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Identification of Systems with Parametric and Nonparametric Uncertainty

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Abstract A method is presented for parameter set estimation for a system which contains both parametric and nonparametric uncertainty. Prior information is available about both types of uncertainty, but only the parametric type is further refined from the measured data.

1 Introduction

The problem is to transform the measured data

$$\{y, u : t = 1, \ldots, N\} \tag{1}$$

into a model structure suitable for robust control design. The system which produced the data is presumed to have the input/output disturbance-free form

$$y = Gu \tag{2}$$

where u is an applied input, y is a measured output, and G represents an uncertain linear-time-invariant discrete-time system with transfer function $G(z)$. (Depending on the context, we use z to denote either the Z-transform variable or the forward shift operator.)

Prior knowledge about G is given as follows:

- G has the structure

$$G = G_\theta(1 + \Delta W) \tag{3}$$

  where G_θ and Δ are uncertain, and where W is a known stable transfer function.

- G_θ is a parametric transfer function with a known dependence on parameters

$$\theta \in \Theta_{\text{prior}} \subset \mathbb{R}^p \tag{4}$$

- Δ is a stable transfer function which is otherwise unknown but bounded such that,

$$|\Delta(e^{j\omega})| \leq 1, \quad \forall \omega \in [-\pi, \pi] \tag{5}$$

  with ω is defined as the normalized frequency variable.

Since both the parameter set Θ_{prior} and the transfer function W are assumed known, it follows that (2)-(5) describe the prior information about the system (2). The problem we address in this paper is to reduce the uncertainty about the parameters $\theta \in \Theta_{\text{prior}}$ by taking into account the measured data (1) and the prior information (2)-(5).

The class of parametric model structures is further restricted to the standard ARMA model structure [9]:

$$\begin{array}{rcl} G_\theta & = & B_\theta/A_\theta \\ B_\theta & = & b_1 z^{-1} + \cdots + b_m z^{-m} \\ A_\theta & = & 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\ \theta^T & = & [a_1 \cdots a_n \;\; b_1 \cdots b_m] \end{array} \tag{6}$$

Previous work (see Reference list) has examined this type of problem as well as some variations, such as: estimating Δ and θ, inclusion of noise and/or disturbances, other forms for the nonparametric uncertainty, and the use of high order or nonparametric model structures. These results produce a parameter set which, under certain conditions, will contain the true parameter. These conditions are often hard to verify or require excessive computation.

The contribution in this paper is to produce a *worst-case parameter set* which, by definition, is guaranteed to contain the true parameter and is no more difficult to compute than least-squares. The computation of the set is easily made recursive and has a form similar to recursive least-squares, although this is not developed here. The methodology is also easily extensible to forms of nonparametric dynamics other than the multiplicative form(3), *e.g.* ,

$$G = G_\theta + \Delta W \quad \text{or} \quad G = \frac{B_\theta + \Delta_B W_B}{A_\theta + \Delta_A W_A}$$

The addition of disturbances, however, poses some computational problem involving nonlinear programming solutions. This is a subject of current research.

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The ideas presented here have been independently developed by Younce and Rohrs [16,17] for the disturbance free case (2) with the uncertainty structure $G = (B_\theta + \Delta W)/A_\theta$. With this choice of nonparametric dynamics the center of the worst-case set coincides with the least-squares estimate. A similar model structure is described in [7] and is used to motivate a recursive parameter estimator. In [16,17] there is also an in-depth discussion of worst-case frequency response set estimation.

The idea of finding a set of parameters consistent with the data and assumed model structure is not new. The case of parametric uncertainty with bounded energy disturbances is addressed in [13,2]. What we call here the worst-case set is referred to there as a *set-membership estimation.* A precise characterization of this set is given in [3,4] and is shown there to be an ellipsoid whose center is the least-squares estimate. Parameter set bounding procedures are also described in [1] for bounded magnitude disturbances. The contribution here and in [16,17] is to characterize the parameter set for the disturbance free case with bounded nonparametric dynamics. As we show here the set can be either an ellipsoid or an hyperboloid, depending on the measured data.

