A KYP Lemma and Invariance Principle for Systems with Multiple Hysteresis Nonlinearities*

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Abstract

Absolute stability criteria for systems with multiple hysteresis nonlinearities are given in this paper. It is shown that the stability guarantee is achieved with a simple two part test on the linear subsystem. If the linear subsystem satisfies a particular linear matrix inequality and a simple residue condition, then, as is proven, the nonlinear system will be asymptotically stable. The main stability theorem is developed using a combination of passivity, Lyapunov, and Popov stability theories to show that the state describing the linear system dynamics must converge to an equilibrium position of the nonlinear closed loop system. The invariant sets that contain all such possible equilibrium points are described in detail for several common types of hystereses. The class of nonlinearities covered by the analysis is very general and includes multiple slope-restricted memoryless nonlinearities as a special case. Simple numerical examples are used to demonstrate the effectiveness of the new analysis in comparison to other recent results, and graphically illustrate state asymptotic stability.

Keywords: Hysteresis, slope-restricted nonlinearity, passivity, Lyapunov functions, Popov indirect control form, linear matrix inequalities, asymptotic stability.

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1 Introduction

The Popov stability criteria [Popov, 1961] has long been the standard analytical tool for systems having memoryless, sector bounded nonlinearities. Details of Popov's analytical approach can be found in the standard texts by Desoer, Vidyasagar and Khalil [Desoer and Vidyasagar, 1975, Vidyasagar, 1993, Khalil, 1996]. When nonlinearities, in addition to being sector bounded, are also monotonic and slope restricted, Zames and Falb [Zames and Falb, 1968] proved that the Popov analysis can be further sharpened by employing a more general type of multiplier, often called the Zames-Falb multiplier. Subsequently, Cho and Narendra [Cho and Narendra, 1968] found that the existence of such multipliers could be established with an off-axis circle test in the Nyquist plane. While this early work was limited to a scalar nonlinearity, an extension by Safonov [Safonov, 1984] considered multiple nonlinearities and established criteria through loop shifting and diagonal frequency dependent matrix multipliers, as is now common in the μ/K_m-analysis approach, introduced by Doyle and Safonov [Doyle, 1982, Safonov, 1982]. An alternate approach for the slope restricted case pursued by Singh [Singh, 1984] and Rasvan [Rasvan, 1988] utilized a multiplier first introduced by Yakubovich [Yakubovich, 1965] for systems with differentiable nonlinearities. Although not as general as the Zames-Falb multiplier, the simple form of the Yakubovich multiplier makes it a valuable complement to the Popov analysis. More recently, Haddad and Kapila [Haddad and Kapila, 1995], and Park [Park et al., 1998] have attempted to generalize the results in [Singh, 1984, Rasvan, 1988] to the case of multiple slope restricted nonlinearities. The resulting criteria offered, however, restrict the value of the linear system transfer matrix, $G(s)$, in a variety of ways. In both papers, for instance, the systems are restricted to be strictly proper (*i.e.*, the feedthrough term $D = 0$). Also, in [Haddad and Kapila, 1995], the value of the system matrix at $s = 0$, $G(0)$ must be either nonsingular or identically zero, while in [Park et al., 1998] the stability guarantee requires that $G(0) = G(0)^T > 0$. In this paper we generalize the analysis for multiple nonlinearities in several ways. First we provide the extension to non-strictly proper systems $D \neq 0$ and relax the positivity requirement to $G(0) = G(0)^T > -M^{-1}$, where $M > 0$ is the diagonal matrix of the maximum slopes occurring in the vector of nonlinearities. More importantly, we show that the same analysis that applies to the slope restricted case is valid for a class of multiple hysteresis nonlinearities as well. This is a rather significant generalization since hysteresis is not sector bounded and has memory, and thus is functionally very different than a memoryless, slope restricted nonlinearity. With this result we, in effect, generalize the early scalar hysteresis analysis by Yakubovich and Barabanov [Yakubovich, 1967, Barabanov and Yakubovich, 1979] and more recent LMI analysis by the authors [Paré and How, 1998b, Paré and How, 1998a], to the case of multiple hysteresis nonlinearities.

Using an approach similar to Park [Park et al., 1998], we present a linear matrix inequality which, if feasible in a set of free matrix variables, will prove the asymptotic stability of the system. For the slope restricted nonlinearity, asymptotic stability means the state converges to the origin, which is assumed to be the unique equilibrium point of the nonlinear system. Since a typical hysteresis is in general multivalued, convergence is not to a single point, but rather to an *stationary set*, defined by the intersection of the nonlinearity and the DC value of the system matrix. We define these sets explicitly for some commonly occurring types of hystereses. In contrast to the previous work of Haddad, Kapila [Haddad and Kapila, 1995] and Park [Park et al., 1998], our Lyapunov function will be a function of the system state, and not its time derivative. This difference results in a more straightforward conclusion of asymptotic stability.

1.1 Approach Overview

The original general form of Popov's stability criterion [Popov, 1961] requires the linear portion of the system to be stable and strictly proper. However, the general form does allow for a single pure integrator in the system. This is sometimes referred to as the *indirect form* or the *indirect control form* of Popov's criteria (see texts [Aizerman and Gantmacher, 1964, Narendra and Taylor, 1973, Vidyasagar, 1993] for scalar versions), and it commonly has associated with it a three term Lyapunov function. In this paper we will extend this form to the vector case using, as a guide, the procedure of Narendra and Taylor [Narendra and Taylor, 1973, p. 100] for the single nonlinearity, which we summarize in three simple steps. First, we apply a loop transformation that changes the slope sector bounds, differentiates the output of the nonlinearity, and results in an integrator state in the transformed linear subsystem, $\tilde{G}(s)$. Provided the original linear subsystem $G(s)$ is stable, $\tilde{G}(s)$ is then cast in Popov's indirect control form. Secondly, we form a three part Lyapunov functional, $V(t)$ that is quadratic in the state of $\tilde{G}(s)$ and includes a particular integral of the nonlinearity. When the nonlinearity is a hysteresis, having memory, the value of the integral is *path dependent*; while in the memoryless case, it is not. Lastly, the requirement that $\dot{V} \leq 0$ is enforced by the existence of a certain LMI, and, subsequently, this condition is used to conclude asymptotic stability of certain stationary sets.

The outline of the paper is as follows. First, we characterize the class of nonlinearities in the next section, and in particular, limit the hysteresis class to multi-valued functions having an input-output relationship with characteristic loops that circulate in a strict direction. Following that, in §3, the nonlinear system is defined and the loop transform used for the analysis is given. The stationary, or equilibrium sets, for the various nonlinear systems are in general polytopic regions of state space, and are detailed in §4. This leads directly to the main stability theorem, which is proved in §5. Frequency domain and passivity interpretations of the Lyapunov result are discussed in §6. Simple numerical examples are then presented in §7 which confirm the benefits of our approach with respect to prior stability criteria and give a graphical illustration of the asymptotic stability to the stationary sets.

2 Nonlinearities and Sector Transformations

2.1 Memoryless, Slope Restricted

Following the definition given by Haddad and Kapila [Haddad and Kapila, 1995], we define the class of nonlinearities as

$$\Phi = \left\{ \phi : R^m \to R^m \;\middle|\; \begin{array}{l} \phi(y) = [\phi_1(y_1), \ldots, \phi_m(y_m)]^T \\ \phi \text{ is differentiable a.e. } \in R^m \\ 0 \le \phi_i' < \mu_i, \quad i = 1, \ldots, m \\ \phi(0) = 0 \end{array} \right\} \tag{1}$$

The set Φ consists of m decoupled scalar nonlinearities, with each scalar component locally slope sector bounded obeying the slope restriction:

$$0 \le \frac{\phi_i(y_i^a) - \phi_i(y_i^b)}{y_i^a - y_i^b} \le \mu_i, \tag{2}$$

for any y_i^a, $y_i^b \in R$. This sector property is sometimes denoted as $\phi_i' \in \text{sector}[0, \mu_i)$, or given the discrete representation [Narendra and Taylor, 1973]:

$$\Delta\phi_i(y_i)/\Delta y_i \in \text{sector}[0, \mu_i). \tag{3}$$

The slope restriction (3) on a function is a stronger than the standard sector bound condition on a function. This idea is formalized with the following proposition.

Proposition 2.1 *(Sector Bound Property) A function $\phi_i : R \to R$ satisfying the conditions $\phi(0) = 0$ and (3) is necessarily sector bounded, with the same bounds. That is, $\phi_i \in sector[0, \mu_i)$.*

Proof: Simply set $y_i^b = 0$ in (2) and multiply through by $(y_i^a)^2$ to get the relation:

$$0 \leq \phi_i(y_i^a)y_i^a < \mu_i(y_i^a)^2,$$

and thus $\phi_i \in \text{sector}[0, \mu_i)$, which is the standard sector bound condition on ϕ_i. □

Using the approach of [Narendra and Taylor, 1973] and [Zames and Falb, 1968], we note that a nonlinearity with local slope confined to a finite sector can be converted to a nonlinearity with infinite sector width. The transformation requires a positive feedback around the nonlinearity, as depicted in Figure 1.

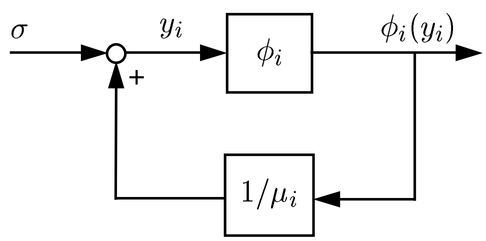


Fig. 1: Sector Transformation $\tilde{\Phi} \in \text{sector}[0, \infty)$

Lemma 2.2 *(Finite/Infinite Sector Transform) A slope restricted function $\phi_i : R \to R$ with $\Delta\phi_i(y_i)/\Delta y_i \in sector[0, \mu_i)$ under positive feedback with gain $1/\mu_i$, as depicted in Figure 1, is is converted to a nonlinearity $\tilde{\phi}_i : R \to R$ with the infinite slope bounds satisfying $\Delta\tilde{\phi}_i(\sigma)/\Delta\sigma \in sector[0, \infty)$.*

Proof: See [Narendra and Taylor, 1973, pp. 108–109][1]. □

A consequence of Lemma 2.2 is that the scalar slope functions are nonnegative:

$$0 \leq \tilde{\phi}_i'(\sigma) < \infty, \tag{4}$$

[1]Note that the sector is half-open, and essentially does not include infinity. More precisely, the transformation should have positive feedback of $1/(\mu - \epsilon)$, where $0 < \epsilon \ll \mu$. This is the approach taken in Ref. [Zames and Falb, 1968], and likewise, we assume this adjustment is included in the sector transform, but for simplicity this will not be expressed explicitly.

