Strategic Asset Allocation with Illiquid Alternatives

Eric Sager Luxenberg
Stanford University
Stanford, California, USA
BlackRock
Palo Alto, California, USA
ericlux@stanford.edu

Stephen Boyd
Stanford University
Stanford, California, USA
BlackRock
Palo Alto, California, USA
boyd@stanford.edu

Misha van Beek
BlackRock
New York, New York, USA
misha.vanbeek@blackrock.com

Wen Cao
BlackRock
New York, New York, USA
wen.cao@blackrock.com

Mykel Kochenderfer
Stanford University
Stanford, California, USA
BlackRock
Palo Alto, California, USA
mykel@stanford.edu

ABSTRACT

We address the problem of strategic asset allocation (SAA) with portfolios that include illiquid alternative asset classes. The main challenge in portfolio construction with illiquid asset classes is that we do not have direct control over our positions, as we do in liquid asset classes. Instead we can only make commitments; the position builds up over time as capital calls come in, and reduces over time as distributions occur, neither of which the investor has direct control over. The effect on positions of our commitments is subject to a delay, typically of a few years, and is also unknown or stochastic. A further challenge is the requirement that we can meet the capital calls, with very high probability, with our liquid assets.

We formulate the illiquid dynamics as a random linear system, and propose a convex optimization based model predictive control (MPC) policy for allocating liquid assets and making new illiquid commitments in each period. Despite the challenges of time delay and uncertainty, we show that this policy attains performance surprisingly close to a fictional setting where we pretend the illiquid asset classes are completely liquid, and we can arbitrarily and immediately adjust our positions. In this paper we focus on the growth problem, with no external liabilities or income, but the method is readily extended to handle this case.

CCS CONCEPTS

• **Applied computing** → **Multi-criterion optimization and decision-making**.

KEYWORDS

convex optimization, illiquid alternatives, model predictive control, endowment model

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1 INTRODUCTION

There is considerable investor interest across several financial contexts in constructing portfolios which mix liquid and illiquid assets, especially illiquid alternative investments. We wish to perform strategic asset allocation to asset classes that include illiquid alternative assets, as well as more liquid asset classes. Several challenges arise. First, we can only augment our illiquid positions by making capital commitments. Moreover, these commitments only indirectly affect our illiquid position through uncertain and delayed capital calls, that we have no direct control over. A further challenge is the solvency requirement: we should be able to fund the capital calls from our liquid positions with very high probability. A simple strategy to guarantee coverage of capital calls is to keep an amount equal to the uncalled capital commitments in cash. However this creates significant cash drag, since this cash could be invested in higher returning liquid assets. The method we describe in this paper addresses all of these issues.

2 PREVIOUS WORK

There is a rich history of studying portfolio construction. Our work helps extend the modern portfolio theory framework developed by Markowitz [15] and Merton [16], which focuses on liquid assets. We contribute to the further study of illiquidity and multi-period planning. While this work takes as an input a stochastic model which describes the risk and return of illiquid investments, calibrating such models is a nuanced and well studied problem. For a guide to the literature on the risks and returns of private equity investments, see Kortweg [13].

Continuous time. There is a breadth of work on modeling portfolio construction with illiquid assets. Many authors consider continuous time stochastic processes. Dimmock et al. study the endowment model, under which university endowments hold high allocations

in illiquid alternative assets, via a continuous time dynamic choice model with deterministic-in-time discrete liquidity shocks every T periods [7]. They allow the investor to increase the position in the illiquid asset instantaneously, not modeling the delayed nature of capital calls. Ang et al. also study a continuous time problem, but model the timing of liquidity events of the illiquid asset as an independent Poisson process [1]. Optimal solutions are assumed to have almost surely non-negative liquid wealth, meaning that the investor must always be able to cover the effects of illiquidity. Sorensen, Wang, and Yang [18] study the commitment risk of a fixed alternative's commitment by focusing on an investor who can modify their positions in stocks and bonds, taking an investment in an illiquid asset as given and held to maturity.

Discrete time. The discrete time case is also well studied. Takahashi and Alexander first introduced what amounts to a deterministic linear system to model an illiquid asset's calls, distributions, and asset value [19]. This model posits that calls are a time-varying fraction of uncalled commitments, and that distributions are a time-varying fraction of the illiquid asset value, and returns are constant. Our model is similar, but differs in two important ways. First, our model is time-invariant. Second our model incorporates randomness in these fractions as well as the returns. Giommetti and Sorensen use the Takahashi and Alexander model in a standard, discrete-time, infinite-horizon, partial-equilibrium portfolio model to determine optimal allocation to private equity [9]. Here the calls and distributions are deterministic fractions of the uncalled commitments and illiquid asset value, but the illiquid asset value grows with stochastic returns.

Optimal allocation to illiquid assets. Across the literature we have reviewed, the reported optimal allocations to illiquid assets are strikingly low compared to the de facto wants and need of institutional investors who are increasingly allocating larger and larger weights to illiquid alternatives. In their extensive survey of Illiquidity and investment decisions, Tédongap and Tafolong [20] report that recommended illiquid allocations range from the low single digits to around 20% on the upper end, much lower than the target levels observed in practice. For example, the National Association of College and University Business Officers (NACUBO) provide data showing the allocation weights of illiquid alternatives in University endowments reaching 52% in 2010. Unlike other analyses, our method does not require investors be able to cover calls with probability one, and instead provides a tool for maintaining an optimized target asset allocation under uncertain calls, distributions, returns, and growth.

