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Maximum Torque-per-Current Control of Induction Motors via Semidefinite Programming

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***Abstract*— We present a method for finding current waveforms for induction motors that minimize resistive loss while achieving a desired average torque output. Our method is not based on reference-frame theory for electric machines, and therefore directly handles induction motors with asymmetric winding patterns, nonsinusoidally distributed windings, and a general winding connection. We do not explicitly handle torque ripple or voltage constraints. Our method is based on converting the torque control problem to a nonconvex linear-quadratic control problem, which can be solved by using a (tight) semidefinite programming relaxation.**

I. Introduction

Alternating-current electric motors are traditionally driven by applying symmetric, multiphase sinusoidal voltage waveforms to their terminals. However, due to the wide availability of switching power converters and microcontrollers for controlling them, it is possible to drive motors using specialized, nonsinusoidal waveforms, allowing us to design the drive voltage waveform along with the motor.

In this paper, we give a general method for computing steady-state voltage and current waveforms that produce a specified average torque from an induction motor while minimizing resistive loss. (Equivalently, we maximize the average torque for a specified resistive loss; this is known as *maximum-torque-per-current control*.) Our method is general, and is applicable to induction motors with nonsinusoidally distributed windings in both the stator and rotor, such as those arising from concentrated stator windings; asymmetric winding layouts, such as those arising from winding fault conditions or unconventionally designed motors; a general connection topology, such as star- or mesh-connected motors; and reluctance torque caused by changing self-inductance of the windings with rotor position.

Our method does *not* apply to motors with a nonlinear relationship between the winding currents and the magnetic flux through these windings; voltage limits imposed by the converter, such as those arising in high-speed operation; and rotor shaft speed that changes (significantly) over time (this amounts to an assumption of a "high impedance" mechanical load, *i.e.*, the mechanical load can absorb the torque ripple without the rotor speed changing excessively). We also note that our formulation maximizes the *average* torque produced and does not explicitly penalize torque ripple.

The optimal waveform design problem is solved by converting it to a nonconvex linear-quadratic control problem, *i.e.*, an optimal control problem with linear dynamics, quadratic cost function, and an arbitrary number of integral-quadratic constraints. Despite nonconvexity of such problems, we can solve a tight semidefinite programming relaxation, and we give a simple method to recover optimal waveforms from a solution to the relaxed problem. Our numerical solution method scales gracefully, and can compute optimal waveforms for motors with dozens of windings (arising, for example, by considering the individual bars in a squirrel-cage rotor) in a matter of minutes.

A. Previous work

Maximum torque-per-current control: For multiphase induction motors with symmetric, sinusoidally distributed windings, the maximum-torque-per-current waveforms are sinusoidal and generate no torque ripple. These waveforms can be found using the Clarke transformation, which reduces the optimization problem to a problem with only a few variables (for example, see [KG83]). Many attempts to generalize these results to induction motors with asymmetric or nonsinusoidally distributed windings are based on extensions of this idea, and typically consider adding only a few higher-order harmonics, while keeping fundamental frequency fixed. For some examples of this approach, see the survey paper [LBP$^+$07].

In the case of permanent-magnet motors, maximum-torque-per-current waveforms are much simpler to compute, and simple analytical solutions exist when the terminal voltages are not limited (*i.e.*, for low-speed

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operation), even if this waveform is nonsinusoidal. For example, if the phases are independently driven, the optimal current waveforms are proportional to the phase back-emf waveforms [YWWL04]. In fact, even for the more complicated case that terminal input voltages are limited, the maximum torque-per-current waveforms can be found by convex optimization [MB15].

Nonconvex linear-quadratic control: We reduce the maximum torque-per-current problem to a nonconvex linear-quadratic control problem, *i.e.*, a problem of minimizing a quadratic cost function over the state and input of a linear dynamical system, subject to nonconvex integral-quadratic constraints on the state and input, Because strong duality holds for such problems, they can be solved exactly by solving a tight semidefinite programming relaxation. A quick introduction to these nonconvex linear-quadratic control can be found in [BGFB94, §10.6], while a more technical introduction can be found in [MY00]. Formulations similar to the one given in the current paper can be found in [Gat06], [Gat10].