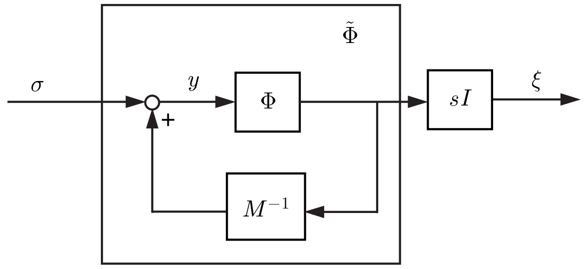


Fig. 2: Sector Transformation $\tilde{\Phi} \in \text{sector}[0, \infty)$

which is equivalent to the sector condition between the time derivatives of the input-output pair:

$$0 \leq \dot{\tilde{\phi}}_i \dot{\sigma} < \infty. \tag{5}$$

Returning to the vector case, we now apply the same sector transform to each scalar component of ϕ and define a new operator by differentiating the vector output, as depicted in Figure 2, where $M = \text{diag}(\mu_1, \ldots, \mu_m) > 0$ is the diagonal matrix of maximum slopes occurring in ϕ. The input-output relation from σ to ξ, as defined in Figure 2, is passive, as detailed by the following lemma.

Lemma 2.3 *(Passive Operator) Consider a slope restricted nonlinearity $\tilde{\Phi} : R^m \to R^m$ with decoupled scalar components satisfying $0 \leq \tilde{\phi}_i'(\sigma) < \infty$. Then the input-output relation defined with $\sigma(t)$ as the input to $\tilde{\Phi}$ and output $\xi(t) = \frac{d}{dt}\tilde{\Phi}(\sigma)$, the time derivative of $\tilde{\Phi}(\sigma)$ (as depicted in Figure 2) is passive.*

Proof: For all $T \geq 0$ we have

$$\begin{aligned}
\int_0^T \sigma^T \xi \, dt &= \sum_{i=0}^{m} \int_0^T \sigma_i \xi_i \, dt && \text{(6a)} \\
&= \sum_{i=0}^{m} \int_0^T \sigma_i \frac{d}{dt} \tilde{\phi}_i(\sigma_i) \, dt && \text{(6b)} \\
&= \sum_{i=0}^{m} \int_0^T \sigma_i \tilde{\phi}_i'(\sigma_i) \dot{\sigma}_i \, dt && \text{(6c)} \\
&= \sum_{i=0}^{m} \int_{\sigma_i(0)}^{\sigma_i(T)} \sigma_i \tilde{\phi}_i'(\sigma_i) \, d\sigma_i(t) && \text{(6d)} \\
&= \sum_{i=0}^{m} \left\{ -\int_0^{\sigma_i(0)} \sigma_i \tilde{\phi}_i'(\sigma_i) \, d\sigma_i(t) + \int_0^{\sigma_i(T)} \sigma_i \tilde{\phi}_i'(\sigma_i) \, d\sigma_i(t) \right\} && \text{(6e)} \\
&\geq -\beta(\sigma(0)) && \text{(6f)}
\end{aligned}$$

where

$$\beta(\sigma(0)) = \sum_{i=0}^{m} \int_0^{\sigma_i(0)} \sigma_i \tilde{\phi}_i'(\sigma_i)\, d\sigma_i(t) \geq 0, \tag{7}$$

since each scalar kernel, $k_i(\sigma_i) = \sigma_i \tilde{\phi}_i'(\sigma_i)$, is a memoryless, sector bounded function, with $k_i \in$ sector$[0, \infty)$. Therefore, the input-output relation is passive, by the definition given in reference [Desoer and Vidyasagar, 1975, p. 173]. □

Having now defined the passive transformation for the memoryless class of slope restricted nonlinearities, we consider the hysteresis case. In the next section we describe the properties of the hysteresis class and show the very same transformation used for the memoryless case will also convert a vector hysteresis into a passive operator.

2.2 Hysteresis

Hysteresis is a property of a wide range of physical systems and devices, such as electro-magnetic fields, mechanical stress-strain elements, and electronic relay circuits. The term *hysteresis* typically refers to the input-output relation between two time-dependent quantities that can not be expressed as a single-valued function. Instead, the relationship usually takes the form of loops that are traversed either in a *clockwise* or *counter-clockwise* direction. A hysteresis with counter-clockwise loops is sometimes referred to as a *passive hysteresis* (see [Hsu and Meyer, 1968, p. 366], for example). In general, the output at any given time is a function of the entire past history of the input, and thus unlike the preceding case, hysteresis nonlinearities have memory. The memory and loop characteristics of hysteresis complicate the analysis to some extent, especially since in practice hysteresis loops can take many forms [Brokate and Sprekels, 1996]. To simplify matters in this section, we assume some additional hysteresis characteristics and thus limit the scope of nonlinearities we consider. The class we define, however, still includes many models that occur in practice, such as the hysteretic relay, backlash, and Preisach hysteresis [Mayergoyz, 1991], which are depicted in Figures 3, 4 and 5, respectively. A characteristic common among these nonlinearities is counter-clockwise circulation of the input-output relation[2]. In the next section, the assumed characteristics of the scalar nonlinearities are detailed, and an example using backlash is given to illustrate the application of the properties. Following that, the vector class of multiple hysteresis nonlinearities is defined using the scalar properties.

[2]While counter-clockwise circulation is an assumed property of the class, it is possible to include clockwise behaviour by employing a coordinate transformation that effectively reverses the circulation, as discussed in [Hsu and Meyer, 1968, p. 366].

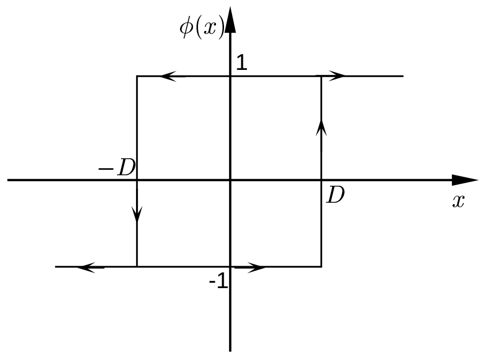


Fig. 3: Relay: switch width = $2D$.

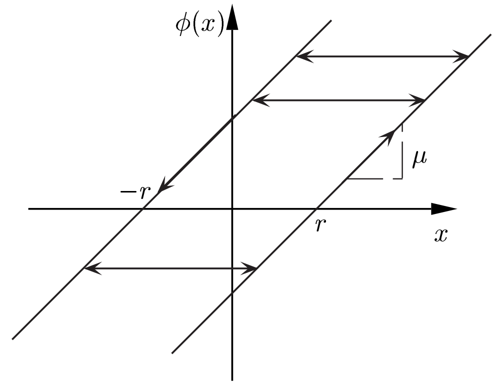


Fig. 4: Backlash: deadzone width = $2r$.

2.2.1 Properties of Hysteresis

Prop. 1 *Non-local memory.* Unlike memoryless nonlinearities, hysteresis output at any given time is a function of the entire history of the input, and the initial condition of the output, ϕ_0. So we define the output $w(t)$ as

$$w(t) = \phi(\phi_0, x([0,t])) \quad (8)$$

$$= \Phi[x, \phi_0](t) \quad (9)$$

At times we will drop the dependence on ϕ_0 for simplicity.

Prop. 2 *Causality, time invariance and rate independence.* The hystereses considered are causal and time-invariant operators, as given by the standard definitions [Desoer and Vidyasagar, 1975]. They are also rate-independent, which essentially means that the input-output relation, as depicted on a graph such as Figure 5, is unchanged for an arbitrary time scaling of the input function. For instance, the input-output relation describe by the relation (x, y) is invariant for changes of the input rate, such as changes in the frequency of cycling. This assumption precludes rate-dependent hysteresis such as the Chua-Stromsmoe model, considered by Safonov and Karimlou [Safonov and Karimlou, 1983], for which the local slope varies with the frequency of the input signal.

Prop. 3 *Counterclockwise circulation.* Closed loops that occur on the input-output characteristic are strictly counterclockwise. That is, a periodic input $x(t)$, with period $T > 0$, will result in a

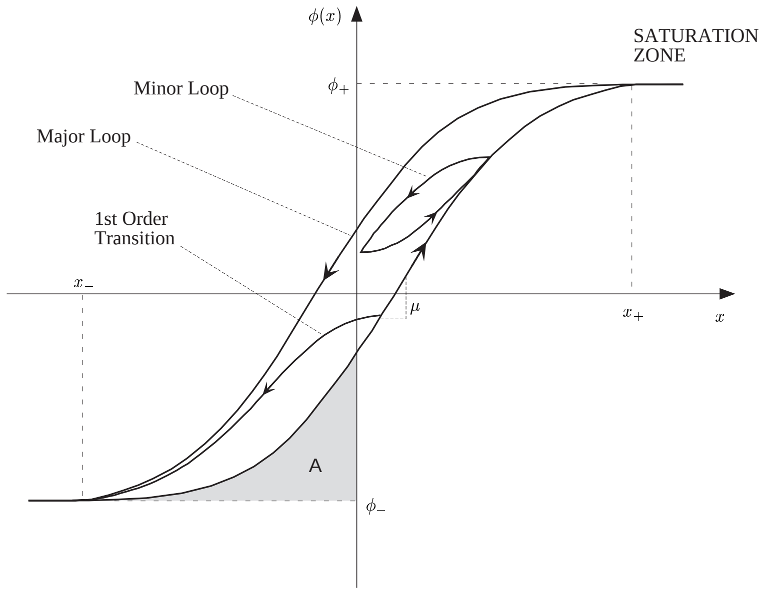


Fig. 5: Typical Preisach hysteresis characteristic.

closed curve relation

$$\int_0^T x(s)\Phi[x]'(s)ds = \int_t^{t+T} x(s)w'(s)\,ds = \oint_{w(t)}^{w(t+T)} x(s)\,dw(s) \geq 0, \tag{10}$$

with equality achieved, for the backlash example, when $x(t)$ remains in the backlash deadzone. The value of the integral (10), when the path is closed, is equal to the area enclosed by the hysteresis loop. For partial, unclosed loops, the integral represents the area between the path traversed and the hysteresis output axis (cf. ϕ-axis in Figs. 3–5).