Hayes, Primbs, and Chiquoine propose a penalty cost approach to asset allocation whereby an additional term is added to the traditional mean-variance optimization (MVO) problem to compensate for the introduction of illiquidity [11]. They solicit a user provided marginal cost curve which captures the return premium needed for an illiquid asset to be preferred over a theoretically equivalent liquid alternative. This leads to a formulation nearly identical to the standard MVO problem, with a liquidity-adjusted expected return (a function of the allocation). In their work the notion of liquidity is captured in a scalar between 0 and 1.

Multi-period optimization. Our policy is based on solving a multi-period optimization problem. Dantzig and Infanger [6] introduce a multi-stage stochastic linear programming approach to multi-period portfolio optimization. Mulvey, Pauling, and Madey survey the advantages of multi-period portfolio models, including the potential for variance reduction and increased return, as well as the ability to analyze the probability of achieving or missing goals [17]. Boyd et al. [3] describe a general framework for multi-period convex optimization. This framework focuses on planning a sequence of trades over a set of periods trades given return forecasts, trading costs, and holding costs. Our framework also solves a multi-period convex optimization problem, but we do not make an approximation of the dynamics, which is more appropriate for the longer time horizons and thus more significant growth observed in strategic asset allocation.

Model predictive control. Our method falls under the category of Model Predictive Control (MPC), which is both widely studied in academia and used in industry. For a survey of MPC, see for example the books *Model Predictive Control* [5] or García et al. [8]. Herzog et al. [12] use an MPC approach for multi-period portfolio optimization, but only consider normally distributed returns and standard liquid assets. They do include a factor model of returns, as well as a conditional value at risk (CVaR) constraint which is different in interpretation but takes the same form as our insolvency constraint. The closest work we have identified to our own is the thesis of Lee, who uses a very similar multi-period optimization problem with linear illiquid dynamics [14]. We both use a quadratic risk, and use certainty equivalent planning to solve an open loop control problem. Lee's problem is multi-period, but the objective is a function of only the final period wealth, whereas in our case we have stage costs, as well as constraints on the solvency of our portfolio. Additionally, in our stochastic model we use random call and distribution intensities.

Contributions. The linear dynamics of the illiquid wealth motivate model predictive control (MPC) as a solution method. To the best of our knowledge, there do not exist multi-period optimization-based policies for constructing portfolios with both liquid and illiquid alternative assets. We believe our contributions include incorporating random intensities with the classic linear model of the illiquid asset's calls and distributions, formulating a multi-period optimization problem to perform strategic asset allocation with liquid and illiquid assets, using liquidity/insolvency constraints to ensure calls are covered with high probability, and obtaining a performance bound for the problem by considering a stylized liquid world where the illiquid asset is completely liquid.

3 STOCHASTIC DYNAMIC MODEL FOR AN ILLIQUID ASSET

In this section we describe our stochastic dynamic model of one illiquid asset. Our model is closely related to the linear system proposed by Takahashi and Alexander [19], with the addition of uncertainty in the capital calls and distributions. We note a straightforward extension of our model which would include the Takahashi model in §7. We consider a discrete-time setting, with period denoted by $t = 1, 2, 3 \ldots$, which could represent months, quarters, years, or

any other period. Our model involves the following quantities, all denominated in dollars.

- $I_t \geq 0$ is the illiquid wealth (or position in or NAV of the illiquid asset) at period t.
- $K_t \geq 0$ is the total uncalled commitments at period t.
- $C_t \geq 0$ is the capital call at period t.
- $D_t \geq 0$ is the distribution at period t.
- $n_t \geq 0$ is the amount newly committed to the illiquid asset at period t.

The commitment n_t is the only variable we can directly control or choose. All the others are affected indirectly by n_t.

Dynamics. Here we describe how the variables evolve over time. At period t,

- we make a new capital commitment n_t (which we can choose)
- we receive capital call C_t (which is not under our control)
- we receive distribution D_t (which is not under our control)

The uncalled commitment in period $t+1$ is

$$K_{t+1} = K_t + n_t - C_t,$$

and the illiquid wealth in period $t+1$ is

$$I_{t+1} = I_t R_t + C_t - D_t,$$

where $R_t \geq 0$ is a random total return on the illiquid asset.

Calls and distributions. We model calls and distributions as random fractions of K_t, I_t, and n_t. We model calls as

$$C_t = \lambda_t^0 n_t + \lambda_t^1 K_t,$$

where $\lambda_t^0 \in [0,1]$ is the random immediate commitment call intensity and $\lambda_t^1 \in [0,1]$ is the random existing commitment call intensity. Similarly, we model distributions as

$$D_t = I_t R_t \delta_t,$$

where $\delta_t \in [0,1]$ is the random distribution intensity.

We assume the random variables $(R_t, \lambda_t^0, \lambda_t^1, \delta_t) \in \mathbf{R} \times [0,1]^3$ are I.I.D., i.e., independent across time and identically distributed. (But for fixed period t, the components R_t, λ_t^0, λ_t^1, and δ_t need not be independent.) We do not know these random variables when we choose the current commitment n_t. Formally, we assume that $n_t \perp\!\!\!\perp (R_t, \lambda_t^0, \lambda_t^1, \delta_t)$. The current commitment can depend on anything known at the beginning of period t (including for example past values of returns and intensities), but the current period return and intensities are independent of the commitment.