In general, there are two main limitations with this formulation for the case considered here, namely: (i) The measured data may be informative with respect to the nonparametric dynamics, and since this uncertainty type is not refined from the data, there is an inherent conservatism in the worst-case set. (ii) There is no unique worst-case set; two are described here. Both these limitations, however, can be almost completely eliminated by appropriate choice of data filters and input spectrum.

2 Worst-Case Set Estimation

In this section we describe three worst-case sets: an equation error set, an output error set, and a limit set which is data independent.

Combining (6) with (2) and (3) gives the equivalent system description

$$A_\theta y - B_\theta u = \Delta W B_\theta u \quad (7)$$

The left hand side above is the usual *equation error* [9]. Observe that $\{y, u : t = 1, \ldots, N\}$ satisfying (2) also satisfies (7) if the system is initially at rest.

To incorporate the prior information expressed by (5), we have from [7] that

$$\sup_\Delta \|\Delta x\|_{N2} = \|x\|_{N2} \quad (8)$$

where $\|x\|_{N2}^2 = \sum_{t=1}^N x(t)^2$ and $\sup_\Delta$ means the supremum over all Δ satisfying (5). Applying (8) to (7) gives the

Worst-Case Equation Error Set

$$\Theta_{\text{wc\_ee}} = \{\theta \in \mathbb{R}^p : \|A_\theta y - B_\theta u\|_{N2} \le \|B_\theta W u\|_{N2}\} \quad (9)$$

The above inequality describes a set of parameter values that depend on the data set (1) and the prior information about Δ in (5). Thus:

> *All parameter values which are consistent with the measured data and the prior information are in the set*
>
> $$\Theta_{\text{prior}} \cap \Theta_{\text{wc\_ee}} \quad (10)$$

Because of the underlying assumptions (2)-(6), the set $\Theta_{\text{wc\_ee}}$ must contain the true parameter value.

There are other worst-case sets that adhere to the governing assumptions. For example, in addition to (7), the system (2) is obviously also equivalent to

$$y - G_\theta u = \Delta W G_\theta u \quad (11)$$

In comparison with (7), the left hand side above is the usual *output error.* Applying (8) to the above gives the

Worst-Case Output Error Set

$$\Theta_{\text{wc\_oe}} = \{\theta \in \mathbb{R}^p : \|y - G_\theta u\|_{N2} \le \|W G_\theta u\|_{N2}\} \quad (12)$$

The sets $\Theta_{\text{wc\_ee}}$ and $\Theta_{\text{wc\_oe}}$ are both worst-case estimates, both contain the true parameter, and both are computable from the measured data, but, they are not necessarily the same. However, when the identification experiment is well designed, meaning an appropriate choice of test signals and data filters, the two sets are indeed very similar. (The effect of data filtering is to replace (y, u) with (Fy, Fu) where F is a filter.) One other major difference is that both sides of the inequalities in $\Theta_{\text{wc\_ee}}$ are affine in θ, whereas in $\Theta_{\text{wc\_oe}}$ they are linear fractional in θ. The former property, as will be shown here, makes it very easy to compute $\Theta_{\text{wc\_ee}}$.

Both $\Theta_{\text{wc\_ee}}$ and $\Theta_{\text{wc\_oe}}$ are potentially conservative set estimates because the particular nonparametric dynamics Δ which actually generated the data set may not achieve the bound (5). It is possible, however, to extract more information from the data set (1). Recall that (5) implies that

$$\sup_\Delta \|\Delta x\|_{k2} = \|x\|_{k2} \quad \forall k \in [1, N] \quad (13)$$

provided there are no non-zero initial conditions. Consequently, we have the worst-case set:

$$\Theta^*_{\text{wc\_ee}} = \{\theta : \|A_\theta y - B_\theta u\|_{k2} \le \|W B_\theta u\|_{k2}, \ \forall k \in [1, N]\} \quad (14)$$

This set is an *intersection* of N sets, and is obviously more informative than $\Theta_{\text{wc\_ee}}$ which is computed for only $k = N$.