Prop. 4 *Positive Path Integral.* Let Ψ be the intersection of the output ϕ-axis and the hysteresis characteristic curves[3]. Any input-output path $\mathbf{\Gamma} = \{(x(t), w(t)) \,|\, t \in [0,T]\}$, originating in Ψ, the path integral $\int_\Gamma x\,dw$ is non-negative. That is, if $x(t), t \in [0,T]$ with $x(0) = 0$ generates the path $\mathbf{\Gamma}$, joining points $p \in \Psi$ and some arbitrary b, we have

$$\int_0^T x(s)\Phi[x]'(s)\,ds = \int_0^T x(s)w'(s)\,ds = \int_{\Gamma_{p\to b}} x\,dw \geq 0. \tag{11}$$

[3]For the unit relay, Fig. 3, this set consists of two points: $\Psi = \{(0,1),(0,-1)\}$, for the backlash and Preisach models, Ψ is the corresponding line segment on the ϕ-axis.

Similarly, now let $\mathbf{\Gamma}$ denote the path joining any two points on the hysteresis graph, and note that this path may involve many complete cycles, as in (10) above. Let $\mathbf{\Gamma}_{ab}$ denote the shortest path joining the two points a and b, not containing any complete cycles. Assuming $\mathbf{\Gamma}$ results from input $x(t), t \in [0,T]$ and taking a third point $p \in \Psi$, we have that

$$\begin{aligned} \int_0^T x\Phi[x]'(t)\,dt &= \int_0^T x(t)w'(t)\,dt && \text{(12a)} \\ &= \int_{\Gamma} x(t)\,dw(t) && \text{(12b)} \\ &\geq \int_{\Gamma_{ab}} x(t)\,dw(t) && \text{(12c)} \\ &= -\int_{\Gamma_{p\to a}} x(t)\,dw(t) + \int_{\Gamma_{p\to b}} x(t)\,dw(t) && \text{(12d)} \\ &\geq -\beta(x(0),\phi_0) && \text{(12e)} \end{aligned}$$

where

$$\beta(x(0),\phi_0) = \int_{\Gamma_{p\to a}} x(t)\,dw(t) \geq 0. \tag{13}$$

The first inequality (12c) holds from the circulation condition (10), while the second inequality (12e), and the positivity of β is a result of (11).

Prop. 5 Finally, we require the property that the above Properties 3 and 4 hold when the nonlinearity is sector transformed in accordance with Lemma 2.2. In essence it is required that, under this transformation, the new hysteresis maintains the circulation and positivity properties, but has infinite slope sector bound: $0 \leq \tilde{\phi}'(\sigma) < \infty$.

Remarks: The constant β in (21) has the interpretation of the maximum energy that can be extracted (available energy) from the nonlinear operator with a given set of initial conditions [Willems, 1972]. While the properties we assume may appear overly restrictive, many common hysteresis have these properties. It can be seen by inspection that the simple relay has properties 1–4, and, in a trivial manner, it satisfies property 5 since it is unaffected by the sector transformation. Under the transformation, the Preisach model is re-shaped, with the saturation region maintained and the region of maximum slope μ (see Fig. 5) becoming vertical; but the circulation and positivity properties still hold. The backlash is a simple analytical model useful to demonstrate these properties, as shown in the following section.

2.2.2 Energy storage and dissipation for the Backlash hysteresis

Here we show the common backlash nonlinearity conforms to the properties 1–5 given in the previous section. In particular, we give a simple mathematical representation for the nonlinearity, and then show that the positivity constraint (12) holds under the sector transformation indicated by property 5.

The input-output behaviour of a backlash (Fig. 4) can be described by two modes of operation, as either tracking or in the deadzone, for which we define:

$$\begin{aligned} &\text{Tracking:} \quad \dot{w} = \mu\dot{y} \quad \begin{cases} \dot{y} > 0, \quad w = \mu(y - r) \quad \text{or} \\ \dot{y} < 0, \quad w = \mu(y + r); \end{cases} \\ &\text{Deadzone:} \quad \dot{w} = 0 \qquad |w - \mu y| \leq \mu r, \end{aligned} \tag{14}$$

where $2r$ is the deadzone width and μ is the slope of the tracking region, as indicated in Fig. 4. Applying the sector transformation, shown in Fig. 1, we have, when tracking with positive velocity

$$\sigma\dot{\phi} = \sigma\dot{w} = (y - \frac{1}{\mu}w)\dot{w} = \mu r\dot{y} \tag{15}$$

and, similarly for negative tracking: $\sigma\dot{w} = -\mu r\dot{y}$. This quantity is then expressed for all times as

$$\sigma\dot{w} = \begin{cases} \mu r|\dot{y}| & \text{when tracking;} \\ 0 & \text{in deadzone.} \end{cases} \tag{16}$$

Defining the interval $\mathcal{I} = [0, T]$, for some $T \geq 0$, and $T_{trk} \subseteq \mathcal{I}$ encompassing all the subintervals in $\mathcal{I}$ for which tracking occurs, the integral (12) for the backlash becomes

$$\int_0^T \sigma(t)w'(t)\,dt = \mu r \int_{t \in T_{trk}} |\dot{y}(t)|\,dt \geq 0. \tag{17}$$

Thus, $\beta = 0$, which means that the sector transformed nonlinearity has zero stored (or available) energy. In this case, it can be shown that the transformation induces a dissipation *equality*. In particular, the energy balance, as noted by Brokate and Sprekels [Brokate and Sprekels, 1996, p. 69], is given as

$$\mathcal{M}'(t) - \mathcal{U}'(t) = |\mathcal{D}'(t)| \tag{18}$$

where the terms from (15–16) are identified with: $\mathcal{M}'(t) = \dot{w}y$ as the mechanical work rate;

$\mathcal{U}'(t) = \frac{1}{\mu} w\dot{w}$, the rate of *hysteresis potential* [Brokate and Sprekels, 1996] energy storage; and $\mathcal{D}'(t) = \mu r \dot{y}$ as the energy dissipation into the hysteretic element. The transformation (Fig. 2) strips the energy potential and leaves only the energy dissipation term in the integrand in (17). Expressed this way, we can exactly account for all energy components associated with the nonlinear operator. Explicit potential, work and dissipation expressions for more complicated hysteresis operators, such as the Preisach and Prandtl models, is discussed in [Brokate and Sprekels, 1996]. While being very powerful analytical tools, they are not pursued further herein.

2.2.3 Multiple Hysteresis Nonlinearities

Having defined all the properties of the scalar hysteresis nonlinearities, defining the class for the vector case is straightforward. We define Φ_h, the multiple hysteresis class as:

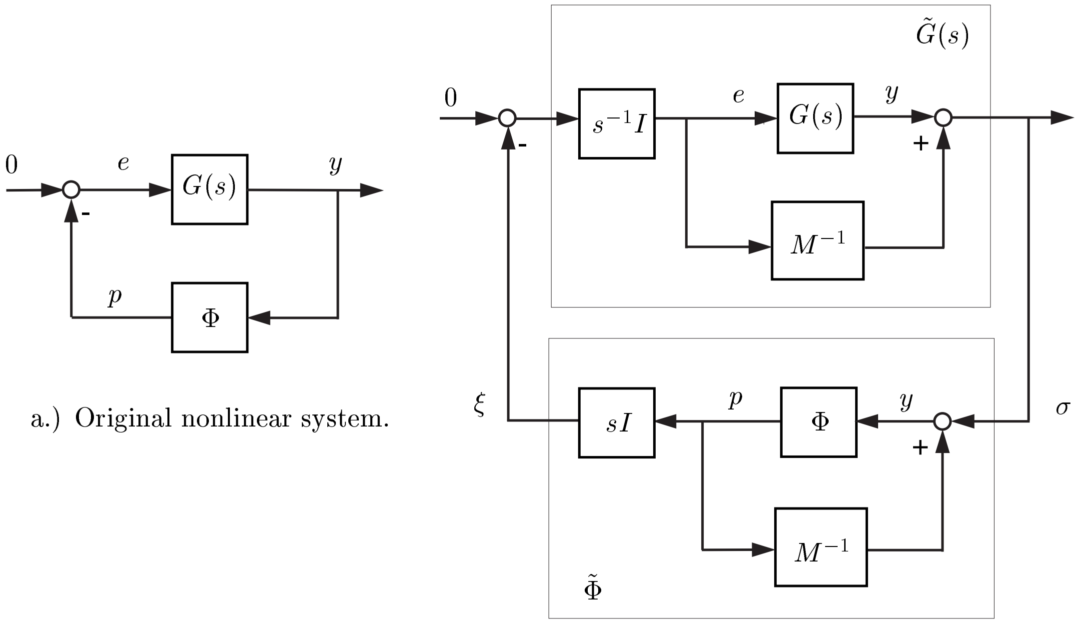
$$\Phi_h = \left\{ \phi : R^m \to R^m \;\middle|\; \begin{array}{l} \phi(y) = [\phi_1(y_1), \ldots, \phi_m(y_m)]^T \\ \phi_i \text{ is differentiable a.e. in } R \\ 0 \le \phi_i' < \mu_i, \quad i = 1, \ldots, m \\ \phi_i \text{ has Properties 1–5} \end{array} \right\}. \tag{19}$$

The set Φ_h consists of m decoupled scalar nonlinearities, with each scalar component locally slope bounded (wherever the nonlinearity is differentiable) and conforming to the properties detailed in the previous section.

Lemma 2.4 *(Passive Operator, Hysteresis case) Consider a vector hysteresis nonlinearity $\Phi_h : R^m \to R^m$ in the class defined (19). Then the input-output relation of the sector transformed operator $\tilde{\Phi}_h$ defined with $\sigma(t)$ as the input to $\tilde{\Phi}_h$ and output $\xi(t) = \frac{d}{dt}\tilde{\Phi}_h(\sigma)$, the time derivative of $\tilde{\Phi}_h(\sigma)$ (as depicted in Figure 2) is passive.*

Proof: For all $T \ge 0$

$$\begin{aligned} \int_0^T \sigma^T \xi \, dt &= \sum_{i=0}^{m} \int_0^T \sigma_i \frac{d}{dt} \tilde{\phi}_i(\sigma_i) \, dt && \text{(20a)} \\ &= \sum_{i=0}^{m} \int_0^T \sigma_i w_i'(t) \, dt && \text{(20b)} \\ &= \sum_{i=0}^{m} \int_{\Gamma_i} \sigma_i(t) \, dw_i'(t) && \text{(20c)} \\ &\ge \sum_{i=0}^{m} \int_{\Gamma_{ab_i}} \sigma_i(t) \, dw_i'(t) && \text{(20d)} \end{aligned}$$



a.) Original nonlinear system.

b.) Loop transformed system.