3.1 Stochastic linear system model

The model above can be expressed as a linear dynamical system with random dynamics and input matrices. With state $x_t = (I_t, K_t) \in \mathbf{R}^2$ and the control or input $u_t = n_t \in \mathbf{R}$, the dynamics are given by

$$x_{t+1} = A_t x_t + B_t u_t,$$

where

$$A_t = \begin{bmatrix} R_t(1-\delta_t) & \lambda_t^1 \\ 0 & 1-\lambda_t^1 \end{bmatrix}, \qquad B_t = \begin{bmatrix} \lambda_t^0 \\ 1-\lambda_t^0 \end{bmatrix}. \tag{1}$$

With output $y_t = (I_t, K_t, C_t, D_t) \in \mathbf{R}^4$, we have

$$y_t = F_t x_t + G_t u_t,$$

where

$$F_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \lambda_t^1 \\ R_t\delta_t & 0 \end{bmatrix}, \qquad G_t = \begin{bmatrix} 0 \\ 0 \\ \lambda_t^0 \\ 0 \end{bmatrix}. \tag{2}$$

We assume the initial state is known. We observe that $x_t \perp\!\!\!\perp (F_t, G_t)$, since the former depends on the initial state, n_t, and (F_τ, G_τ) for $\tau < t$, and these are all independent of (F_t, G_t).

A careful reader might notice that these linear dynamics mean that the commitments and distributions asymptotically approach zero but never terminate. However, the fractions of calls and distributions relative to the initial amounts are minuscule after several periods, and are negligible in the presence of new commitments coming in each period. Additionally, Gupta and Van Nieuwerburgh [10] found in analyzing long-term private equity behavior that often funds have activity even fifteen years after inception, further justifying the persisting calls and distributions in the linear systems model.

3.2 Mean dynamics

Let $\overline{x}_t = \mathbf{E}x_t$ denote the mean of the state, $\overline{u}_t = \mathbf{E}u_t$ denote the mean of the input or control, and $\overline{y}_t = \mathbf{E}y_t$ denote the mean of the output. We define the mean matrices

$$\overline{A} = \mathbf{E}A_t, \qquad \overline{B} = \mathbf{E}B_t, \qquad \overline{F} = \mathbf{E}F_t, \qquad \overline{G} = \mathbf{E}G_t$$

(which do not depend on t). We then have

$$\overline{x}_{t+1} = \overline{A}\overline{x}_t + \overline{B}\overline{u}_t, \qquad \overline{y}_t = \overline{F}\overline{x}_t + \overline{G}\overline{u}_t, \tag{3}$$

which states that the mean state and output is described by the same linear dynamical system, with the random matrices replaced with their expectations. The mean dynamics is a time-invariant deterministic linear dynamical system.

3.3 A particular return and intensity distribution

We suggest the following parametric joint distribution for the random vector $(\lambda_t^1, \lambda_t^0, \delta_t, R_t)$. They are generated from a random 3-vector

$$z_t \sim \mathcal{N}(\mu, \Sigma) \in \mathbf{R}^3. \tag{4}$$

From these we obtain

$$\lambda_t^1 = \frac{1}{1+\exp(z_t)_1}, \ \lambda_t^0 = \frac{1}{2}\lambda_t^1, \ \delta_t = \frac{1}{1+\exp(z_t)_2}, \ R_t = \exp(z_t)_3. \tag{5}$$

With this model, the return is log-normally distributed while the call and distribution intensities are logit-normally distributed. Dependency among the return and the intensities are modeled by the off-diagonal entries of Σ.

3.4 Example

Here we describe a particular instance of the distribution described above, that we will use in various numerical examples in the sequel.

Example return and intensity distribution. In this example we use the following parameters for the distribution of $(\lambda_t^1, \lambda_t^0, \delta_t, R_t)$ specified in (4):

$$\mu = \begin{bmatrix} -0.700 \\ -0.423 \\ 0.158 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 0.068 & 0.072 & 0.006 \\ 0.073 & 0.271 & 0.043 \\ 0.006 & 0.043 & 0.079 \end{bmatrix}. \tag{6}$$

This example is based on yearly periods. The mean return of the illiquid asset is derived from the BlackRock Capital Market Assumptions as of July 2021, which reports one private equity asset, Buyout, with a mean annual return of 15.8% [2]. The call and distribution mean intensities are calibrated from private equity data for the eFront Buyout fund. The mean values of the intensities are we report the empirical means

$$\overline{\lambda}_t^1 = .26, \qquad \overline{\lambda}_t^0 = .128, \qquad \overline{\delta}_t = .33.$$

(These are found by Monte Carlo simulation, since the mean of a logit-normal distribution doesn't have an analytical expression.) The covariance matrix is calibrated from the same data.

3.5 Comparison with the Takahashi and Alexander model

Our stochastic model of an illiquid asset is closely related to that of Takahashi and Alexander [19], but it differs in to key ways. The most important difference is that our model is Markovian; the calls, distributions, and returns at time t are conditionally independent of the all previous quantities, given the state at time t. In comparison, Takahashi and Alexander's model specifies time varying call and distribution intensity parameters. These time varying intensities mean that the final intensities can be set to 1, meaning calls and distributions can have deterministic end times, and the exposure will not geometrically decline. In §7 we describe how to modify our model to depend on arbitrarily many previous time periods. This means we can exactly capture the original Takahashi and Alexander model with this extension of our model. We emphasize that this generalization remains fully tractable from the portfolio optimization standpoint described in this paper. The second difference between our model and that of Takahashi and Alexander is that ours is a stochastic model, with random intensities, whereas theirs is deterministic.