The question arises as to whether there is a "limit set" corresponding to the plant uncetainty description alone. From the assumptions we have

$$\Delta(e^{j\omega}) = \frac{G(e^{j\omega}) - G_\theta(e^{j\omega})}{W(e^{j\omega})G_\theta(e^{j\omega})}, \ \ \forall\omega \in [-\pi, \pi] \tag{15}$$

Using (5) and (6) gives the

Worst-Case Limit Set

$$\Theta_{\text{wc\_lim}} = \{\theta \in \mathbb{R}^p : f_{\text{lim}}(\theta, \omega) \leq 0 \ \ \forall\omega \in [-\pi, \pi]\} \tag{16}$$

where

$$f_{\text{lim}}(\theta, \omega) = |A_\theta(e^{j\omega})G(e^{j\omega}) - B_\theta(e^{j\omega})|^2 - |W(e^{j\omega})B_\theta(e^{j\omega})|^2$$

Because $\Theta_{\text{wc\_lim}}$ depends on the true, but unknown, frequency response $G(e^{j\omega})$, it cannot be computed from the data. However, it is a useful set for analytic purposes since it defines the maximum range of θ for a given representative frequency response $G(e^{j\omega})$ and any $\Delta(e^{j\omega})$ satisfying (5). In Section 4 we show that for sufficiently large data length N, $\Theta_{\text{wc\_lim}}$ is always contained in $\Theta_{\text{wc\_ee}}$ or $\Theta_{\text{wc\_oe}}$. However, this is not necessarily the case for finite N. Section 5 contains an example of this phenomena. The reason is that the set $\Theta_{\text{wc\_lim}}$ contains equivalent *frequency respones*, that is, for any parameter in the limit set, there is a corresponding Δ satisfying (5) such that for any periodic input u the periodic part of the output remains unchanged. Thus, for N much larger than the time constants of the system, the periodic part will dominate. However, the transient can be significantly different for those choices of $\theta \in \Theta_{\text{wc\_lim}}$ and Δ satisfying (5) which leave $G(e^{j\omega})$ unchanged. Hence, for a finite data length N which just captures the transient, there can be considerably more information for extracting the parametric model. This possibility is not included in the set $\Theta_{\text{wc\_lim}}$.

To capture the transient information we can describe a limit set corresponding to equivalent *transfer functions* $G(z)$ rather than frequency responses $G(e^{j\omega})$. To do this, observe that since Δ is stable, (5) is equivalent to

$$\sup_{|z|\leq 1} |\Delta(z)| \leq 1 \tag{17}$$

This gives the limit set:

$$\Theta^*_{\text{wc\_lim}} = \{\theta : |A_\theta(z)G(z) - B_\theta(z)| \leq |W(z)B_\theta(z)|, \ \forall |z| \leq 1\} \tag{18}$$

One cautionary note: although $\Theta_{\text{wc\_ee}}$ is easy to compute, $\Theta_{\text{wc\_lim}}$ or $\Theta^*_{\text{wc\_lim}}$ involves an infinite search. These sets are infinite intersections of sets at every (possibly complex) frequency in $|z| \leq 1$. For practical purposes these sets can be approximated by computing over a finite set of frequencies.

The form of (16) strongly suggests replacing the unknown frequency response $\{G(e^{j\omega}) : \omega \in [-\pi, \pi]\}$ with any good frequency response estimate $\{G_{\text{freq}}(\omega) : \omega \in \Omega\}$ where Ω is typically a set of discrete values. For example, if G_{freq} is obtained from standard spectral techniques, then $\Omega = \{\omega_k = 2\pi k/N : k = 0, \ldots, N-1\}$. Techniques for frequency response set estimation are explored in [16].

In the remainder of the paper we concentrate on computing $\Theta_{\text{wc\_ee}}$ and also show the effect of test signals and data filters.