B. Contributions

We give a general method for determining current and voltage waveforms that maximize the average torque produced by an induction motor, while minimizing the resistive losses. Our model assumes magnetic linearity and ignores potential input voltage constraints, but includes general, nonsinusoidal, asymmetric winding patterns and general connection topology. To our knowledge, this is the first time this problem has been addressed at this level of generality. To the best of our knowledge, our specific formulation of the nonconvex linear-quadratic control problem, and the corresponding solution method, are novel.

C. Outline

In section II, we describe our motor model. In section III, we define the maximum torque-per-current optimal control problem, and show how it can be converted to a nonconvex linear-quadratic control problem. In section IV, we give a practical method for solving the nonconvex linear-quadratic control problem using semidefinite programming. In section V, we give a numerical example.

II. Induction motor model

We consider an abstract electromechanical device with n electrical ports and one mechanical shaft (*i.e.*, one rotational mechanical port). The device contains n magnetically coupled resistive-inductive (RL) circuits, called *windings*, with currents $i(t) \in \mathbf{R}^n$ and voltages $v(t) \in \mathbf{R}^n$. The magnetic flux through the coils is $\lambda(t) \in \mathbf{R}^n$. The mechanical shaft has angular position $\theta(t)$, angular speed $\omega(t)$, and torque $\tau(t)$, all of which are real numbers. Although we focus on devices with one mechanical coordinate, our model applies to more general devices (*e.g.*, planar motors) with minor modifications.

Where convenient, we will use a single dot to denote a time derivative, and a single prime to denote a derivative of a function of a single variable. We will often omit dependence on time for notational convenience.

Torque output: We assume the stored magnetic energy is path independent and has the form

$$E(\lambda, \theta) = \frac{1}{2}\lambda^{\mathsf{T}} L(\theta)^{-1}\lambda$$

The inductance matrix $L(\theta) \in \mathbf{S}_{++}$ is assumed to be periodic, so that $L(\theta + 2\pi/N_{\mathrm{pp}}) = L(\theta)$. The positive integer N_{pp} is the pole-pair number, which is the number of "electrical cycles" per rotation of the mechanical shaft.

Taking partial derivatives with respect to λ, we recover the (multivariate) inductor characteristic:

$$L(\theta)i = \lambda. \tag{1}$$

Taking a partial derivative of the energy with respect to θ, and using the inductor characteristic, we obtain the electromagnetic torque

$$\tau = -\frac{1}{2} i^{\mathsf{T}} L'(\theta) i.$$

The average torque, starting at $t = 0$, is

$$\bar{\tau}(i) = \lim_{T\to\infty} \frac{1}{T}\int_0^T -\frac{1}{2} i^{\mathsf{T}} L'(\theta) i \; dt. \tag{2}$$

Circuit dynamics: The voltage drop across the resistors is Ri, where $R \in \mathbf{S}^n_{++}$ is the winding resistance matrix. By Faraday's law, the voltage drop across the coupled inductors is $\frac{d\lambda}{dt}$. Combining this with Kirchoff's voltage law, we have

$$v = Ri + \frac{d\lambda}{dt}.$$

By evaluating the derivative, and using the inductor characteristic (1), we obtain the circuit dynamics equation

$$v = Ri + L(\theta)\frac{di}{dt} + \omega L'(\theta) i. \tag{3}$$

Average power loss: The average resistive loss in the windings, starting at $t = 0$, is

$$P_{\mathrm{loss}}(i) = \lim_{\tau\to\infty} \frac{1}{T}\int_0^T i^{\mathsf{T}} R i \; dt. \tag{4}$$

