dots, k_{ij}^{(n_{ij})}, m_{ij}^{(n_{ij})})$ such that $\alpha_i$ and $\beta_i$ remain constant between the nodes $(m_{ij}^{(l)}, k_{ij}^{(l+1)})$ along $\ell_j$. In other words, there is an arc labeled $c_i^T x = \text{constant}$ entering node $m_{ij}^{(l)}$ from node $k_{ij}^{(l)}$ and there is no arc labeled $c_i^T x =$ constant along $\ell_j$ from node $m_{ij}^{(l)}$ to node $k_{ij}^{(l+1)}$ (cf. [19]). If $s(t_l) = m_{ij}^{(l)}$ and $s(t_{l+1}) = m_{ij}^{(l+1)}$ we have

$$\int_{t_l}^{t_{l+1}} \Big(\alpha_i(\tau) c_i^T x(\tau) + \beta_i(\tau)\Big) c_i^T \dot{x}(\tau)\, d\tau =$$
$$\int_{\sigma_{ij}^{(l)}}^{\sigma_{ij}^{(l+1)}} (\alpha_{ik_{ij}^{(l)} m_{ij}^{(l)}} \sigma + \beta_{ik_{ij}^{(l)} m_{ij}^{(l)}})\, d\sigma =$$
$$\tfrac{1}{2}\alpha_{ik_{ij}^{(l)} m_{ij}^{(l)}} \left(\sigma_{ij}^{(l+1)2} - \sigma_{ij}^{(l)2}\right) + \beta_{ik_{ij}^{(l)} m_{ij}^{(l)}} \left(\sigma_{ij}^{(l+1)} - \sigma_{ij}^{(l)}\right).$$

where $(k_{ij}^{(l)}, m_{ij}^{(l)}, \sigma_{ij}^{(l)}) \in S_i$. Therefore, the net amount of the integral terms of $V$ around $\ell_j$ is non-negative if and only if

$$\sum_{i=1}^{a} \sum_{l=1}^{n_{ij}} \Bigg( \frac{1}{2}\alpha_{ik_{ij}^{(l)} m_{ij}^{(l)}} \left(\sigma_{ij}^{(l+1)2} - \sigma_{ij}^{(l)2}\right) + \beta_{ik_{ij}^{(l)} m_{ij}^{(l)}} \left(\sigma_{ij}^{(l+1)} - \sigma_{ij}^{(l)}\right)\Bigg) \geq 0. \tag{7}$$

(Note that we define $\sigma_{ij}^{(n_{ij}+1)} = \sigma_{ij}^{(1)}$ in the summation).

To guarantee that the integral terms remain bounded below in any node $m$, if not all arcs exiting $m$ are labeled $c_i^T x =$ constant we require that

$$\alpha_{ikm} \begin{cases} \geq 0 & \text{if } c_i^T x > \sigma \text{ for } x \in R_{\mathcal{I}(m)}, \\ \leq 0 & \text{if } c_i^T x < \sigma \text{ for } x \in R_{\mathcal{I}(m)}, \end{cases} \tag{8}$$

for all such $(k, m, \sigma) \in S_i$ and $i = 1, \ldots, a$. Roughly, $\alpha_{ikm}$ results in a quadratic term $\alpha_{ikm} x^T c_i c_i^T x$ in $V$ and condition (8) ensures that it can never become $-\infty$ as $\|x(t)\| \to \infty$ for $s(t) = m$. No constraint is required on $\beta_{ikm}$ since it corresponds to a first order term in $x$ and is therefore "overwhelmed" by the quadratic part of $V$ for large $\|x(t)\|$.

(7) and (8) are linear inequality constraints in the design parameters $\alpha_{ikm}$ and $\beta_{ikm}$.

### 5.3 Negativity of $\dot{V}$

For each node $s \in \{1, \ldots, N\}$ define

$$T_s = \left\{ w \;\middle|\; \begin{array}{l} w \text{ is a forward walk of } \mathcal{G}(\mathcal{H}) \text{ ending at} \\ \text{node } s \text{ and traversing all arcs, with at} \\ \text{least one arc traversed only once.} \end{array} \right\}.$$

($T_s$ can be computed using basic network optimization.) Now suppose that for a given $w \in T_s$, $n_{sw}$ is the closest node to $s$ along $w$ satisfying $(k_{sw}, n_{sw}) \in S_i$ for some $k_{sw}$. This means that $\alpha_{ik_{sw}n_{sw}}$ and $\beta_{ik_{sw}n_{sw}}$ are possible values for $\alpha_i$ and $\beta_i$ in $V$ when the discrete state is equal to $s$.