3 Computing $\Theta_{\text{wc\_ee}}$

A convenient form for computing (and interpreting) $\Theta_{\text{wc\_ee}}$ now follows. Observe first that:

$$\begin{aligned} A_\theta y - B_\theta u &= y - \theta^T\phi & (19)\\ B_\theta W u &= \theta^T\psi & (20)\end{aligned}$$

where ϕ and ψ have the form

$$\phi = \begin{bmatrix} \phi_y \\ \phi_u \end{bmatrix} \quad \psi = \begin{bmatrix} 0 \\ W\phi_u \end{bmatrix} \tag{21}$$

with

$$\begin{aligned} \phi_y^T &= [-z^{-1}y \cdots - z^{-n}y] & (22)\\ \phi_u^T &= [z^{-1}u \cdots z^{-m}u] & (23)\end{aligned}$$

Using the above definitions we can express (9) as

$$\Theta_{\text{wc\_ee}} = \{\theta \in \mathbb{R}^p : \|y - \theta^T\phi\|_{N2} \leq \|\theta^T\psi\|_{N2}\} \tag{24}$$

It is convenient to introduce the averaging operator $\mathcal{E}_N\{\cdot\}$ defined by

$$\mathcal{E}_N\{x\} = \frac{1}{N}\sum_{t=1}^{N} x(t) \tag{25}$$

It follows that $\frac{1}{N}\|x\|_{N2}^2 = \mathcal{E}_N\{x^T x\}$. Thus, (9) is given by the quadratic form:

$$\Theta_{\text{wc\_ee}} = \{\theta \in \mathbb{R}^p : \theta^T\Gamma\theta - 2\beta^T\theta + \alpha \leq 0\} \tag{26}$$

where

$$\begin{aligned} \Gamma &= \mathcal{E}_N\{\phi\phi^T - \psi\psi^T\} & (27)\\ \beta &= \mathcal{E}_N\{\phi y\} & (28)\\ \alpha &= \mathcal{E}_N\{y^2\} & (29)\end{aligned}$$

Provided Γ^{-1} exists, another convenient expression is:

$$\begin{aligned} \Theta_{\text{wc\_ee}} &= \{\theta \in \mathbb{R}^p : (\theta - \theta_{\text{c\_ee}})^T\Gamma(\theta - \theta_{\text{c\_ee}}) \leq V\}\\ \theta_{\text{c\_ee}} &= \Gamma^{-1}\beta\\ V &= \beta^T\Gamma^{-1}\beta - \alpha\end{aligned}$$

If $\Gamma > 0$, then $V > 0$, for otherwise the data is not consistent with the prior assumptions. Hence, $\Gamma > 0$ implies that $\Theta_{\text{wc\_ee}}$ is an ellipsoid in R^p with center at $\theta_{\text{c\_ee}}$. The largest radius of the ellipsoid, denoted by ρ_{wc}, bounds the Euclidean norm of $\theta - \theta_{c\_ee}$ as follows:

$$||\theta - \theta_{c\_ee}|| \leq \rho_{wc} = \sqrt{V/\min\lambda(\Gamma)} \qquad (30)$$

But Γ not > 0 is possible. To see this, we can express Γ as follows:

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_{22} \end{bmatrix} \qquad (31)$$

where

$$\Gamma_{11} = \mathcal{E}_N\left\{\phi_y\phi_y^T\right\} \qquad (32)$$

$$\Gamma_{12} = \mathcal{E}_N\left\{\phi_y\phi_u^T\right\} \qquad (33)$$

$$\Gamma_{22} = \mathcal{E}_N\left\{\phi_u\phi_u^T - (W\phi_u)(W\phi_u)^T\right\} \qquad (34)$$

The Γ_{22} matrix subblock can have negative eigenvalues if the spectrum of u is concentrated at those frequencies where $|W(e^{j\omega})|$ is large. We will illustrate this later in an example.

In summary: (i) All the eigenvalues of Γ are real with at most m being negative, these latter arising whenever the 22-subblock has negative eigenvalues. (ii) When $\Gamma > 0$, Θ_{wc_ee} is an ellipsoid in $\mathbb{R}^p$. (iii) When Γ not > 0, Θ_{wc_ee} is an hyperboloid in $\mathbb{R}^p$.