Fig. 6: Nonlinear system and loop transformation.

$$= \sum_{i=0}^{m} \left\{ -\int_{\Gamma_{p_i \to a_i}} \sigma_i(t)\, dw_i'(t) + \int_{\Gamma_{p_i \to b_i}} \sigma_i(t)\, dw_i'(t) \right\} \tag{20e}$$

$$\geq -\beta(\sigma(0)) \tag{20f}$$

where

$$\beta(\sigma(0), w(0)) = \beta(y(0), \phi_0) = \sum_{i=0}^{m} \int_{\Gamma_{p_i \to a_i}} \sigma_i(t)\, dw_i'(t) \geq 0, \tag{21}$$

according to properties 4–5 of the class. Hence, the input-output relation is passive, by the definition given in [Desoer and Vidyasagar, 1975, p. 173]. □

Note, that the proof is structured in a way analogous to the memoryless case. Instead of positive (sector bounded, path independent) line integrals, the corresponding steps here involve positive path integrals.

3 System Description and Loop Transformation

As in the standard absolute stability analysis framework, it is assumed that the nonlinearity can be isolated from the linear dynamics and placed into a feedback path, as is shown in Fig. 6a. Assuming the linear dynamics $G(s)$ has a minimal state space representation (A, B, C, D), with A Hurwitz, the nonlinear (Lur'e) system is described as

$$\begin{aligned} \dot{x} &= Ax + Be \\ y &= Cx + De \\ p_i(t) &= \phi_i(y_i(t)), \quad i = 1, \dots, m, \end{aligned} \tag{22}$$

where $p(t) \in R^m$ and $\phi \in \Phi$, as defined by either the multiple memoryless or hysteresis class, as before. In order to convert the nonlinearity into a passive operator, in accordance with Lemma 2.3-4, we introduce the loop transform, as described in Fig. 2, to give the equivalent system shown in Fig. 6b. Note that $\tilde{\Phi}$ is now passive, and that the transformed linear system:

$$\tilde{G}(s) = (G(s) + M^{-1})(s^{-1}I), \tag{23}$$

has the state space representation:

$$\tilde{G} \stackrel{s}{=} \left[\begin{array}{c|c} \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} & \begin{bmatrix} B \\ D + M^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array} \right]. \tag{24}$$

By the Hurwitz assumption, we have that A is invertible, and thus by introducing the similarity transform:

$$T = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix},$$

the augmented system $\tilde{G}(s)$ can be decomposed into its stable and constant dynamic components as:

$$\tilde{G}(s) = \tilde{G}_r(s) + s^{-1}R, \tag{25}$$

where $R = G(0) + M^{-1}$ with $G(0) = -CA^{-1}B + D$, and the stable component G_r is reduced by the integrator states and has the state space description:

$$\tilde{G}_r \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline CA^{-1} & 0 \end{array} \right]. \tag{26}$$

With the linear dynamics decomposed in this way, the nonlinear, closed loop system can then be

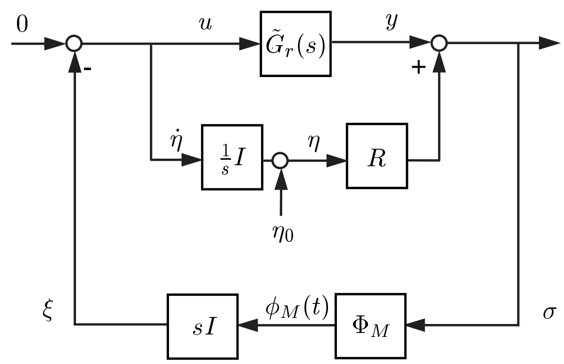


Fig. 7: Popov indirect control form.

expressed in the vector version of Popov's indirect control form (see [Vidyasagar, 1993, p. 231], for example), as is depicted in Fig. 7. The dynamics of the original Lur'e system (22) corresponding now to the Popov form are equivalently given as:

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax - B\xi \\ \dot{\eta} &= -\xi, \quad \eta(0) = -\phi_M(0) \\ \sigma &= CA^{-1}x + R\eta \\ \xi &= \dot{\phi}_M(t) \end{aligned} \tag{27}$$

Proper initialization of the integral state η, as shown in Fig. 7, leads to the identities:

$$\eta(t) = -\phi_M(t) \tag{28a}$$

$$\dot{\eta}(t) = -\xi = -\dot{\phi}_M(t). \tag{28b}$$

The stable (equilibrium) conditions for the hysteresis case differs from the memoryless, slope-restricted because the hysteresis is multivalued. As a result, while the equilibrium point for the memoryless nonlinear system is unique, convergence for the hysteresis system is to an invariant set, which may consist of an infinite number of points. The next section provides explicit descriptions of these stability sets.

4 Stationary Sets and Stability Definitions

Stability theory is often used to determine whether or not an autonomous system will achieve some sort of steady state condition. Generally speaking, in steady state, the system state may be at an equilibrium point (at rest with $\dot{x} = 0$), or in a limit cycle. In either case, the state $x(t)$ belongs to an invariant set [Hahn, 1963, Vidyasagar, 1993]. The *largest* invariant set $\mathsf{M} \subset R^n$, for a particular system, is the union of all equilibrium points and the sets containing all possible limit cycles. The equilibrium, or stationary, set $\mathsf{E} \subseteq \mathsf{M}$, for the nonlinear system (22) is defined as:

$$\mathsf{E} = \left\{ \; x \in R_n \;\middle|\; \text{such that (30) is satisfied} \; \right\}, \tag{29}$$

where (30) is the set of algebraic conditions:

$$\begin{aligned} y_{ss} &= [-CA^{-1}B + D]e_{ss} = G(0)e_{ss} && \text{(30a)} \\ e_{ss} &= -\phi(y_{ss}) && \text{(30b)} \\ x_{ss} &= -A^{-1}Be_{ss}. && \text{(30c)} \end{aligned}$$

Naturally, E is unique to each system (22) and, in particular, depends on the type of nonlinearity present. Various stationary sets are given below.

4.1 Stationary Set for Memoryless Nonlinearity

For the slope-restricted nonlinearity, we assume there exists a unique equilibrium point $x = 0$, for the closed loop system (22). That is, E_m is a singleton:

$$\mathsf{E}_m = \{0\}\,. \tag{31}$$

This result is consistent with the sector bounded property of the class Φ, and the assumption $G(0) > M^{-1}$. Geometrically, this condition means that the graph of *i-th* nonlinearity $\phi_i(y_i)$ and the line $\phi_i = -y_i/G_{ii}(0)$ intersect only once, at the origin. This intersection is necessarily non-unique in the hysteresis case, and as a result, E_h is comprised of finite regions in state space. These sets are defined below for various special cases.

4.2 Stationary Sets for Hysteresis Nonlinearities

The stationary sets for multiple hysteresis can be defined with a simple extension of the graphical technique for the scalar case originally detailed by [Barabanov and Yakubovich, 1979][4]. To proceed, consider a generic Preisach nonlinearity, and note that conditions (30a–b) together can be depicted graphically, as shown in Fig. 8, as the intersection of the line $\phi_i = -y_i/G_{ii}(0)$ and the graph of the hysteresis. This intersection defines the range of outputs for each nonlinearity $\phi_i \in [\underline{\phi}_i, \overline{\phi}_i]$

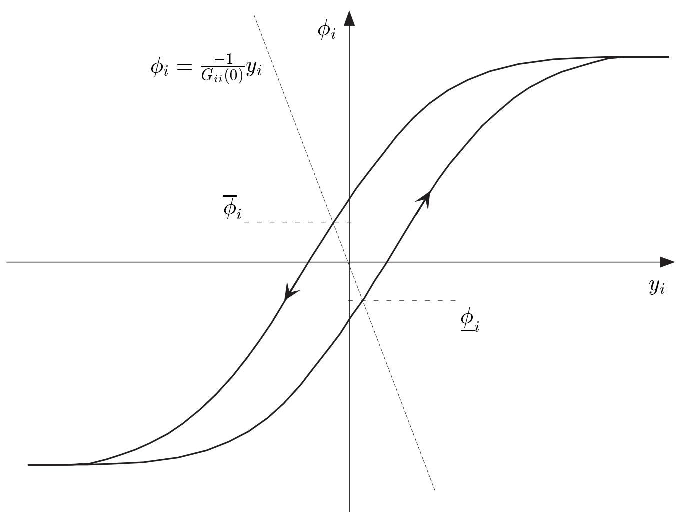


Fig. 8: Graphical criteria for determining E.

which must be satisfied simultaneously for each $\phi_i,\ i = 1, \ldots, m$. Then letting each ϕ_i vary over the allowed range maps out the invariant set E, according to the condition (30c) $x = -A^{-1}Be$, where $e = -\phi$. Note that if $G_{ii}(0) = 0$, then the corresponding limits $\underline{\phi}_i, \overline{\phi}_i$ are simply the extreme values of intersection of the hysteresis with the ϕ–axis. The stationary sets for the relay, backlash, and Preisach hysteresis nonlinearities are given next.

4.2.1 Hysteretic Relay

For a system with a bank of m unit relays, as shown in Fig. 3, the stationary set is given by:

$$\mathsf{E}_{relay} = \left\{ x \in R^n \;\middle|\; \begin{array}{l} x = -A^{-1}Be \\ e \in R^m, \quad e_i \in \{-1, 1\}, \quad i = 1, \ldots, m \end{array} \right\} \tag{32}$$

[4] A similar definition for (29) is given in Ref. [Jönsson, 1998].

E_{relay} consists of 2^m discrete points in R^n. Each point is essentially the steady state solution of the open loop system $G(s)$ in response to a particular constant input vector e consisting of elements $e_i = +1, \text{ or } -1$.

4.2.2 Backlash and Preisach Nonlinearities

The equilibrium sets for these two types of nonlinearities are defined in the same way, since both operators admit outputs that range continuously over a prescribed interval. Once the output limits are defined, the stationary set is completely determined.

$$\mathsf{E}_{backlash},\ \mathsf{E}_{Preisach} = \left\{ x \in R^n \;\middle|\; \begin{array}{l} x = -A^{-1}Be \\ e \in R^m, \quad e_i \in [\underline{\phi}_i, \overline{\phi}_i], \quad i = 1, \ldots, m \end{array} \right\} \tag{33}$$

Note that these sets are polytopic regions, and are equivalently defined as the convex hull of the corresponding set of limiting vectors:

$$\mathsf{E}_{backlash},\ \mathsf{E}_{Preisach} = \mathbf{Co}\left\{\underline{v}_i, \overline{v}_i, \ldots, \underline{v}_m, \overline{v}_m\right\}, \tag{34}$$

where $\underline{v}_i, \overline{v}_i \in R^n$, with

$$\underline{v}_i = -A^{-1}B\underline{z}, \quad \text{where } \underline{z}_j = \begin{cases} \underline{\phi}_i, & j = i \\ 0, & \text{else,} \end{cases}$$

and $\overline{v}_i$ defined similarly.