Now using the continuous dynamics equation $\dot{x} = A_s x + b_s$ at discrete state $s$, it is possible to find expressions for $\dot{V}$ at discrete state $s$ depending on the possible values for $\alpha_i$ and $\beta_i$. It can be shown that $\dot{V} \leq 0$ if (cf. [19])

$$\begin{bmatrix} \begin{pmatrix} A_s^T P_s + P_s A_s \\ +A_s^T \sum_i \alpha_{ik_{sw}n_{sw}} c_i c_i^T \\ +\sum_i \alpha_{ik_{sw}n_{sw}} c_i c_i^T A_s \\ +H_{\mathcal{I}(s)}^T \Lambda_s H_{\mathcal{I}(s)} \end{pmatrix} & \begin{pmatrix} P_s b_s + A_s^T q_s \\ +\sum_i \alpha_{ik_{sw}n_{sw}} c_i c_i^T b_s \\ +\sum_i \beta_{ik_{sw}n_{sw}} A_s^T c_i \\ -H_{\mathcal{I}(s)}^T \Lambda_s g_{\mathcal{I}(s)} \end{pmatrix} \\ \begin{pmatrix} b_s^T P_s + q_s^T A_s \\ +\sum_i \alpha_{ik_{sw}n_{sw}} b_s^T c_i c_i^T \\ +\sum_i \beta_{ik_{sw}n_{sw}} c_i^T A_s \\ -g_{\mathcal{I}(s)}^T \Lambda_s H_{\mathcal{I}(s)} \end{pmatrix} & \begin{pmatrix} 2q_s^T b_s + g_{\mathcal{I}(s)}^T \Lambda_s \\ 2\sum_i \beta_{ik_{sw}n_{sw}} c_i^T b_s \end{pmatrix} \end{bmatrix} \preceq 0,$$

$$\Lambda_s = \Lambda_s^T, \quad (\Lambda_s)_{k_1k_2} = 0, \quad (\Lambda_s)_{k_1k_2} \geq 0, \; k_1, k_2 = 1, \ldots, J_s, \tag{9}$$

for $s = 1, \ldots, N$ and $w \in T_s$. (9) is an LMI in the variables $P_s$, $q_s$, $r_s$, $\alpha_{ikm}$, and $\beta_{ikm}$.

### 5.4 Summary and remarks

To summarize, (5)-(9) are the conditions for $V$ in (4) to be a Lyapunov functional that proves Lagrange stability for $\mathcal{H}$ according to Theorem 1. These conditions are linear equality, linear inequality, or linear matrix inequality constraints in the design parameters $P_s$, $q_s$, $r_s$, $\alpha_{ikm}$, and $\beta_{ikm}$. Therefore, a feasible $V$, if any, can be computed efficiently using SDP and basic network optimization [20] (to compute the sets $S_i$ and $T_s$). If $\mathcal{H}$ is given by a labeled transition diagram instead of the tuple (1), the invariant regions $R_i$ and invariant region function $\mathcal{I}$ can be computed using linear programming (cf. [19]). Note that when discrete states and transitions (events) have hierarchy, the sets $R_i$, $S_i$, and $T_s$ can be computed much easier, often by inspection (cf. §6.2). Hierarchical organization of states such that states can exist within other states produces neat, manageable diagrams, and are encouraged whenever possible.

In this paper we have assumed that $\alpha_i$ and $\beta_i$ only change when $x$ hits a $c_i^T x =$const. hyperplane and are functions of the previous and current discrete state. It is possible to consider more general cases in which, for example, $\alpha_i$ and $\beta_i$ are functions of the previous, current and next states or loops in the transition diagram (corresponding to a non-causal $V$). In many practical cases, however, it may not be necessary to assume this extra degree of freedom in choosing $\alpha_i$ and $\beta_i$. To make things even simpler we may assume $P_s = P$, $q_s = q$, and $r_s = r$ to be constant throughout the state-space, which results in a significant reduction in the number of variables in the SDP for computing $V$, and hopefully would not sacrifice much performance (cf. §6).