Winding pattern: In many cases, the winding voltages are controlled only indirectly by manipulating a vector of input voltages $u(t) \in \mathbf{R}^m$, which affect $v(t)$ according to the connection topology of the device. In addition, the winding currents may also be constrained by the circuit topology. We assume that we have

$$Ci = 0, \qquad v = C^\mathsf{T} e + Du, \tag{5}$$

where $C \in \mathbf{R}^{p \times n}$ is the *connection topology matrix*, and $D \in \mathbf{R}^{m \times n}$ is the *voltage input matrix*, and $e(t) \in \mathbf{R}^p$ are floating node voltages. Note that the first equation in (5) is Kirchoff's current law for the circuit (see [DK69] for details).

III. Average torque problem

We consider the problem of choosing the winding voltages to achieve a desired average torque while minimizing the resistive power loss, for a constant rotor shaft speed.

$$\begin{array}{ll} \text{minimize} & P_{\text{loss}}(i) \\ \text{subject to} & \bar{\tau}(i) = \tau_{\text{des}}, \\ & \text{circuit dynamics (3)}, \\ & \text{winding pattern (5)}. \end{array} \tag{6}$$

The problem parameters are the resistance matrix $R \in \mathbf{S}^n_{++}$, the inductance matrix $L : \mathbf{R} \to \mathbf{S}^n_{++}$ as a function of position, the constant rotor speed $\omega \in \mathbf{R}$, and the desired torque $\tau_{\text{des}} \in \mathbf{R}$. The variables are the current waveform $i : \mathbf{R}_+ \to \mathbf{R}^n$, and the voltage waveforms $u : \mathbf{R}_+ \to \mathbf{R}^m$, $v : \mathbf{R}_+ \to \mathbf{R}^n$, and $e : \mathbf{R}_+ \to \mathbf{R}^p$. Note that with the rotor shaft speed constant, the shaft position θ in (2) and (3) can be replaced by ωt, a function of time, making the dynamics and torque function time varying and periodic with period $2\pi/N_{\text{pp}}$,

In the motor control community, problem (6) is known as the maximum torque-per-current control problem. We note that this problem does *not* address torque ripple.

Conversion to nonconvex linear-quadratic control problem: Because of the quadratic torque constraint $\bar{\tau}(i) = \tau_{\text{des}}$, problem (6) is not convex. However, because the objective and first constraint of (6) are integral-quadratic, and the remaining constraints describe a periodic, linear dynamical system, it is possible to solve it by transforming it into a nonconvex linear-quadratic control problem.

To do this, we write $i = Fx$ for some x, where the columns of F form a basis for the nullspace of C. Then the dynamics are

$$\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t), \tag{7}$$

where $A(\theta) = (F^\mathsf{T} L(\theta) F)^{-1} F^\mathsf{T}(R + \omega L'(\theta))F$, $B(\theta) = (F^\mathsf{T} L(\theta) F)^{-1} F^\mathsf{T} D$, and the integrands of (2) and (4) are $x^\mathsf{T} F^\mathsf{T} L'(\theta) F x$ and $x F^\mathsf{T} R F x$, respectively. For a constant rotor speed ω, we have $\theta = \omega t$, and system (7) is a periodic, continuous-time linear dynamical system. We then discretize this system. The resulting periodic, discrete-time dynamical system can be converted into a linear, time-invariant, discrete-time system. The details of the full problem transformation are complicated, but standard (for example, see [Hes09]). We note that the last step, involving conversion from a periodic linear system into a time-invariant linear system is not strictly necessary, as an equivalent formulation of the nonconvex linear-quadratic control problem can can be given directly for periodic, discrete-time systems. However, because the time-invariant case is pedagogically simpler, and we restrict our attention to this case.