4 JOINT LIQUID AND ILLIQUID MODEL

We now describe a model for an investment universe consisting of multiple illiquid alternative and liquid assets. First, we extend to a universe of n^{ill} illiquid assets.

Multiple illiquids. We extend the same quantities as in §3 from scalars to vectors of dimension n^{ill}.

$$K_t, I_t, C_t, D_t \in \mathbf{R}^{n^{\text{ill}}}, \qquad n_t \in \mathbf{R}^{n^{\text{ill}}}, \qquad R_t^{\text{ill}}, \lambda_t^1, \lambda_t^0, \delta_t \in \mathbf{R}^{n^{\text{ill}}}.$$

We have the exact same dynamics as before, duplicated for each illiquid asset. Each has its own states for exposure and uncalled commitment, and its own control for its new commitments. The illiquid calls, distributions, and returns are now part of a joint distribution. The illiquid dynamics extend in vectorized form to

$$K_{t+1} = K_t + n_t - C_t, \qquad I_{t+1} = \mathbf{diag}(R_t) I_t + C_t - D_t,$$

with

$$C_t = \mathbf{diag}(\lambda_t^0) n_t + \mathbf{diag}(\lambda_t^1) K_t, \qquad D_t = \mathbf{diag}(R_t)\, \mathbf{diag}(\delta_t) I_t.$$

We emphasize that while the return, call, and distribution dynamics here are separable across the illiquid assets, the underling random variables $((R_t)_j, (\lambda_t^0)_j, (\lambda_t^1)_j, (\delta_t)_j)$ can be modeled jointly. We continue with our assumption that these random variables are independent across time.

Multiple liquids. There are now a set of n^{liq} liquid assets available to us. The liquid assets are simple: we can buy and sell them at will at each period; they suffer none of the complex dynamics of the illiquid assets. We add one new state, L_t, the (total) liquid wealth at period t. In addition to new commitments for each illiquid asset, at each time t we now control how we allocate our liquid wealth each period, as well as how much outside cash to inject into our liquid wealth. Thus we have the additional quantities, which we can control:

- $h_t \geq 0$ ($\in \mathbf{R}^{n^{\text{liq}}}$) is the allocation in dollars invested in liquid assets at period t
- $s_t \geq 0$ ($\in \mathbf{R}$) is the outside cash injected at period t

At the beginning of period t, we invest (or allocate) our liquid wealth in liquid assets. This corresponds to the constraint $L_t = \mathbf{1}^T h_t$. We receive multiplicative liquid returns $(R_t^{\text{liq}})_j \in \mathbf{R}$ on liquid asset j, yielding total return $h_t^T R_t^{\text{liq}}$. We then pay out capital calls from and receive distributions to our liquid wealth, for all illiquid assets. This corresponds to a net increase in liquid wealth given by $-\mathbf{1}^T C_t + \mathbf{1}^T D_t$. Lastly, if at this stage our liquid wealth is negative, we are forced to add outside cash s_t to at least bring our liquid wealth to zero. Compactly, the liquid dynamics are

$$L_{t+1} = h_t^T R_t^{\text{liq}} - \mathbf{1}^T C_t + \mathbf{1}^T D_t + s_t,$$

with constraints

$$h_t, n_t, s_t \geq 0, \qquad L_t \geq 0, \qquad L_t = \mathbf{1}^T h_t.$$

4.1 Stochastic linear system model

We can again represent these dynamics as a stochastic linear system. Let $x_t = (L_t, I_t, K_t) \in \mathbf{R}^{1+2n^{\text{ill}}}$ be the state vector. The control is $u_t = (h_t, n_t, s_t) \in \mathbf{R}^{1+n^{\text{liq}}+n^{\text{ill}}}$. Extending the A and B matrices from §1, define

$$A_t = \begin{bmatrix} 0 & (\delta_t \circ R_t^{\text{ill}})^T & -{\lambda_t^1}^T \\ 0 & \mathbf{diag}((\mathbf{1} - \delta_t) R_t^{\text{ill}}) & \mathbf{diag}(\lambda_t^1) \\ 0 & \mathbf{diag}(\mathbf{0}) & \mathbf{diag}(\mathbf{1} - \lambda_t^1) \end{bmatrix} \tag{7}$$

$$B_t = \begin{bmatrix} {R_t^{\text{liq}}}^T & -{\lambda_t^0}^T & 1 \\ \mathbf{0}^T & \mathbf{diag}(\lambda_t^0) & 0 \\ \mathbf{0}^T & \mathbf{diag}(\mathbf{1} - \lambda_t^0) & 0 \end{bmatrix}. \tag{8}$$

Then the random linear dynamics with multiple illiquids and liquids are

$$x_{t+1} = A_t x_t + B_t u_t,$$

with constraints

$$h_t, n_t, s_t \geq 0, \qquad L_t \geq 0, \qquad L_t = \mathbf{1}^T h_t.$$

The presence of the outside cash control s_t implies that a feasible control exists for any feasible value of the states, since s_t prevents the liquid wealth from ever being negative.