4 Experiment Design

As discussed in [9], it is critical to a successful identification that data filters and inputs are well selected. Let $F(z)$ denote a (stable) transfer function which is used to modify the data set (1) to now read

$$\{Fy, Fu : t = 1, \ldots, N\} \qquad (35)$$

We refer to F as the data filter. It follows that the worst case set Θ_{wc_ee} has exactly the same form as in (26) but now

$$\Gamma = \mathcal{E}_N\left\{(F\phi)(F\phi)^T - (F\psi)(F\psi)^T\right\} \qquad (36)$$

$$\beta = \mathcal{E}_N\left\{(F\phi)(Fy)\right\} \qquad (37)$$

$$\alpha = \mathcal{E}_N\left\{(Fy)^2\right\} \qquad (38)$$

To see the effect of the data filter and the input selection, suppose that u is a deterministic sequence with spectral density $S_{uu}(\omega)$. Then,

$$\lim_{N\to\infty} \Theta_{wc\_ee} = \{\theta \in \mathbb{R}^p : \int_{-\pi}^{\pi} f_{ee}(\theta,\omega)\,d\omega \leq 0\} \qquad (39)$$

where

$$f_{ee}(\theta,\omega) = f_{lim}(\theta,\omega)|F(e^{j\omega})|^2 S_{uu}(\omega)$$

with $f_{lim}(\theta,\omega)$ as defined in (16). Comparing this set with Θ_{wc_lim} in (16), it is clear that

$$\Theta_{wc\_lim} \subseteq \lim_{N\to\infty} \Theta_{wc\_ee} \qquad (40)$$

The question arises as to whether there are choices of F and u such that $\lim_{N\to\infty}\Theta_{wc\_ee} \approx \Theta_{wc\_lim}$. To see this, suppose that u consisits of sinusoids at K frequencies $\{\omega_1, \ldots, \omega_K\}$. Then,

$$\lim_{N\to\infty} \Theta_{wc\_ee} = \{\theta \in \mathbb{R}^p : \sum_{k=1}^{K} \mu_k f_{lim}(\theta,\omega_k) \leq 0\} \qquad (41)$$

where

$$\mu_k = |F(e^{j\omega_k})|^2 S_{uu}(\omega_k)$$

If μ_k is small whenever $|W(e^{j\omega_k})|$ is large, then the non-parametric dynamics will contribute only a small correction to the worst-case set. The dominant errors will be soley due to identifiability issues, *i.e.* , how well the set Θ_{wc_lim} is approximated by (41). As mentioned in Section 2, for small values of N the limit set may not be completely contained within Θ_{wc_ee} . This is seen in the simulation example discussed next.

5 Simulation Example

The parametric transfer function is

$$G_\theta(z) = \frac{bz^{-1}}{1 + az^{-1}} \qquad \theta^T = [a\ b]$$

The true system which generated the data set has the transfer function

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s}P(s)\right\}$$

$$P(s) = \frac{10}{s+1}\ \frac{(10)^2}{s^2 + 2(.005)(10)s + (10)^2}$$

$$G_{\theta_{true}} = (1 - z^{-1})\mathcal{Z}\left\{\frac{10}{s+1}\right\}$$

The sampling frequency is chosen as 10 hz (62.83 rad/sec) which makes the true parameter,

$$\theta_{true}^T = [a_{true}\ \ b_{true}] = [-.9048\ \ .9516]$$

The Bode magnitude plot of $|W(e^{j\omega})|$ is shown in Figure 1 along with the true response $|\Delta_{true}(e^{j\omega})W(e^{j\omega})|$. The chosen W reflects a low frequency uncertainty of 10% relative error, and anticipates a rather large resonance at frequencies beyond about 10 rad/sec. The true system has a resonance at 10 rad/sec. The bandwidth of the true parametric model is about 1 rad/sec.

Two series of simulation experiments were performed. The first series of experiments used a pure sinewave input at several frequencies, *i.e.* , $u = sin(\omega_{sin}t)$ for $\omega_{sin} \in \{8, 6, 4, 2, 1\}$ rad/sec. In the second series of experiments the input was a log spaced sine- sweep from .1 to 31 rad/sec over 102.3 sec, thus, $N = 1024$ data samples. In this case several low pass data filters were used, where each had a different corner frequency. Specifically, we used 8th order Butterworth filters with corner frequencies $\omega_f \in \{8,6,4,2,1\}$ rad/sec. We also processed the data with no filters.