The definitions for the stationary sets E provide a clear idea of the position of $x \in R^n$ should the system achieve the equilibrium condition defined by $\dot{x} = 0$. Before providing the stability criteria that guarantees the system is indeed stable, we give precise definitions of what it means for a system to be stable with respect to an invariant set.

4.3 Definitions of Stability

Using standard notation (as by [Hahn, 1963], for example), define the trajectory of motion for an initial condition $x(0) = x_0$ of some arbitrary system as $q(x_0, t)$. For an invariant set M of the system, the distance to the set from any arbitrary point is given by:

$$\text{dist}(x, \mathsf{M}) = \inf |x - y|, \quad y \in \mathsf{M},$$

with $\mathrm{dist}(x, \mathsf{M}) = 0$ for $x \in \mathsf{M}$. A closed invariant set M is called *stable*, if for every $\epsilon > 0$ a number $\delta > 0$ can be found such that for all $t > 0$,

$$\mathrm{dist}(q(x_0, t), \mathsf{M}) < \epsilon$$

provided

$$\mathrm{dist}(x_0, \mathsf{M}) < \delta.$$

If in addition,

$$\mathrm{dist}(q(x_0, t), \mathsf{M}) \to 0, \quad \text{as } t \to 0,$$

then M is said to be *asymptotically stable.*

5 Stability Theorem

This section provides a Lyapunov-based asymptotic stability theory for the systems with either slope-restricted (memoryless) or hysteresis nonlinearities. The Lyapunov function used refers to the transformed system defined in §3 and includes the integral of the nonlinearity that is positive, as a result of the passive properties defined in §2. Negativity of the Lyapunov derivative is enforced by a certain matrix inequality of a form similar to that associated with the well-known KYP lemma (see [Boyd et al., 1994, p. 120], for one treatment). The theorem then concludes asymptotic stability of the origin in the case of the memoryless, slope-restricted nonlinearities, and for the equilibrium sets given §4.2 in the hysteresis case by using the Lyapunov conditions and employing basic analytical results.

Theorem 5.1 *(Asymptotic Stability) If there exists constants P, N, Δ, with*

$$\begin{aligned} &P \in R^{n\times n}, \quad P = P^T > 0 \\ &\Delta \in R^{m\times m}, \quad \Delta = \Delta^T > 0 \\ &N = diag(n_1, \ldots, n_m), \quad n_i > 0, \ i = 1, \ldots, m \end{aligned} \tag{35}$$

such that

$$\begin{bmatrix} -A^TP - PA & C^TN + A^{-T}C^T - PB \\ (\cdot)^T_{12} & ND + D^TN + 2NM^{-1} - \Delta \end{bmatrix} \geq 0, \tag{36}$$

and $R = R^T > 0$, $R = G(0) + M^{-1}$, then the closed loop system (27) is asymptotically stable. In *this case, the Lyapunov functional:*

$$V(x(t),\eta(t),t) = x(t)^T Px(t) + 2\int_0^t \sigma^T(\tau)\xi(\tau)\,d\tau + \beta(\sigma_0,\xi_0) + \eta^T(t)R\eta(t) \tag{37}$$

proves stability.

Proof: Choosing β as (7) for the slope-restricted nonlinearity, or as (21) when the nonlinearity is a multiple hysteresis[5], then $V \geq 0$; and since $P, R > 0$, $V \to \infty$ whenever $(x,\eta) \to \infty$, so V is positive definite and hence, a valid Lyapunov candidate. In order to assert $\dot{V} \leq 0$, first note that matrix inequality (36) implies, for all $x \in R^n,\ u \in R^m$

$$\begin{aligned} x^T Px &\leq 2x^T(C^T N + A^{-T}C^T - PB)u + u^T(ND + D^T N + 2NM^{-1} - \Delta)u \\ &= 2x^T(C^T N + A^{-T}C^T - PB)u + u^T M_{22} u, \end{aligned}$$

where M_{22} is the $(2,2)$ entry of the LMI (36). Using this fact, and (28) we have

$$\begin{aligned} \dot{V}(x,\eta) &= x^T(PA + A^T P)x - 2x^T PB\dot{\phi}_M + 2\sigma^T\dot{\phi}_M + 2\eta^T R\dot{\eta} \\ &\leq -2x^T(C^T N + A^{-T}C^T)\dot{\phi}_M + 2\sigma^T\dot{\phi}_M + 2\phi_M^T R\dot{\phi}_M + \dot{\phi}_M^T M_{22}\dot{\phi}_M && \text{(39a)} \\ &= -2(\dot{\sigma} + (D + M^{-1})\dot{\phi}_M)^T N\dot{\phi}_M - 2(\sigma + R\phi_M)^T\dot{\phi}_M + 2\sigma^T\dot{\phi}_M \\ &\quad + 2\phi_M^T R\dot{\phi}_M + \dot{\phi}_M^T M_{22}\dot{\phi}_M \\ &= -2\dot{\sigma}^T N\dot{\phi}_M - \dot{\phi}_M^T\Delta\dot{\phi}_M \\ &\leq -\dot{\phi}_M^T\Delta\dot{\phi}_M && \text{(39b)} \\ &\leq -\delta|\dot{\phi}_M|^2 && \text{(39c)} \\ &\leq 0, && \text{(39d)} \end{aligned}$$

where the first inequality (39a) is due to the LMI condition, the second (39b) a result of the time-derivative sector condition (5), and the last two (39c-39d) follow from the constraint $\Delta > 0$ (35), and the assumption that δ is the minimum eigenvalue of Δ. Now since V is positive definite in x, η and $\dot{V} \leq 0$, we conclude the closed loop system is stable, or, simply that x and η are bounded. To find asymptotic stability, first note that,

$$\dot{V} \leq -\delta|\dot{\phi}_M|^2 \quad \Rightarrow \quad \dot{\phi}_M(t) \to 0 \text{ as } t \to \infty, \tag{40}$$

[5] In the particular case when the nonlinearity is of the multiple backlash type, $\beta = 0$, as discussed in §2.2.2.

since $V(t)$ is bounded below. Further, using (39c), we have

$$V(t) - V(0) \leq -\delta \int_0^t |\dot{\phi}_M|^2 \, dt, \tag{41}$$

which, can be rewritten as

$$\int_0^t |\dot{\phi}_M|^2 \, dt \leq \frac{1}{\delta}(V(0) - V(t)) \leq V(0)/\delta, \tag{42}$$

which implies $\dot{\phi} \in \mathcal{L}_2$, and as a result $y(t) \in \mathcal{L}_2$ as well since $\tilde{G}_r$ is $\mathcal{L}_2$-stable (*i.e.*, A Hurwitz). Using the system dynamics (27), the signal y and its derivative are expressed as

$$y(t) = CA^{-1}\left[\exp(At)x(0) + \int_0^t \exp(A(t-\tau))Bu(\tau)\, d\tau\right] \tag{43a}$$

$$\dot{y}(t) = CA^{-1}\left[A\exp(At)x(0) - \exp(At)\dot{\phi}_M(t)\right]. \tag{43b}$$

Assuming Lipschitz continuous nonlinearities so that $\dot{\phi}_M(t)$ exists, we have that $\dot{y} \in \mathcal{L}_\infty$.[6] In this case, the two conditions $y(t) \in \mathcal{L}_2$, $\dot{y} \in \mathcal{L}_\infty$ imply that $y(t) \to 0$ as $t \to \infty$ (see, for example, [Narendra and Annaswamy, 1989, Lemma 2.1.2]). The asymptotic conditions $y(t), \dot{\phi}(t) \to 0$ together require that the closed loop system must approach an equilibrium condition as $t \to 0$. To see this, note that the conditions $\xi = \dot{\phi}_M \to 0$ and $y \to 0$ imply that all signals of the Popov system (27) contained in the shaded region of the block diagram in Fig. 9a approach zero asymptotically. Recall that the initialization of variable $\eta(0) = -\phi_M(0)$ implies that $\eta(t) = -\phi_M(t)\, \forall t \geq 0$, as given by Eqn. (28). Thus, in the limit, the zero signals can be eliminated and the system reduced to that shown in Fig. 9b, where the signal equivalence mentioned above is indicated by the dashed line. Reversing the sector transformation further simplifies the diagram to that in Fig. 9c, which corresponds to the equivalent algebraic conditions:

$$y_{ss} = G(0)u_{ss} \tag{44a}$$

$$u_{ss} = -\phi(y_{ss}) \tag{44b}$$

$$\bar{x} = -A^{-1}Bu_{ss}, \tag{44c}$$

[6] See Remark 2 below concerning continuous approximations for discontinuous nonlinearities such as the relay hysteresis.