The equilibrium points of $\mathcal{H}$ correspond to the stationary points of the Lyapunov functional $V$. Hence if $x_{\mathrm{eq},s}$ is an equilibrium point of $\mathcal{H}$ at discrete state $s$ we have

$$(P_s + \textstyle\sum_i \alpha_{iks} c_i c_i^T) x_{\mathrm{eq},s} + (q_s + \sum_i \beta_{iks} c_i) = 0.$$

If a stationary point of $V$ is a minimum, maximum, or saddle point, the corresponding equilibrium point will be a stable node, unstable node, or saddle point respectively.

To see how the class of Lyapunov functionals introduced generalizes the Lyapunov functional of Yakubovich (3) for (piecewise-linear) hysteresis nonlinearities consider the system of Figure 1. The Lyapunov functional $V$ in (4) coincides with that of Yakubovich with the values

$$a = 1, \quad c_1 = c, \quad P_s = P, \quad q_s = 0, \quad r_s = 0,$$
$$\beta_{121} = \beta_{134} = \beta_{141} = 1/2, \quad \beta_{112} = \beta_{123} = \beta_{134} = 0.$$

Finally, note that the proposed class of Lyapunov functionals includes piecewise-quadratic Lyapunov functions as a special case, and as shown in §6.1 it performs strictly better than piecewise-quadratic Lyapunov functions.

## 6 Examples

### 6.1 Linear system with hysteresis element

Consider the linear system with hysteresis of Figure 1 and Figure 2 with

$$A = \begin{bmatrix} -0.1 & -1 \\ 0 & -0.2 \end{bmatrix}, \quad b = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The goal is to construct a Lyapunov functional to prove stability for this system using the method of this paper.

We consider the Lyapunov functional

$$V(x) = x(t)^T P x(t) + 2\int_0^t \Big(\alpha(\tau) c^T x(\tau) + \beta(\tau)\Big) c^T \dot{x}(\tau)\, d\tau$$

as described in §5 (here $P \in \mathbf{SR}^{2\times 2}$ is assumed to be constant). Conditions (7) and (8) become

$$1.5(\alpha_{34} - \alpha_{12}) + \beta_{34} - \beta_{12} \geq 0,$$
$$\alpha_{23} \leq 0, \quad \alpha_{43} \leq 0, \quad \alpha_{41} \geq 0, \quad \alpha_{21} \geq 0,$$

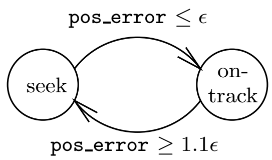


**Figure 3:** Transition diagram for disk drive controller.

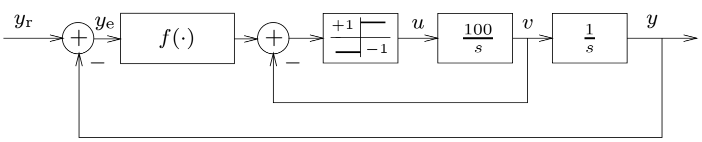


**Figure 4:** Time-optimal controller for double integrator plant.

These with $P \succ 0$ and the LMI (9) are conditions for stability of the system using the Lyapunov functional given above. Solving the corresponding SDP for a $P$ with minimum condition number (using the SDP parser/solver `sdpsol` [7]) we get

$$\alpha_{43} = \alpha_{23} = -0.674, \quad \alpha_{34} = -0.242, \quad \alpha_{12} = -0.521,$$
$$\alpha_{41} = \alpha_{21} = 0, \quad \beta_{43} = \beta_{23} = 0, \quad \beta_{34} = 11.614,$$
$$\beta_{12} = 0.0141, \quad \beta_{41} = 10.397, \quad \beta_{21} = 10.397,$$
$$P = \begin{bmatrix} 1.137 & -1.264 \\ -1.264 & 12.636 \end{bmatrix}.$$

It should be noted that a piecewise-quadratic Lyapunov function cannot prove stability for this example (the resulting SDP is infeasible). This is not unexpected of course, because a piecewise-quadratic Lyapunov function cannot distinguish between a clockwise (passive) or counter-clockwise (active) hysteresis nonlinearity since it does not take into account the direction of loops in the transition diagram. Therefore, piecewise-quadratic Lyapunov functions are very conservative in dealing with such problems.