Results with sine wave inputs

Figure 2 shows Θ_{wc_ee} for each of the input frequencies $\omega_{sin} \in \{8,6,4,2,1\}$ rad/ sec. Observe that at 8 and 7 rad/sec the sets are hyperboloids. Since the system is stable we know that the pole a must be negative, so only that portion is shown. Figure 3 shows the sets for 2 and 1 rad/ sec as well as points in the limit set Θ_{wc_lim} indicated by the shaded region. The * denotes the least-squares estimate for $\omega_{sin} = 1$ rad/sec. The axes in this plot show relative error between the estimate and the true parameter, thus, the (0,0) point is the true parameter value. Observe that the estimation tolerance for the pole a is from -3.5% to +3% whereas for the gain b the tolerance is much worse, from -20% to +40%. The least-squares estimate of the pole a is almost perfect and for the gain b has a +10% tolerance. However, the least-squares estimate would not change if some other value were selected out of the limit set, in which case the least-squares estimate would be inaccurate by as much as $\pm 2\%$ for a and almost 10% for b.

In all the cases we ran with sine inputs, the ellipsoids were concentric, decreased in volume with decreasing frequency, and completely covered the limit set. But this is not always to be expected as seen in the next experiment.

Results with sinesweep and data filters

Figure 4 shows Θ_{wc_ee} for the sinesweep ($N = 1024$) processed with no filter and with each of the data filter corner frequencies $\omega_f \in \{8,6,4,2,1\}$ rad/ sec. Hyperboloids are obtained with no data filter and for $\omega_f = 8$ rad/sec. In this case the remaining sets are ellipsoids but they are not concentric. Figure 5, which uses a relative error scale, shows the bounding ellipsoids for $\omega_f \in \{4,2,1\}$ rad/sec. The 'dots' are points in the limit set. Again, the (0,0) point is the true parameter value. The * denotes the least-squares estimate using the filtered data with $\omega_f = 1$ rad/sec. The estimation accuracy is significantly increased when ω_f is 2 or 1 rad /sec. For 1 rad/sec, both a and b are within $\pm 3\%$ tolerance.

From Figure 5, we see that not all points in the limit set Θ_{wc_lim} are in the set Θ_{wc_ee}, a possibility which was discussed in Section 2. The implication here is that there is significant transient response during the observation period of 102.3 sec with $N = 1024$ points. To verify this we increased the observation samples to $N = 2048$ by appending to the original sine sweep with $N = 1024$ the reverse sine sweep, *i.e.* , the new sweep went from .1 to 31 rad/sec and then from 31 to .1 rad/sec over 204.7 sec. Figure 6 shows the sets for $\omega_f = 1$ rad/sec with $N = 1024$ and $N = 2048$ samples. The 'dots' are points in the limit set, which of course remains unchanged; it is not affected by the data. But now, the limit set is completely contained in the bounding ellipsoid for $N = 2048$. This which verifies the theoretical result in Section 4 that $\Theta_{wc\_lim} \subseteq \lim_{N\to\infty} \Theta_{wc\_ee}$. It is clear from Figure 6 that lower observation times which capture transients can significantly reduce set uncertainty. Observe also that the least-squares estimate (denoted by *) remins almost unaffected by increasing N from 1024 to 2048.

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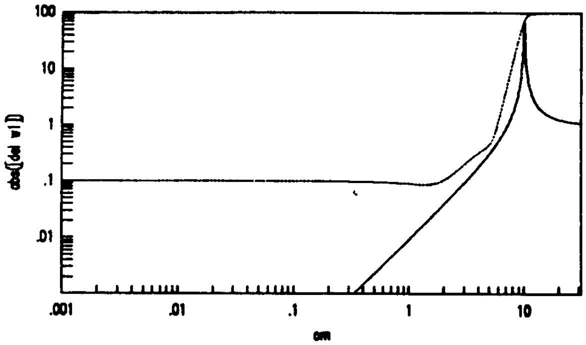

Figure 1: Bode magnitude plots of $\Delta_{\text{true}}W$ and W.