As in §3.2, we let $\overline{x}_t = \mathbf{E}x_t$ denote the mean of the state, $\overline{u}_t = \mathbf{E}u_t$ denote the mean of the input or control, define the mean system matrices as

$$\overline{A} = \mathbf{E}A_t, \qquad \overline{B} = \mathbf{E}B_t,$$

and recover the same mean dynamics

$$\overline{x}_{t+1} = \overline{A}\overline{x}_t + \overline{B}\overline{u}_t. \tag{9}$$

4.2 Return and intensity distribution

We extend the previous generative model specified in (4) and (5) to include liquid returns,

$$z_t = \begin{bmatrix} z_t^{\text{int}} \\ z_t^{\text{ret}} \end{bmatrix} \sim \mathcal{N}(\mu, \Sigma) \in \mathbf{R}^{3n^{\text{ill}}+n^{\text{liq}}}, \ \mu = \begin{bmatrix} \mu^{\text{int}} \\ \mu^{\text{ret}} \end{bmatrix}, \ \Sigma = \begin{bmatrix} \Sigma_{\text{int}} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{\text{ret}} \end{bmatrix}. \tag{10}$$

From these we obtain delayed and immediate call intensities

$$\lambda_t^1 = \frac{1}{1+\exp(z_t)_{1:n^{\text{ill}}}}, \qquad \lambda_t^0 = \frac{1}{2}\lambda_t^1.$$

The distribution intensities and returns are, respectively

$$\delta_t = \frac{1}{1+\exp+(z_t)_{n^{\text{ill}}+1:2n^{\text{ill}}}}, \quad \begin{bmatrix} R_t^{\text{ill}} \\ R_t^{\text{liq}} \end{bmatrix} = \begin{bmatrix} \exp(z_t)_{2n^{\text{ill}}+1:3n^{\text{ill}}} \\ \exp(z_t)_{3n^{\text{ill}}:} \end{bmatrix}.$$

5 STRATEGIC ASSET ALLOCATION UNDER THE RELAXED LIQUID MODEL

In this section we introduce a highly simplified model, where all of the challenges of illiquid alternative assets are swept under the rug. This model is definitely not realistic, but we can use it to develop an unattainable benchmark for performance that can be obtained with the more accurate model.

5.1 Relaxed liquid model

As a thought experiment, we imagine the illiquid assets are completely liquid: we have arbitrary control of illiquid asset positions (immediate increase or decrease). This is a relaxation of the actual problem setting, where we must face stochastic and only indirectly controllable calls and distributions. The idea of a relaxed liquid model is not new; for example, Giommetti et al. [9] consider the target allocations resulting from treating illiquid assets as fully liquid for comparison, but do not evaluate stochastic control policies trying to achieve these allocations in an illiquid world. The relaxed liquid model is also implicitly behind various Captial Market Assumptions, where return ranges, and correlations, are given for both liquid and illiquid assets.

The relaxed liquid model is very simple. There is only one state, the total wealth W_t. The quantities we have control over are the allocations to liquid and illiquid assets, denoted $h_t^{\text{liq}} \in \mathbf{R}^{n^{\text{liq}}}$ and $h_t^{\text{ill}} \in \mathbf{R}^{n^{\text{ill}}}$. The wealth evolves according to the dynamics

$$W_{t+1} = u_t^T r_t, \qquad u_t^T \mathbf{1} = W_t, \qquad u = \begin{bmatrix} h_t^{\text{liq}} \\ h_t^{\text{ill}} \end{bmatrix},$$

where $r_t = z_t^{\text{ret}}$ is defined in (10).

We use the standard trick of working with the weights of the allocations in each period, denoted w_t, instead of u_t. This is defined as $w_t = u_t/W_t$, so $\mathbf{1}^T w_t = 1$. We recover the dollar allocations as $u_t = W_t w_t$.

5.2 Markowitz allocation and policy

A standard way to choose a portfolio allocation is to solve the one period risk-constrained Markowitz problem,

$$\begin{array}{ll} \text{maximize} & \mu^T w \\ \text{subject to} & \mathbf{1}^T w = 1, \quad w \geq 0 \\ & \|\Sigma^{1/2} w\|_2 \leq \sigma, \end{array} \tag{11}$$

where σ is the maximum tolerable return standard deviation, and μ and Σ are the expected return and return covariance, respectively. We denote the optimal allocation as $w^\star$. The natural policy associated with solving the Markowitz problem simply rebalances to $w^\star$: it sets $u_t = W_t w^\star$ for each period t. This simple rebalancing is of course not possible under the accurate model that includes the challenges of alternative assets, but it is under the relaxed liquid model.

5.3 Example

Liquid performance. Under the assumptions of §5.1, we solve the one period Markowitz problem with $\mu_{\text{ret}}, \Sigma_{\text{ret}}$ the mean and covariance of the joint distribution of liquid, illiquid asset returns. Using this relaxed Markowitz target, we simulate the fantasy performance achieved by being able to perfectly rebalance both liquids and illiquids to the Markowitz target each period, for multiple periods, using the policy described earlier in 5.2.