IV. Nonconvex linear-quadratic control

After discretization, the maximum-torque-per-current problem (6) can be converted to the following abstract form:

$$\begin{array}{ll} \text{minimize} & g_0(x, u) \\ \text{subject to} & g_1(x, u) = \gamma_1 \\ & x_{t+1} = Ax_t + Bu_t, \quad t = 0, 1, \ldots \end{array} \tag{8}$$

The variables are x and u, which are (bounded) sequences of state and input vectors, so that $x = (x_0, x_1, \ldots)$ and $u = (u_0, u_1, \ldots)$, where $x_t \in \mathbf{R}^n$ and $u_t \in \mathbf{R}^m$. The functions g_i are given by

$$g_i(x, u) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q^{(i)} & S^{(i)} \\ S^{(i)\mathsf{T}} & R^{(i)} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}.$$

The matrix that appears in the sum is not required to be positive semidefinite, so problem (8) is not convex in general. For simplicity, we restrict our search to points for which the limit

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \begin{bmatrix} x_t \\ u_t \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^\mathsf{T}$$

exists. This guarantees that the limits in the definition of g_0 and g_1 exist.

Note that here, the objective corresponds to P_{loss}, the first constraint corresponds to the torque constraint $\bar{\tau}(i) = \tau_{\text{des}}$, and the final (dynamics) constraint corresponds to the circuit dynamics (3) and the winding constraints (5). Depending on the method used to discretize the continuous-time dynamics, $S^{(i)}$ and $R^{(i)}$ may be zero.

We note that much more general versions of (8) can be formulated and solved using methods similar to those described below. (For example, we can include an arbitrary number of integral-quadratic equality or inequality constraints involving both the state and input; see [MY00], [Gat06], [Gat10].)

A. Relaxed problem

We consider the following relaxation of (8):

$$\begin{array}{ll} \text{minimize} & \tilde{g}_0(X,U,Y) \\ \text{subject to} & \tilde{g}_1(X,U,Y) = \gamma_1 \\ & X = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X & Y \\ Y^\mathsf{T} & U \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix}^\mathsf{T} \\ & \begin{bmatrix} X & Y \\ Y^\mathsf{T} & U \end{bmatrix} \succeq 0. \end{array} \tag{9}$$

with variables $X \in \mathbf{S}^n$, $U \in \mathbf{S}^m$, and $Y \in \mathbf{R}^{n\times m}$. The functions $\tilde{g}_i$ are given by

$$\tilde{g}_i(X,U,Y) = \mathbf{Tr} \begin{bmatrix} X & Y \\ Y^\mathsf{T} & U \end{bmatrix} \begin{bmatrix} Q^{(i)} & S^{(i)} \\ S^{(i)\mathsf{T}} & R^{(i)} \end{bmatrix}.$$

To see that problem (9) is indeed a relaxation of (8), take a feasible point (x,u) for (8). Then define a point for (9) as the average second moment of the state-input vector:

$$\begin{bmatrix} X & Y \\ Y^\mathsf{T} & U \end{bmatrix} = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \begin{bmatrix} x_t \\ u_t \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^\mathsf{T}. \tag{10}$$

It is straightforward to verify that this new point satisfies all constraints of (9) and obtains the same objective value as the original point.

Solution in expectation: Given a solution to (9) with objective value γ_0, it is possible to construct a stationary random process that solves (8) "in expectation"; that is, all realizations of the random process satisfy the dynamics constraint

$$x_{t+1} = Ax_t + Bu_t \tag{11}$$

for all t, and additionally we have

$$\mathbf{E}\, g_i(x_t,u_t) = \gamma_i. \tag{12}$$

for all i.

The random variables x_t and u_t are defined inductively: we first take x_0 to be a normally distributed zero-mean random variable with covariance matrix X. Then for any t, after defining x_t, u_t is chosen so that (x_t,u_t) as normally distributed with zero mean and covariance matrix

$$\mathbf{E} \begin{bmatrix} x_t \\ u_t \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^\mathsf{T} = \begin{bmatrix} X & Y \\ Y^\mathsf{T} & U \end{bmatrix}. \tag{13}$$

To do this, the conditional distribution of u_t given x_t is normal with mean $Y^\mathsf{T} X^\dagger x_t$ and covariance $U - Y^\mathsf{T} X^\dagger Y$. (Here $X^\dagger$ denotes the pseudoinverse of X.) Then x_{t+1} is given by the dynamics constraint (11); feasibility of X, Y, and U for (9) implies that x_{t+1} has second moment matrix X. Because (13) holds, we have

$$\mathbf{E} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q^{(i)} & S^{(i)} \\ S^{(i)\mathsf{T}} & R^{(i)} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \gamma_i,$$

for all t, and consequently $\mathbf{E}\, g_i(x,u) = \gamma_i$.