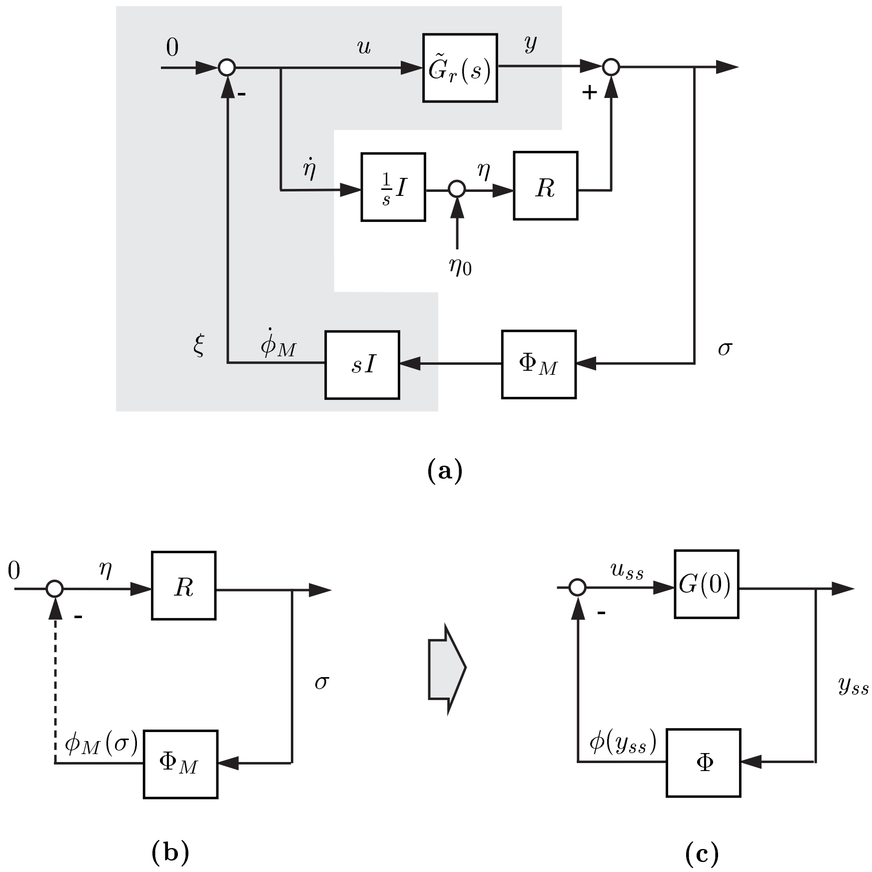


Fig. 9: The condition $\dot{V} \equiv 0$ implies steady state condition on the Popov system.

which are identical to the conditions (30) that describe the stationary set E. Therefore, in the hysteresis case, we conclude global asymptotic stability of the set E. In the special case of the system with multiple slope-restricted nonlinearities, the set E is simply the origin, as noted by Eqn. (31). □

Remarks:

1. This proof utilizes a combination of Lyapunov and input-output stability theories. Of course, connections between Lyapunov and input-output stability concepts have been well established [Willems, 1971b, Hill and Moylan, 1980, Boyd and Yang, 1989]. In this case, passivity conditions are used to establish Lyapunov stability arguments for slope restricted/hysteresis nonlinear systems, all within the analytical framework of Popov's indirect control form. An alternate approach could proceed using passivity (as is done in [Paré and How, 1998b]) or Popov's hyperstability theorem [Popov, 1973], exclusively. However, the Lyapunov compo-

nent included here enables the additional conclusion of asymptotic stability of the set E. Positive real and passivity interpretations of the analysis are further explored in the following section.

2. Note that the hysteresis set Φ_h (19) includes the hysteretic relay, which has a discontinuous input-output mapping. Strictly speaking, the proof given does not apply to such nonlinearities directly. In order to maintain simplicity, we will assume in these cases that discontinuities can be replaced with a reasonably smooth approximations so that Lipschitz conditions are satisfied (see [Visintin, 1988] for a similar approximation for the hysteretic relay). A more rigorous approach, could be developed for these discontinuous nonlinearities using Filippov [Filippov, 1988] state solutions, one-sided Lyapunov derivatives as described by [Hahn, 1963, Clarke, 1983], and the generalized version of LaSalles Invariance Principle [LaSalle, 1976].

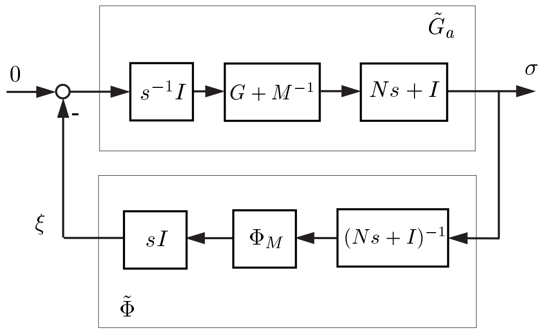
3. The condition $R = R^T > 0$ is not overly restrictive. For instance, the off-diagonal elements $G(s)$ can often be arbitrarily scaled using diagonal scaling matrices. In this way the matrix $G(0)$ can be made symmetric with the necessary gain adjustments incorporated into the nonlinearity. The condition $R = R^T > 0$ is less restrictive than the condition $G(0) = 0$ given by [Haddad and Kapila, 1995], and the criterion $G(0) = G(0)^T > 0$ required by [Park et al., 1998], whenever the nonlinearity has finite maximum slope. The criteria in Ref. [Park et al., 1998] includes the additional constraint that $NG(0) = G(0)^T N$, which limits N to a scalar quantity in the case when $G(0)$ is a full matrix. This can further restrict the analysis, as is illustrated with a simple example in §7.

6 Passivity and Frequency Domain Interpretations

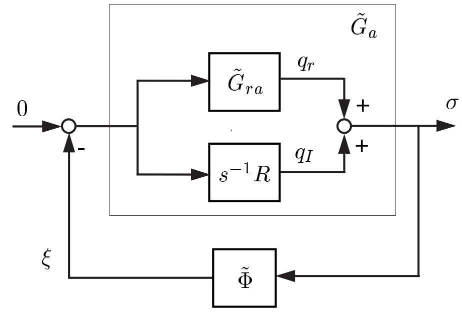
The LMI (36) is recognized as a strict passivity condition on the linear system:

$$\tilde{G}_{ra} = \left[\begin{array}{c|c} A & B \\ \hline NC + CA^{-1} & N(D + M^{-1}) \end{array}\right] \tag{45}$$

which is an augmented version of the reduced system $\tilde{G}_r$. Strict passivity of this augmented system is a requirement for stability that could have been derived with an equivalent analysis of the system



a.) Augmented, transformed nonlinear system.



b.) System in Popov Indirect form.

Fig. 10: Augmented, passive system.

in Fig. 6 that employs noncausal multipliers, as is detailed in Ref. [Willems, 1971a, Ch. 6]. A robust stability analysis using passivity and multipliers for specific case of systems with a single hysteresis was recently done by the authors [Paré and How, 1998b]. To proceed, introduce the multiplier $W(s) = Ns + I$ (with N as defined in (35)) into the transformed system, as shown in Fig. 10a. In this case, premultiplying the hysteresis Φ_M with W^{-1} as shown results in a new nonlinearity $\tilde{\Phi}$ in the feedback path which is passive. This passivity condition is ascertained using the steps in the proofs of Lemma 2.3–4 and using the additional time derivative constraint (5). Introducing the multiplier similarly leads to the transformed linear system:

$$\tilde{G}_a(s) = W(s)(G(s) + M^{-1})(s^{-1}I) = (Ns + I)(G(s) + M^{-1})(s^{-1}I). \tag{46}$$

This decomposes, as was done in Eqs. (23-25), to

$$\tilde{G}_a(s) = \tilde{G}_{ra}(s) + s^{-1}R, \tag{47}$$

where again $R = G(0) + M^{-1}$, and $G_{ra}(s)$ is the augmented system (46) reduced by the integrator states and has the state space representation (45). This leads directly to the Popov indirect form, with a parallel combination of the augmented system $\tilde{G}_{ra}$ and the constant dynamics $s^{-1}R$, as depicted in Fig. 10b. In the passivity framework, stability requires either the feedforward or feedback operator be strictly passive. In this case, strict passivity is achieved by conditions on the reduced system $\tilde{G}_{ra}$ and strict positivity of R, as detailed in the following corollary.

Corollary 6.1 *(Strict Passivity) If there exists* $N = diag(n_1, \ldots, n_m) \geq 0,\ \Delta = \Delta^T > 0$ *such that* *the following two conditions:*

1. $R = R^T, \ R = G(0) + M^{-1} > 0$

2. *The reduced system G_{ra}, given by (45) is dissipative with respect to the supply rate:*

$$r(p,q) = p^T q - p^T \Delta p, \tag{48}$$

are satisfied, then the system $\tilde{G}_a(s)$ is strictly passive. In this case, the closed loop system (22) is asymptotically stable.

Proof: Let $\eta : R_+ \to R^m$ represent the integrator state with $\eta(0) = \eta_0$, $V : R^n \to R_+$ be a storage function for G_r and q_I, q_r be the outputs of the integrator and G_r, respectively, as depicted in Fig. 10b. Then for any $T \geq 0$ we have

$$\begin{aligned} \int_0^T q^T p\, dt &= \int_0^T (q_I + q_r)^T p\, dt \\ &= \int_0^T \eta^T(t) R^T \frac{d}{dt}\eta(t)\, dt + \int_0^T q_r^T p\, dt \\ &\geq \frac{1}{2}(\eta_T^T R \eta_T - \eta_0^T R \eta_0) + V(x_T) - V(x_0) + \langle p, \Delta p\rangle_T^2 \\ &\geq -\beta(\eta_0, x_0) + \delta \|p_T\|_2^2, \end{aligned}$$

where $\beta(\eta_0, x_0) = \eta_0^T R \eta_0 / 2 + V(x_0) \geq 0$ and $\delta > 0$ is the minimum eigenvalue of Δ. Thus, $\tilde{G}_a$ is strictly passive by the definition given in [Desoer and Vidyasagar, 1975]. Then, since the loop transformed system consists of a passive (transformed) nonlinearity in feedback with a strictly passive linear system, we thus conclude the closed loop system will converge asymptotically to the equilibrium conditions. □

Corollary 6.1 essentially uses the idea that an operator consisting of the parallel combination of passive (nonstrict) and a strictly passive operators is strictly passive. Condition 1 ensures the passivity (nonstrict) of the integrator component, while condition 2 enforces the strict passivity of the reduced system $\tilde{G}_{ra}$. The necessary dissipation for the parallel system is ultimately guaranteed by the existence of some $\Delta > 0$. Naturally, the scalar analogy for the positivity condition: $R = R^T > 0$ on the integrator term is the simple capacitor, which is passive[7] provided the capacitance value is positive. The notion that a linear system can be strictly passive even though it has zero eigenvalues is not intuitive, but similar results are available in the literature, and usually

[7]Assuming the input/output relation across the capacitor terminals is current/voltage.

involve decomposing the system into its stable and constant dynamic components, as is done here for the indirect Popov criterion. In Ref. [Anderson and Vongpanitlerd, 1973, p. 216], for example, it is shown that systems with purely imaginary poles are positive real only if the associated residue matrices are nonnegative definite Hermitian. A similar state space diagonalization is used to establish Lyapunov stability criteria in [Boyd et al., 1994, pp. 20–22] for systems having eigenvalues with a zero real part. In essence, Thm. 4.1 is an extension of these ideas to a particular version of the KYP Lemma, and in effect could be called the *Indirect Control KYP Lemma,* for the historical reasons cited in §1.