### 6.2 Simplified disk drive controller

In this section we briefly sketch how the method of this paper can be used to prove stability for a simplified disk drive controller.

In our simplified version, the disk drive controller operates in two modes. In *seek* mode, the controller needs to bring the head to (the vicinity of) any desired position or track as fast as possible. In *on-track* mode, the controller needs to maintain some level of disturbance rejection for read/write at that desired position or track. This is shown by the transition diagram in Figure 3 where the controller switches to on-track mode whenever the head position error is less than $\epsilon$ and switches back whenever the position error exceeds $1.1\epsilon$.

A very simple model for the dynamics of a disk drive head is the double integrator plant with bounded input

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} u, \quad |u(t)| \le 1.$$

The ("bang-bang") control law that steers the state to $[y_{\rm r}\ 0]^T$ in minimum time is shown in Figure 4 where $f(y_{\rm e}) = \mathrm{sgn}(y_{\rm e})\sqrt{200|y_{\rm e}|}$ (see, *e.g.*, [21]). This control law, although time-optimal, can cause difficulties due to "chattering" and robustness issues. Even the smallest system process or measurement noise will cause the control to chatter between the maximum and minimum values resulting in a very wide-band controller. In practice, one could replace the hard-limiter by a hysteresis or saturation to reduce the chatter. Figure 5 shows

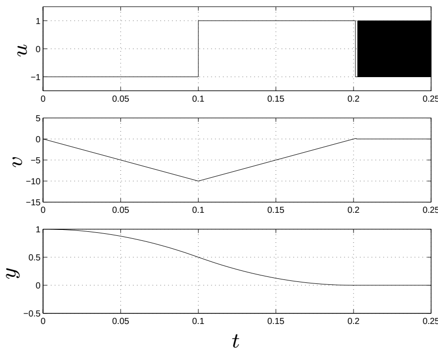


**Figure 5:** Control input and state trajectories for time-optimal controller when state is steered from $[1\ 0]^T$ to $[0\ 0]^T$.

the control input $u$, and state trajectories $v$ and $y$ when the state is steered from $[1\ 0]^T$ to $[0\ 0]^T$. Note the significant amount of chatter in $u$ for $t \ge 0.2$ when the state is in the vicinity of $[1\ 0]^T$.

For the simplified disk drive controller of this example, we assume that in seek mode the control law is similar to that of Figure 4 but with $f(y_{\rm e}) = 16.9 y_{\rm e}$ (which is a linear approximation to $f(y_{\rm e}) = \mathrm{sgn}(y_{\rm e})\sqrt{200|y_{\rm e}|}$) and the hard-limiter replaced by a hysteresis switch of width $2\delta = 0.2$. In on-track mode, we assume that the control law is $u = \mathrm{sat}(Kx)$ where $K = [-1\ \ -1.4]$ ($K$ is designed using an LQR approach to reject sensor noise). The transition diagram of the system in seek and on-track modes are shown in Figure 6. We also assume that $\epsilon = 0.1$ in Figure 3. A typical trajectory of this controller is shown in Figure 7. Note that for $t \ge 0.2$ the position of the head $y$ lies in the acceptable on-track region $|y - y_{\rm r}| < \epsilon$. However, it is not clear how the system will perform for other arbitrary initial conditions. In particular, it is necessary to verify stability for this system.

Figures 3 and 6 give a two-level hierarchical description for the transition diagram of the LHDS representing the disk drive controller which has $N = 25$ discrete states. At the superlevel there are the superstates 'seek' and 'on-track' (Figure 3). At the sublevel there are 2 and 3 substates within the superststates 'seek' and 'on-track' respectively (Figure 6). Due to the hierarchical structure of the design it is easy to read out the sets $R_i$, $S_i$, $T_i$, and system matrices $A_i$ and $b_i$.