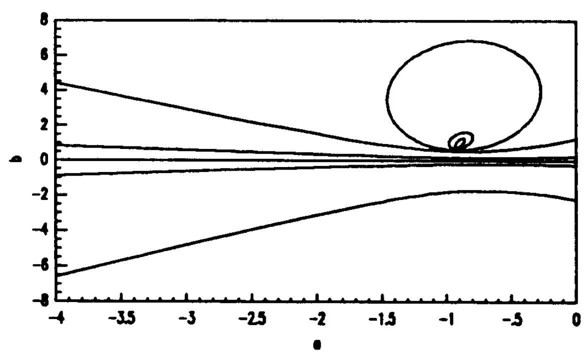

Figure 2: Worst-case sets for $\omega_{sin} = \{8, 6, 4, 2, 1\}$. Hyperboloids for $\omega_{sin} = \{8, 6\}$.

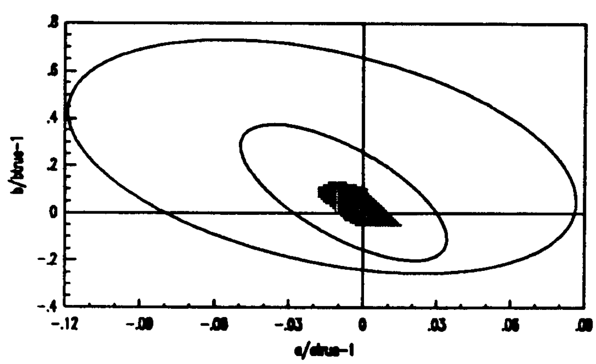

Figure 3: Worst case sets for $\omega_{sin} = \{2, 1\}$; dots are in limit set; * is least-squares estimate for $\omega_{sin} = 1$.

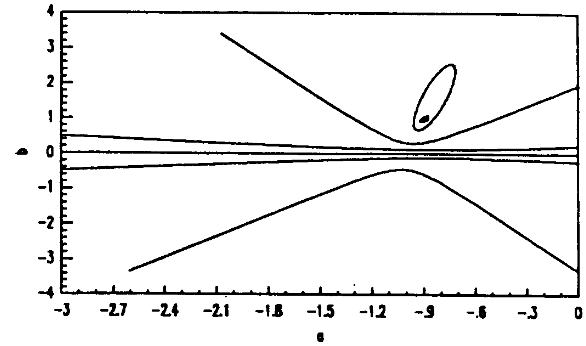

Figure 4: Worst case sets for no filter and $\omega_f = \{8, 6, 4, 2, 1\}$. Hyberboloids for no filter and $\omega_f = 8$.

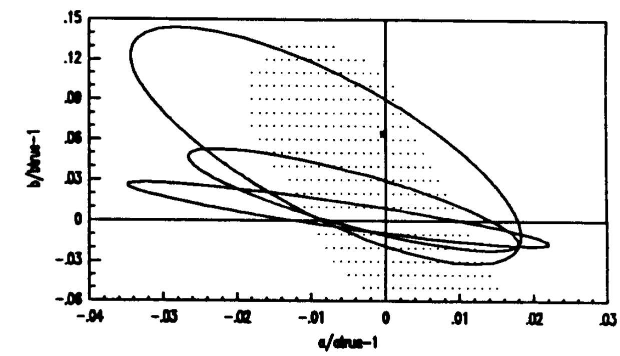

Figure 5: Worst case sets for $\omega_f = \{4, 2, 1\}$; dots are in limit set; * is least-squares estimate for $\omega_f = 1$.

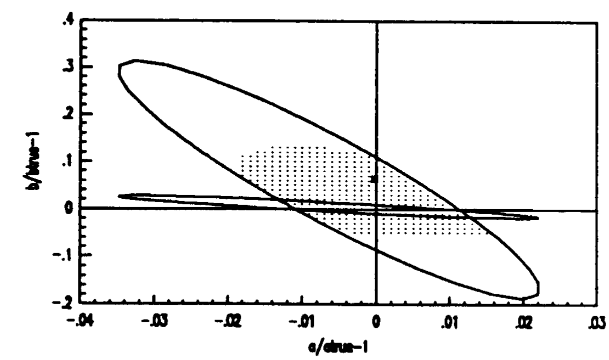

Figure 6: Worst case sets for $\omega_f = 1$; small ellipse for $N = 1024$; large ellipse for $N = 2048$; dots are in limit set; * is least-squares estimate for $\omega_f = 1$ and $N = 2048$.