The inductive definition of x_t and u_t can be interpreted as forward simulation of the dynamical system (11) with random initial condition and linear control law

$$u_t = Kx_t + w_t, \tag{14}$$

where $K = Y^\mathsf{T} X^\dagger$ and w_t is zero-mean, normally distributed noise with covariance matrix $W = U - Y^\mathsf{T} X^\dagger Y$ (which is also uncorrelated across time).

Equivalence of the relaxed problem: It can be shown that the two problems are in fact equivalent, *i.e.*, given a feasible point X, Y, and U to (9), we can construct a feasible point to (8) with the same objective value. Intuitively, this can be done by taking generating a sequence of so-called solutions in expectation described above, and linking them together to get a single trajectory that satisfies the constraints of (8). Due to space limitations, we omit a full proof.

A more practical solution for (8): The proof for equivalence of (8) and (9) is indeed constructive, but the constructed solution is cumbersome for most practical problems. Here we give a simpler procedure that works if $A + BK$ is diagonalizable and has no repeated eigenvalues on the unit disk, and $W = U - Y^\mathsf{T} X^\dagger Y = 0$. (The diagonalizability and $W = 0$ assumptions are not necessary, but simplify the exposition and hold at the solution of most practical problem instances, including all examples in section V.)

Under these assumptions, a optimal trajectory (x,u) for (8) can be constructed by simulating the system forward with deterministic control law $u_t = Kx_t$, where $K = Y^\mathsf{T} X^\dagger$. The initial condition is

$$x_0 = V \left(\mathbf{diag} \left(V^{-1} X V^{-*} \right)^{1/2} \right), \tag{15}$$

where $V\Lambda V^* = A + BK$ is an eigendecomposition of $A + BK$. Such a trajectory will evidently satisfy the dynamics constraint (11); in the remainder of this section we show it also satisfies the quadratic constraints (12).

First we now show that V^{-1} diagonalizes X. We can rewrite the equality constraint of (9) (with $W = 0$) as

$$X = (A + BK)X(A + BK)^\mathsf{T}, \tag{16}$$

or equivalently,

$$V^{-1}XV^{-*} = \Lambda(V^{-1}XV^{-*})\Lambda^{*}. \tag{17}$$

We define $P = V^{-1}XV^{-*}$, so (17) becomes $P = \Lambda P \Lambda^{*}$. The i, j element of (17) is

$$P_{ij} = \lambda_i \lambda_j^{*} P_{ij}.$$

This holds only if either $P_{ij} = 0$ or $\lambda_i \lambda_j^{*} = 1$. In particular, $P_{ii} = 0$ if $|\lambda_i| \neq 1$. Because P is positive semidefinite, we also have $P_{ij} = 0$ if either $|\lambda_i| \neq 1$ or $|\lambda_j| \neq 1$. For $i \neq j$, if $|\lambda_i| = 1$ and $|\lambda_j| = 1$, then $\lambda_i \lambda_j^{*} = 1$ implies $\lambda_i = \lambda_j$. Because this contradicts our assumption of no repeated eigenvalues on the unit disk, we also must have $P_{ij} = 0$. This means P is diagonal and P_{ii} is nonzero only if $|\lambda_i| = 1$.

This has the following dynamical system interpretation: any steady-state distribution of the autonomous linear system $x_{t+1} = (A+BK)x_t$ has nonzero variance only in directions corresponding to modes with unity-magnitude eigenvalues; furthermore, if $A + BK$ has no repeated eigenvalues on the unit disk, the correlation between these modes is zero.