For example, the invariant regions $R_i$ are easily found by intersecting the invariance regions for the superstates and the substates within the superstates. The superlevel defines the transition vector $c_1 = [1\ 0]^T$ (since transitions occur when $x_1 =$ const. hyperplanes are hit), and the sublevel defines the transition vectors $c_2 = [0\ 1]^T$ and $c_3 = K^T$ (since transitions occur when $x_2 =$ const. and $Kx =$ const. hyperplanes are hit). Hence $\alpha_1$, $\beta_1$ only change when the superstates change, and $\alpha_{2,3}$, $\beta_{2,3}$ only change when the corresponding substates in each superstate change. Therefore it is easy to construct $S_i$ for $i = 1, 2, 3$. The hierarchical structure of the controller defines 4 loops around which the net value of the integral terms should be non-negative. One loop is at the superlevel ('seek', 'on-track', 'seek') and three are at the sublevel (1,2,1 in 'seek', and 1,2,1 and 2,3,2 in 'on-track'). $T_i$, $A_i$, and $b_i$ can also be found easily using the hierarchical structure of the controller. For details refer to [19].

Once the sets $R_i$, $S_i$, $T_i$, and system matrices $A_i$ and $b_i$ are found, we can search for a Lyapunov functional $V$ over the class of Lyapunov functionals (4) using SDP to prove stability of the disk drive system. In fact we were able to find such a Lyapunov functional so the system is proved stable. Note that

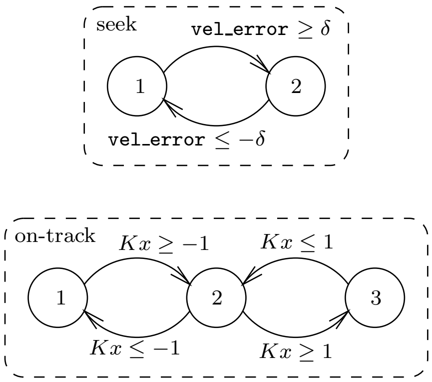


**Figure 6:** Transition diagram in *seek* and *on-track* modes.

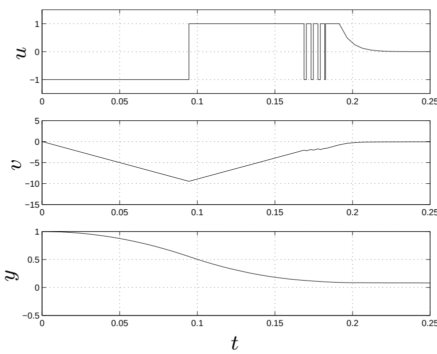


**Figure 7:** Control input and state trajectories for simplified disk drive controller when state is steered from $[1\ 0]^T$ to $[0\ 0]^T$.

the quadratic part of $V$ was assumed to be a simple constant quadratic $x^T P x$ with $P \succ 0$, so the continuity and positivity equations (5) and (6) were not needed in the optimization.

A more realistic disk drive controller would include more discrete states at the superlevel, an observer to estimate $x$, higher order modes in the head dynamics, and stiction. It is still straightforward to analyze such a controller using the method of this paper as long as the nonlinearities (stiction in this case) are approximated by piecewise-linear functions.

## 7 Conclusions and further remarks

In this paper we introduced a class of Lyapunov functionals for analyzing linear hybrid dynamical systems which can be thought of as a generalization of the Lyapunov functional of Yakubovich for analyzing linear systems with hysteresis nonlinearities. This class of Lyapunov functionals is very effective in analyzing linear hybrid dynamical systems as demonstrated in §6, and performs strictly better than piecewise-quadratic Lyapunov functions. These Lyapunov functionals can be computed efficiently by solving semidefinite programs and basic network optimization.

This paper only concentrated on the question of stability in the Lagrange sense. Lagrange stability is more relevant from a practical point of view for linear hybrid dynamical systems that can easily have more than one equilibrium point. Using dissipation theory and adding inputs/outputs to the system it is straightforward to analyze performance measures other than stability. Computing reachable and invariant sets is also possible.

The ideas presented here for analyzing linear hybrid dynamical systems can also be generalized to handle *nondeterministic* transitions. In such cases, for example, the transition label $1 \geq c_i^T x \geq 1 + \delta$ means that the transition can happen nondeterministically for any $x$ satisfying $1 \geq c_i^T x \geq 1 + \delta$. Nondeterministic transitions are very useful for modeling uncertainty in a hybrid dynamical system.

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