For the parameters defined in (10), we use the specific values of

$$\mu_{\text{ret}} = (0.158,\ 0.000,\ 0.072,\ 0.023,\ 0.036,\ 0.046)\,, \tag{12}$$

$$\sigma_{\text{ret}} = (0.281,\ 0.000,\ 0.206,\ 0.046,\ 0.047,\ 0.162)\,, \tag{13}$$

and

$$C_{\text{ret}} = \begin{bmatrix} 1.000 & 0.000 & 0.422 & -0.298 & -0.002 & 0.261 \\ 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.422 & 0.000 & 1.000 & -0.843 & 0.197 & 0.800 \\ -0.298 & 0.000 & -0.843 & 1.000 & -0.018 & -0.739 \\ -0.002 & 0.000 & 0.197 & -0.018 & 1.000 & 0.628 \\ 0.261 & 0.000 & 0.800 & -0.739 & 0.628 & 1.000 \end{bmatrix}, \tag{14}$$

with $\Sigma_{\text{ret}} = \mathbf{diag}(\sigma_{\text{ret}}) C_{\text{ret}} \mathbf{diag}(\sigma_{\text{ret}})$. The liquid return mean and covariance matrix are gathered from the BlackRock Capital Market Assumptions for equities as of July 2021 [2]. The corresponding expected returns are

$$\overline{R}_t^{\text{liq}} = (1.218,\ 1.098,\ 1.024,\ 1.038,\ 1.061,\ 1.000)\,,$$

where the last asset is cash.

Risk-return trade-off. By solving the Markowitz problem with these parameters across a range of values for the risk tolerance σ (which give rise to corresponding Markowitz targets), we can create a risk-return trade-off plot, shown in figure 1 as the "performance ceiling." We should consider this trade-off curve as an unattainable performance benchmark, that we can only strive to attain when the challenges of illiquid alternatives are present.

6 STRATEGIC ASSET ALLOCATION WITH FULL ILLIQUID DYNAMICS

In §5, we describe an approach to strategic asset allocation for portfolios including an imagined class of illiquid alternatives which are rendered completely liquid. In this section, we provide methods to perform strategic asset allocation with mixed liquid and illiquid alternative portfolios where we can only augment our illiquid position by making new commitments, and the effect of this action is random and delayed. First, we describe a method which over time establishes and then maintains a given target allocation under growth. Then, we describe a more sophisticated MPC method which jointly selects a target allocation based on a user's risk tolerance, establishes the target, and maintains the target in growth.

6.1 Steady-state commitment policy

We first describe a simple policy, which seeks to track a target allocation θ^{targ}. It allocates liquid assets proportionate to its desired liquid allocation, and makes new commitments of a target level of illiquid wealth scaled by the asymptotic expected private response to constant commitment. The input is a target allocation θ^{targ}, current liquid wealth L and illiquid wealth I. First, the policy checks if L is negative. If it is, it returns control

$$u = (h, n, s), \qquad h = 0, \quad n = 0, \quad s = |L|.$$

Otherwise, if the liquid wealth is positive, the policy proceeds as follows. First, the policy rebalances the liquid holdings proportionately to θ^{targ},

$$h = L\frac{\theta^{\text{liq}}}{1^T\theta^{\text{liq}}},$$

where $\theta^{\text{liq}}, \theta^{\text{ill}}$ are the liquid and illiquid blocks of the allocation vector $\theta^{\text{targ}} = \begin{bmatrix}\theta^{\text{liq}} \\ \theta^{\text{ill}}\end{bmatrix}$, respectively. Then, with α_I as the 1 dollar private commitment step response (the steady-state level of the illiquid asset resulting from a constant unit commitment $n_1 = 1, n_2 = 1, \ldots$), and I^{targ} as the target illiquid level, $I^{\text{targ}} = \theta^{\text{ill}}(L + I)$, the policy commits

$$n_i = \frac{I_i^{\text{targ}}}{\alpha_{I_i}}$$

and returns control $u = (h, n, 0)$.

6.2 Model predictive control policy

We now describe a more sophisticated policy which plans ahead based on a model of the future, seeking to maximize wealth subject to various risk constraints. For a sequence of prospective actions, the policy forecasts future state variables using the mean dynamics described in (9). The policy then chooses a sequence of actions by optimizing an objective which depends on the planned actions and forecast states. Finally, the policy executes solely the first step of the planned sequence. The impact of that action is observed, and the resulting state is observed, and then this cycle repeats.

The policy selects a planned sequence of actions by trying to maximize the ultimate total liquid and illiquid wealth. However, it is also constrained by a user's risk tolerance, which caps the allowable per period return volatility. Additionally, because capital calls are stochastic in nature, the policy seeks to guarantee that with high probability, all capital calls can be funded from the liquid wealth.

Modified Markowitz constraint. Motivated by the standard one period risk-constrained Markowitz problem (11), we would like to include a risk constraint in our planning problem. However, the Markowitz problem has variables in weight space rather than wealth space. Other multi-period optimization problems based on the Markowitz problem, such as in [3], assume a timescale over which the wealth does not grow significantly over the planning horizon. In our case, since potential application contexts include endowments and insurers, we must handle substantial growth over the investment horizon. Thus, we consider an analogous risk constraint in wealth space rather than weight space,

$$\frac{y^T\Sigma y}{(\mathbf{1}^T y)^2} \le \sigma^2 \iff \|\Sigma^{1/2}y\|_2 \le \sigma \mathbf{1}^T y.$$

$y = (h, I)$ is the liquid and illiquid exposure. Thus, we use the constraint

$$\|\Sigma^{1/2}y\|_2 \le \sigma \mathbf{1}^T y,$$

which is invariant in wealth. It is also convex, which means that problems with such constraints can be reliably solved.