Of course, as is well known, a linear system is strictly passive if and only if its Hermitian form is strictly positive definite for all frequencies [Desoer and Vidyasagar, 1975, p. 174]; that is, a system $H(s)$ is strictly passive if and only if, for some $\delta > 0$,

$$H(j\omega) + H^*(j\omega) > \delta I, \quad \forall \omega \geq 0. \tag{49}$$

Hence, the stability question can be addressed by asking the equivalent question: When is a square, linear system having zero eigenvalues strictly passive? Note that, unlike the approach taken in [Haddad and Kapila, 1995, Park et al., 1998], we do not require the linear system to be strictly positive real (SPR) [Khalil, 1996, Wen, 1988], which is a stronger condition than strict passivity. In fact, the transformed system $\tilde{G}_a$ in general can not be SPR since the multiplier $W(s)$ introduces a zero eigenvalue (see [Khalil, 1996, pp 404-405]); however, it is clear that $\tilde{G}_a$ satisfying the conditions of Thm 4.1 are strictly passive. This follows since,

$$\begin{aligned} \tilde{G}_a(j\omega) + \tilde{G}_a^*(j\omega) &= \tilde{G}_{ra}(j\omega) + \tilde{G}_{ra}^*(j\omega) + \frac{1}{j\omega}(R - R^T) && \text{(50a)} \\ &= \tilde{G}_{ra}(j\omega) + \tilde{G}_{ra}^*(j\omega) && \text{(50b)} \\ &> \Delta && \text{(50c)} \\ &\geq \delta I, && \text{(50d)} \end{aligned}$$

where δ is the minimum eigenvalue of Δ. Therefore the strict passivity condition (49) is achieved. Here again, as in the Corollary 5.1, the role of symmetric R is apparent, this time in the frequency domain.

7 Numerical Examples

7.1 Computing the Maximum Allowed Slope for Nonlinearities

A common engineering problem that often arises is that of finding the maximum sloped nonlinearity that a given system can tolerate before going unstable. This problem was posed in [Park et al., 1998], and an LMI solution was suggested based on the analysis given in that paper. The same problem in terms of the conditions of Theorem 4.1 is stated as:

$$\max\ \mu \quad \text{subject to:} \begin{cases} (35),\ (36) \\ R = R^T > 0 \end{cases} \tag{51}$$

where $M = \mu I_m$. Solving (51) for the arbitrary 2×2 system $G(s)$ given as

$$G_1(s) = \begin{bmatrix} \frac{s^2-0.2s+0.1}{s^3+2s^2+2s+1} & \frac{s^2-0.4s+0.75}{s^3+3s^2+3s+1} \\ \frac{0.1s^2+5s+0.75}{s^3+1.33s^2+2s+1} & \frac{0.15(s^2+s+0.75)}{s^3+s^2+1.1s+1} \end{bmatrix}, \tag{52}$$

by using the LMI solver [Gahinet et al., 1995], yields a maximum allowed slope value of $\mu = 0.940$. By comparison, the equivalent problem using the stability criteria from [Park et al., 1998] results in a maximum slope of 0.392, approximately a factor of 2 smaller. Obviously, Theorem 4.1 is less conservative in guaranteeing stability for this system. The reason for this is that while $G(0)$ is symmetric, and thus satisfies the criteria in [Park et al., 1998], $G(0)$ is a full matrix. As a result of Park's additional constraint, $NG(0) - G(0)^T N = 0$, the multiplier N must reduce to a scalar, positive number. By contrast, our Theorem 4.1 poses no such condition on $G(0)$, and allows N to remain a diagonal matrix with two degrees of freedom, and is thus able to give less conservative stability guarantees. This relative advantage is likely to increase as the number of nonlinearities increases in the case of non-diagonal $G(0)$. This follows since Theorem 4.1 will allow one additional degree of freedom for each nonlinearity, while the criteria from [Park et al., 1998] restricts the multiplier to a single scalar number (*i.e.*, $N = nI_m$) regardless of the problem size.

As a second example, consider

$$G_2(s) = \begin{bmatrix} \frac{s-0.2}{s^3+2s^2+2s+1} & \frac{0.1s^2+1}{s^2+3s+1} \\ \frac{0.1s^2+5s+1}{s^2+1.33s+1} & \frac{0.2(s^2+s+0.75)}{s^3+s^2+1.1s+1} \end{bmatrix}, \tag{53}$$

and note the state space version of this system has a non-zero feedthrough term, $D \neq 0$, and the system matrix at $s = 0$:

$$G_2(0) = \begin{bmatrix} -0.2 & 1.0 \\ 1.0 & 0.15 \end{bmatrix}$$

has a negative eigenvalue. For either of these reasons, the recent results of [Park et al., 1998] and [Haddad and Kapila, 1995] do not apply in this case. Within the context of absolute stability then, it is fair to conclude the criteria in [Park et al., 1998, Haddad and Kapila, 1995] can guarantee stability only for nonlinearities having *zero* maximum slope (*i.e.*, only when nonlinearities are not present). However, Theorem 4.1 does apply and guarantees stability for all nonlinearities in the classes described in §2 that have a maximum slope less than 0.996. The corresponding stability multiplier is $N = \text{diag}(25.327, 11.134)$.

7.2 Asymptotic Stability with Single Hysteretic Relay

As a simple example of an application of Theorem 4.1 for a system with a single nonlinearity, consider a third order system:

$$G(s) = \frac{s^2 + 0.01s + 0.25}{(s+1)(s^2+s+1)} \tag{54}$$

that is attached in negative feedback with a hysteretic relay (Fig. 3). A simple graphical check, as described in §4.2 shows that the line $\phi = -y/G(0)$ intersects the nonlinearity in two stable points, $\phi = \pm 1$, and does not intersect the discontinuous portion of the characteristic. In this case the stationary set is well defined and, according to definition (32), is simply two discrete points in state space:

$$\mathsf{E} = \left\{ \pm \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}. \tag{55}$$

To prove asymptotic stability of E, we solve the LMI (36) by approximating the infinite slope of the relay with the value $\mu = 1 \times 10^6$. Using the LMI toolbox [Gahinet et al., 1995], the stability parameters are found to be

$$P = \begin{bmatrix} 5.0826 & -0.02149 & 0.16304 \\ -0.02149 & 4.7911 & -0.02991 \\ 0.16304 & -0.02991 & 3.038 \end{bmatrix},$$

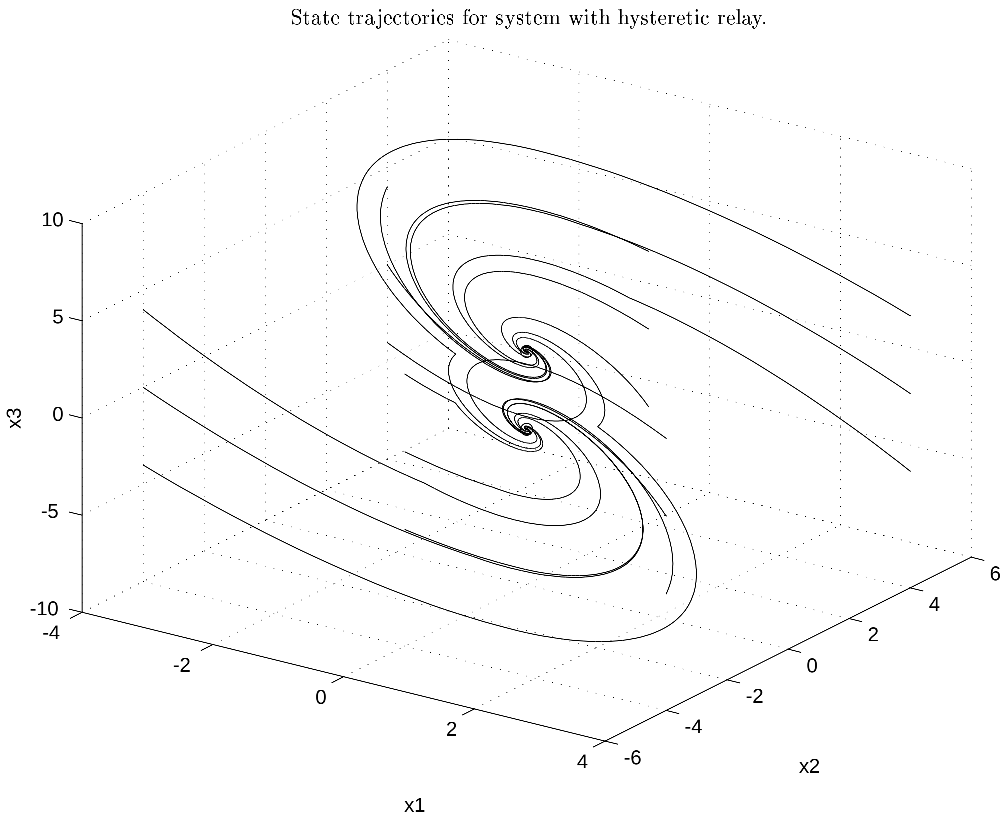


Fig. 11: All solutions converge to the two points in the set E.

$N = 4.7078$, and $\Delta = 4.77 \times 10^{-6}$, which proves the global asymptotic stability of the set E. Note that in this case $G(s)$ is not positive real, and thus an analysis of this hysteretic relay system based on the circle criteria, such as the IQC technique given by [Rantzer and Megretski, 1996], would fail. However, the graph of $G(j\omega),\ w \geq 0$ does not enter the third quadrant of the Nyquist plane and therefore satisfies less restrictive stability criteria for systems with scalar hysteresis nonlinearities, as detailed in [Paré and How, 1998b]. Several simulations of the nonlinear system confirm this result. The set is clearly visible in Fig. 11, as initial conditions at various locations in state space converge to either of the two discrete points. The nonlinear behaviour of the system is evident in Fig. 12, which shows nonsmooth trajectories of the state that result at times when the relay switches. The nonlinear switching is also the cause of the asymmetric pattern of the state trajectories, as seen in the x_1-x_3 plane.