Now we define $\tilde{P}$ as a rank-one matrix whose diagonal elements are those of P (so that $\tilde{P}_{ij} = \sqrt{P_{ii}P_{jj}}$); we can write $\tilde{P}$ as the outer product

$$\tilde{P} = \left(\mathbf{diag}\,(P)^{1/2}\right)\left(\mathbf{diag}\,(P)^{1/2}\right)^{\mathsf{T}}.$$

From (15), we have $x_0 x_0^{\mathsf{T}} = V\tilde{P}V^{*}$.

The average second moment of the state vector is then

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} x_t x_t^{\mathsf{T}} \\
&\qquad = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} (A+BK)^t x_0 x_0^{\mathsf{T}} (A+BK)^{t\mathsf{T}} \\
&\qquad = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} V\Lambda^t V^{-1}(V\tilde{P}V^{*})V^{-*}\Lambda^{*t}V^{*} \\
&\qquad = V\left(\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \Lambda^t \tilde{P} \Lambda^{*t}\right)V^{*}.
\end{aligned}$$

The i, j element of the matrix in parentheses is

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} (\lambda_i\lambda_j^{*})^t \sqrt{P_{ii}P_{jj}},$$

which is zero if $\lambda_i \neq \lambda_j$ or $|\lambda_i| \neq 1$ (in this latter case, this is because $P_{ii} = 0$, as shown above). Therefore all elements are zero except the diagonal elements, which are equal to those of P. Then we have

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} x_t x_t^{\mathsf{T}} = VPV^{*} = X.$$

The quadratic constraints and objective satisfy

$$\begin{aligned}
&g_i(x,u) \\
&\quad = \frac{1}{T}\lim_{T\to\infty}\sum_{t=0}^{T} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q^{(i)} & S^{(i)} \\ S^{(i)\mathsf{T}} & R^{(i)} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}. \\
&\quad = \mathbf{Tr} \begin{bmatrix} Q^{(i)} & S^{(i)} \\ S^{(i)\mathsf{T}} & R^{(i)} \end{bmatrix} \left(\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T} \begin{bmatrix} x_t \\ Kx_t \end{bmatrix} \begin{bmatrix} x_t \\ Kx_t \end{bmatrix}^{\mathsf{T}} \right) \\
&\quad = \mathbf{Tr} \begin{bmatrix} Q^{(i)} & S^{(i)} \\ S^{(i)\mathsf{T}} & R^{(i)} \end{bmatrix} \begin{bmatrix} X & XK^{\mathsf{T}} \\ KX & KXK^{\mathsf{T}} \end{bmatrix} \\
&\quad = \mathbf{Tr} \begin{bmatrix} Q^{(i)} & S^{(i)} \\ S^{(i)\mathsf{T}} & R^{(i)} \end{bmatrix} \begin{bmatrix} X & Y^{\mathsf{T}} \\ Y & U \end{bmatrix} \\
&\quad = \gamma_i
\end{aligned}$$

Note that in substituting Y for $KX = (Y^{\mathsf{T}}X^{\dagger})X$, we use the fact that $\mathcal{N}(Y) \subset \mathcal{N}(X)$.

Because the new trajectory (x, u) satisfies the dynamics constraint (11) and obtains the correct value for the quadratic constraints and objective (12), it is indeed optimal for (8).

V. Example

In this section, we give an example of a five-phase induction motor, under both normal conditions and after an open-winding stator fault.