Insolvency constraint. An important challenge in performing strategic asset allocation with illiquid alternatives is ensuring that the probability of being unable to pay a capital call is extremely low. In our model, this corresponds to requiring

$$P(W_{t+1} < 0 \mid X_t, n_t, h_t) \le \epsilon^{\text{ins}}$$

for a small probability of failure ϵ^{ins}. We make several approximations to facilitate a convex constraint. First, we approximate R_t^{liq} as a multivariate normal random variable,

$$R_t^{\text{liq}} \sim N(\mu_{\text{liq}}, \Sigma_{\text{liq}}).$$

It is important to note that these parameters are the mean and covariance of the liquid returns, rather than the mean and covariance which parameterize the log normal liquid return distribution given by μ_{ret} and Σ_{ret} in (10). Then we assume we receive the expected calls $\bar{c}_t = \mathbf{E}[C_t \mid X_t, n_t, h_t]$, which is a linear function of our controls. They are given by $\bar{c}_t = \bar{\lambda}_t^{1,T} K_t + \bar{\lambda}_t^{0,T} n_t$. Finally, we assume pessimistically there are no distributions or outside cash. With these approximations, we have

$$\begin{aligned} P(W_{t+1} < 0 \mid X_t, n_t, h_t) &\approx P(R_t^{\text{liq}} h_t - \bar{c}_t \le 0) \\ &= P(N(h_t^T\mu_{\text{liq}} - \bar{c}_t, h_t^T\Sigma_{\text{liq}}h_t) < 0) \le \epsilon. \end{aligned}$$

This probabilistic constraint holds if and only if

$$\bar{c}_t - h_t^T\mu_{\text{liq}} \le \Phi^{-1}(\epsilon^{\text{ins}})\|\Sigma_{\text{liq}}^{1/2}h_t\|_2, \tag{15}$$

where Φ is the standard normal cumulative distribution function. This constraint is convex provided $\epsilon^{\text{ins}} \le 1/2$, since then $\Phi^{-1}(\epsilon^{\text{ins}}) \le 0$, and (15) is a second order cone constraint (see [4, §4.4.2]).

As mentioned above, the constraint (15) is pessimistic because it assumes no distribution. An alternative and less pessimistic formulation of the insolvency constraint would consider the distribution, the calls, and the liquid returns all under a joint normal approximation.

Smoothing penalty. Among control sequences with similar objective values, we would like for new commitments to be fairly smooth across time. We can consider a natural commitment smoothing penalty

$$g(n) = \sum_{t=0}^{H-1} \gamma^t \|n_{t+1} - n_t\|^2$$

The time discount γ appears because in a growth context we expect n_t to increase over time. Additionally, it helps account for the increased uncertainty of future planned steps.

MPC planning problem. All objective terms and constraints outlined above are consolidated into one optimization problem. At time t, we plan $\{\hat{x}_{\tau|t}\}_{\tau=t}^{t+H+1}$, $\{\hat{u}_{\tau|t}\}_{\tau=t}^{t+H}$, where H is the planning horizon, by solving the optimization problem

$$\begin{array}{ll} \text{maximize} & \sum_{\tau=t}^{t+H} \gamma^t \left(\hat{L}_{\tau|t} + \mathbf{1}^T \hat{I}_{\tau|t} - \lambda^{\text{cash}} \hat{s}_{\tau|t}\right) - \lambda^{\text{smooth}} g(\hat{n}_{\cdot|t}) \\ \text{subject to} & \hat{x}_{t|t} = x_t \\ & \hat{x}_{\tau+1|t} = \overline{A}\hat{x}_{\tau|t} + \overline{B}\hat{u}_{\tau|t} \\ & \hat{L}_{\tau|t} \geq 0 \\ & \hat{h}_{\tau|t}, \hat{n}_{\tau|t}, \hat{s}_{\tau|t} \geq 0 \\ & \mathbf{1}^T \hat{h}_{\tau|t} = \hat{L}_{\tau|t} \\ & \|\Sigma^{1/2} \hat{y}_{\tau|t}\|_2 \leq \sigma \mathbf{1}^T \hat{y}_{\tau|t} \\ & \overline{\lambda}^{1,T} \hat{K}_\tau + \overline{\lambda}^{0,T} \hat{n}_{\tau|t} - \hat{h}_{\tau|t}^T \mu_{\text{liq}} \leq \Phi^{-1}(\epsilon^{\text{ins}}) \|\Sigma_{\text{liq}}^{1/2} \hat{h}_{\tau|t}\|_2, \end{array} \tag{16}$$

where $\lambda^{\text{cash}} > 0$ is a hyperparameter penalizing outside cash use, and $\tau = t, \ldots, T+H$ for all terms except L, where $\tau = t, \ldots, t+H+1$. Recall that L, I, and K are components of x, and h, n, and s are components of u.

6.3 Example

In this example, we evaluate the performance of the two policies described in §6.1 and §6.2 using the risk return trade off. For the parameters defined in (10), we use the specific values of

$$\mu_{\text{ret}} = (0.158,\ 0.000,\ 0.072,\ 0.023,\ 0.036,\ 0.046)\,, \tag{17}$$

$$\sigma_{\text{ret}} = (0.281,\ 0.000,\ 0.206,\ 0.046,\ 0.047,\ 0.162)\,, \tag{18}$$

$$C_{\text{ret}} = \begin{bmatrix} 1.000 & 0.000 & 0.422 & -0.298 & -0.002 & 0.261 \\ 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.422 & 0.000 & 1.000 & -0.843 & 0.197 & 0.800 \\ -0.298 & 0.000 & -0.843 & 1.000 & -0.018 & -0.739 \\ -0.002 & 0.000 & 0.197 & -0.018 & 1.000 & 0.628 \\ 0.261 & 0.000 & 0.800 & -0.739 & 0.628 & 1.000 \end{bmatrix}, \tag{19}$$

with $\Sigma_{\text{ret}} = \mathbf{diag}(\sigma_{\text{ret}}) C \mathbf{diag}(\sigma_{\text{ret}})$.