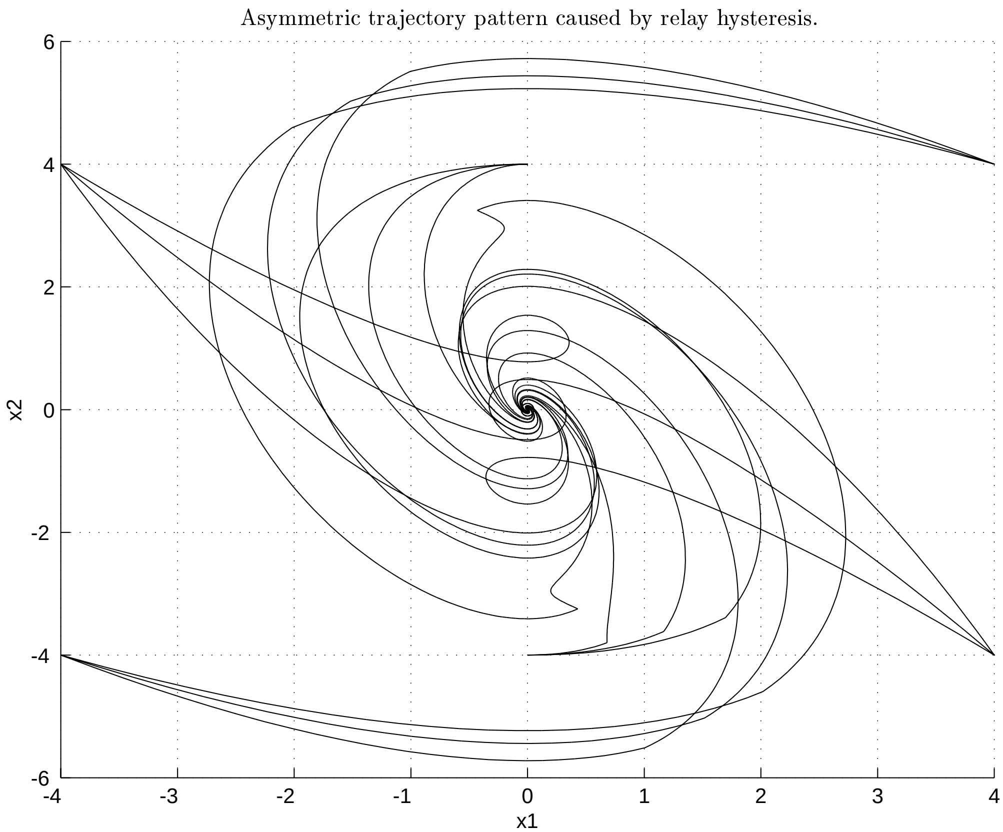


Fig. 12: Trajectories are nonsmooth as a result of relay switching.

7.3 Asymptotic Stability with Multiple Backlash Nonlinearities

Here we investigate the stability of the two-input, two-output system:

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|cc} -2 & -1 & -0.5 & 0.19365 & 0.41312 \\ 2 & 0 & 0 & 0 & -0.41312 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1.875 & -0.1875 & 0.09375 & 0 & 0 \\ 1 & 0.75 & 1 & 0 & 0 \end{array}\right] \tag{56}$$

that is attached in feedback with two backlash nonlinearities, described in Fig. 4, each having unit slope and deadband width ($\mu, D = 1$). The system matrix at $s = 0$:

$$G(0) = \begin{bmatrix} 0.0363 & 0.3873 \\ 0.3873 & 0.20656 \end{bmatrix}$$

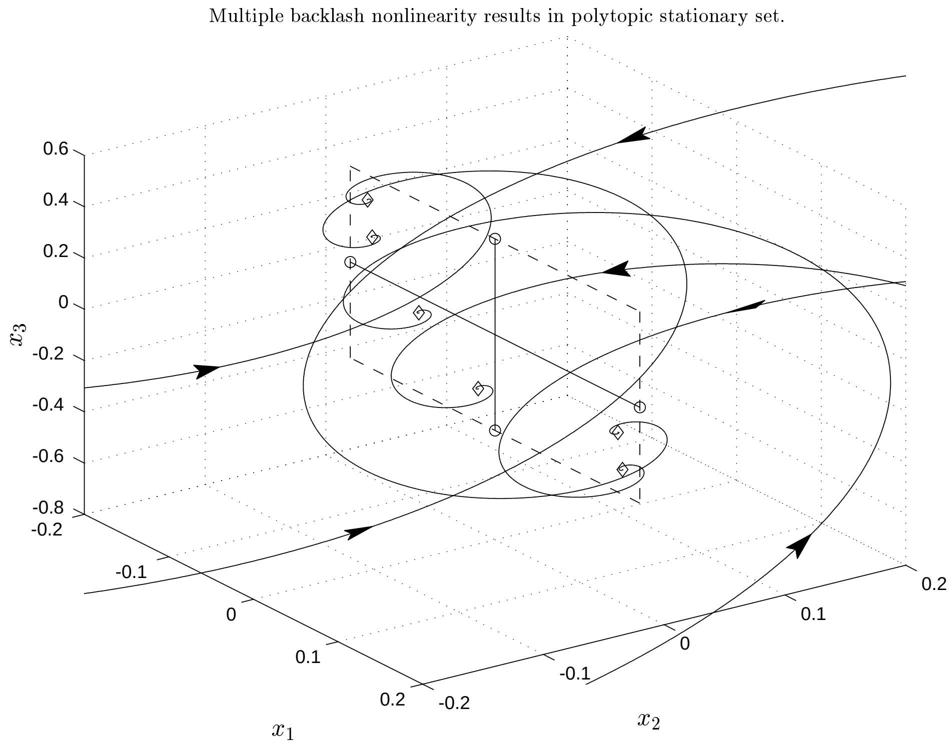


Fig. 13: The stationary set E for a multiple backlash nonlinearity is a rectangular region in the x_1-x_3 plane.

is symmetric, and has eigenvalues $\lambda = -0.275, 0.518$, so that the criteria $R = R^T > 0$, where $R = G(0) + I$ is satisfied. Solving the LMI (36) yields the stability parameters

$$P = \begin{bmatrix} 4.2914 & -1.921 & -3.7638 \\ -1.921 & 7.6573 & -2.3389 \\ -3.7638 & -2.3389 & 18.354 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 1.7292 & 0 \\ 0 & 1.6253 \end{bmatrix} \quad \Delta = \begin{bmatrix} 0.75697 & -0.18976 \\ -0.18976 & 0.69497 \end{bmatrix}.$$

Positivity of these matrices proves global asymptotic stability for the set E, as per Theorem 4.1. In this case, E is a polytopic region, as described for the backlash nonlinearity by Eqn. (34), given by:

$$\mathsf{E} = \mathbf{Co}\left\{ \pm \begin{bmatrix} 0.1712 \\ 0 \\ 0 \end{bmatrix}, \quad \pm \begin{bmatrix} 0 \\ 0 \\ 0.3737 \end{bmatrix} \right\}. \tag{57}$$

The stationary set E (57) is shown dashed in Fig. 13. Simulation of the nonlinear system with six different initial conditions confirms the stability of the set. All trajectories terminate in E, as shown in Fig. 13. The perspective looking down onto the x_2-x_3 plane, given in Fig. 14, confirms that the second component of the state indeed converges to zero, since the various trajectories all end in the corresponding segment of the x_3-axis.

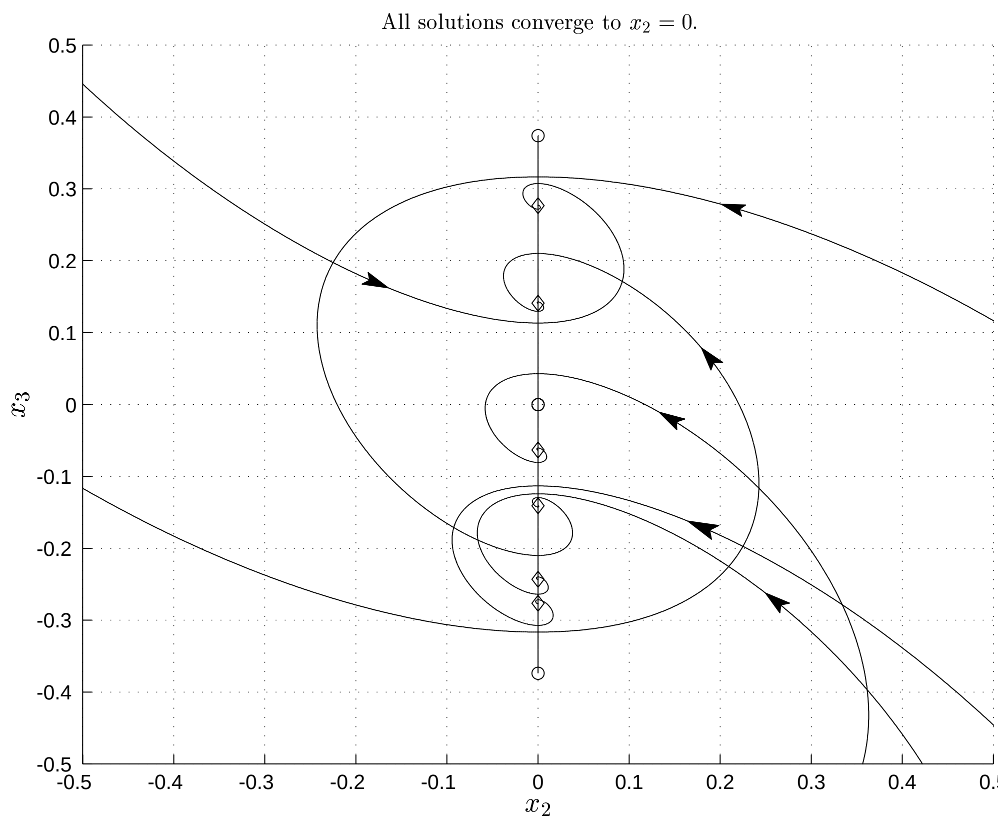


Fig. 14: As viewed in the x_2-x_3 plane, the set E appears as a segment of the x_3-axis.

8 Conclusions

This paper establishes absolute stability criteria for systems with multiple hysteresis and slope-restricted nonlinearities. Using Popov's indirect control form as an analytical framework, a Lyapunov stability proof is developed to guarantee stability for these two classes of nonlinear systems. The analysis for the two different cases is effectively unified by introducing a transformation that converts either nonlinearity into a passive operator. In the hysteresis case, the Lyapunov function includes a integral term that is dependent on the nonlinearity input-output path, while the corresponding Lyapunov term for the memoryless nonlinearity is not. As a result of the new analysis, early work performed by Yakubovich for a scalar hysteresis is extended to handle multiple nonlinearities, and recent work on multiple slope-restricted nonlinearities is further generalized. The stability guarantee is presented in terms of a simple linear matrix inequality (LMI) in the given system matrices, and a certain residue matrix condition that must be satisfied. Asymptotic stability is with respect to a subset of state space that contains all equilibrium positions of the nonlinear system. Descriptions of these stationary sets for several common hysteresis types are given in detail. Simple numerical examples are then used to demonstrate the effectiveness of the new analysis in comparison to other recent results, and graphically illustrate state asymptotic stability. By contrast to the previous work, our analysis allows for non-strictly proper systems and, except for trivial cases such as a diagonal system matrix, the stability multiplier is allowed to be more general and leads to less conservative stability predictions.

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