We use the detailed induction motor model in [TLW91], which we do not reproduce here. The model includes an inductive-resistive winding for each of the five stator windings and 26 rotor bars. The stator winding and rotor bar resistances are 2.6 Ω and 0.41 Ω, respectively, and the leakage inductances for the stator windings and rotor bars (*i.e.*, the additive inductance terms that does not couple across phases or change with rotor position) are 15 mH and 3.6 mH, respectively. The full inductance matrix L is derived from the (ideal) motor geometry; we deviate from [TLW91] only in that we use a sinusoidally stator winding, for simplicity, and to compare our solution with analytical results, where available. We scale the inductance matrix so that each (position-independent) stator winding self-inductance is 165 mH, and the rotor bar self-inductance is 43 mH. For simplicity, we take the rotor end cap resistance and inductance to be zero.

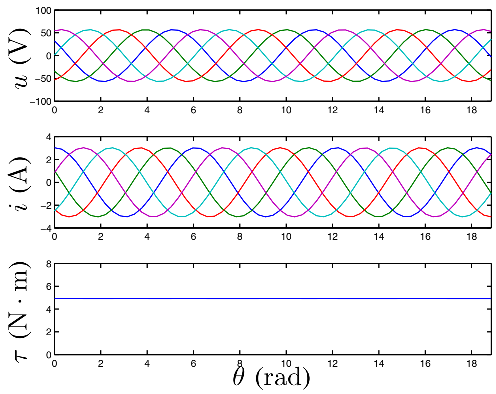

Fig. 1: Optimal input voltage, stator winding currents, rotor winding (bar) currents, and output torque.

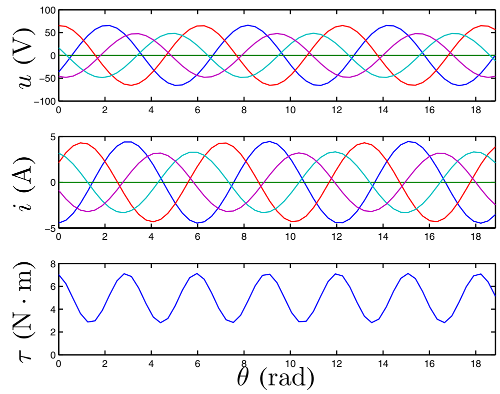

Fig. 2: Optimal input voltage, stator winding currents, rotor winding (bar) currents, and output torque, for the case of a stator open-phase fault.

A. Normal operation

We consider first normal operation at rotor speed $\omega = 50\ \mathrm{rad/s}$ and desired torque $\tau_{\mathrm{des}} = 5\ \mathrm{N \cdot m}$. The resulting stator and rotor winding currents, input voltage, and output torque waveforms are shown in figure 1. The resulting optimal waveforms correspond to the theory for multiphase, symmetric, sinusoidally wound induction motors: the optimal stator current waveforms are sinusoidal and symmetric (*i.e.*, each waveform is identical to the others, when displaced by $2\pi k/5$, for some integer k); furthermore, negligible torque ripple is developed, even though torque ripple is not explicitly penalized in the problem statement. The optimal value of problem (6) is $56\ \mathrm{W}$, which means that for these operating conditions, the resistive power loss is 11 watts per newton-meter of torque produced.

The optimal input voltage frequency can be calculated from the matrix $A + BK$ in (16), by examining the angle of the pair of complex eigenvalues that lie on the unit circle. For the current example, the input voltage frequency is 51.7 rad/s; this matches the analytical calculation of the optimal slip speed given in [KG83].

B. Operation under stator winding fault

We now consider operation when one of the stator windings is in open-phase fault. We model this by adding a row to the matrix A of equation (5) that enforces one stator winding current to be zero. We use the same operating parameters ($\omega = 50\ \mathrm{rad/s}$ and $\tau_{\mathrm{des}} = 5\ \mathrm{N \cdot m}$) as the previous example.

The resulting stator and rotor winding currents, input voltage, and output torque waveforms are shown in figure 1. Perhaps surprisingly, the resulting optimal stator waveforms are still sinusoidal, but are not simply displaced versions of each other. In this case, the optimal value of problem (6) is 72 W, which means that for these operating conditions, the resistive power loss around 14 watts per newton-meter of torque produced, compared to 11 W without a stator open-phase fault.

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