Illiquid dynamics. We consider the actual illiquid world: full call/distribution random dynamics as described in §4.1. We evaluate the two policies described in §6.1 and §6.2 on the same simulated returns as the imaginary Markowitz portfolio.

Example policy specifications. In this case study, we use the steady state commitment policy with parameter $\alpha_I = 3.685$, and values of θ arising from solving the one period Markowitz problem defined in 5.2 for 30 evenly spaced values of σ between 0 and .3, with our specified return distribution parameters defined in (12–14).

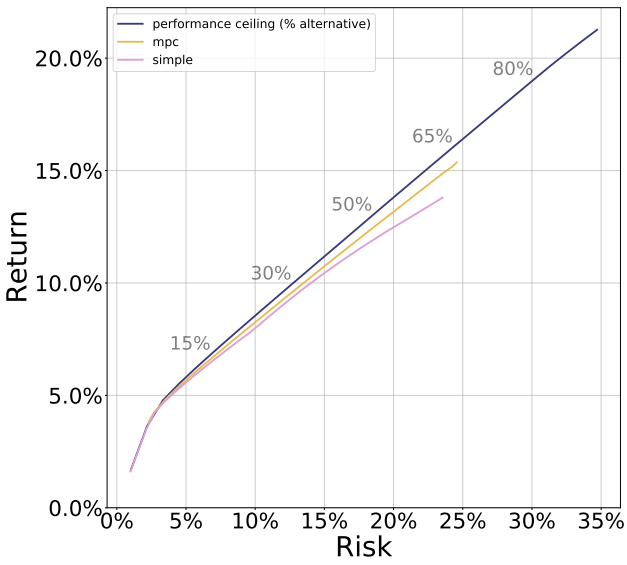

Figure 1: Risk return trade-off, 200 simulations of 20 periods.

For the MPC policy, we use the same σ values described above, but for numerical reasons use the standard trick of moving the risk limit to penalized form by subtracting

$$\lambda^{\text{risk}}(\|\Sigma^{1/2} y_t\|_2 - \sigma \mathbf{1}^T y_t)_+$$

from each term of the objective defined in (16), penalizing excess risk. The parameter values are

$$\gamma = .97,\ H = 10,\ \epsilon^{\text{ins}} = .02,\ \lambda^{\text{risk}} = 10,\ \lambda^{\text{smooth}} = .1,\ \lambda^{\text{cash}} = 1000,$$

with $\overline{A}, \overline{B}$ as defined in (9), with the distributions instantiated in (6) and (17–19).

Results. We see in figure 1 that both the MPC and heuristic policies are extremely close to the risk-return performance of the liquid relaxation, which is an unattainable benchmark. This is despite the challenging illiquid dynamics we face in the non-relaxed setting. The performance stated here is averaged across 20 periods of simulation, for 200 simulated trajectories.

Across a shorter time horizon, there is a larger gap between the MPC policy and the liquid performance ceiling, and also between the MPC and simple policies. This has a perfectly clear interpretation: because there is a roughly 4 period delay before peak illiquid exposure, the impact of the illiquid alternative asset's high returns is delayed. Additionally, by planning ahead, the MPC policy achieves illiquid exposure faster than the simple policy.

7 EXTENSIONS

We list a brief collection of extensions discussed in the longer paper.

- extending the model to allow for liquidation of illiquid alternatives on the secondaries market
- incorporating liabilities
- tracking target portfolio weights, including for the illiquid assets
- planning based on time-varying forecasts

- modeling separate vintages of the illiquid asset

8 CONCLUSION

We have described a flexible stochastic linear system model of liquid and illiquid alternative assets, that takes into account the dynamics of the illiquid assets and the randomness of returns, calls, and distributions. This model allows us to develop an MPC policy that in each time period chooses a liquid wealth allocation, and also new commitments to make in each alternative asset.

We compare the results of this policy with a relaxed liquid model, where we assume that all illiquid assets are fully liquid. This relaxed liquid model is easy to understand, since the challenges of alternative assets have all been swept under the rug. For the relaxed liquid model, we can work out optimal investment policies. The performance with these policies can be thought of as an unattainable benchmark, that we know we cannot achieve or beat when all the challenges of alternative investments are present.

Suprisingly, the performance of the MPC policy under the real model, with all the challenges of alternative assets, is very close to the performance of the relaxed liquid model, under an optimal policy. Roughly speaking, there isn't much room for improvement. This is a strong validation of the MPC policy.

Another interesting conclusion is that the relaxed liquid model is not as useless as one might imagine, since MPC can attain similar performance with all the challenges present. In a sense this validates reasoning based on the relaxed liquid model, where illiquid assets are treated as liquid assets. Roughly speaking, the asset manager can reason about the portfolio using the simple relaxed liquid model; feedback control with the MPC policy handles the challenges of illiquid alternative assets.

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