# Advances in Convex Optimization: Theory, Algorithms, and Applications

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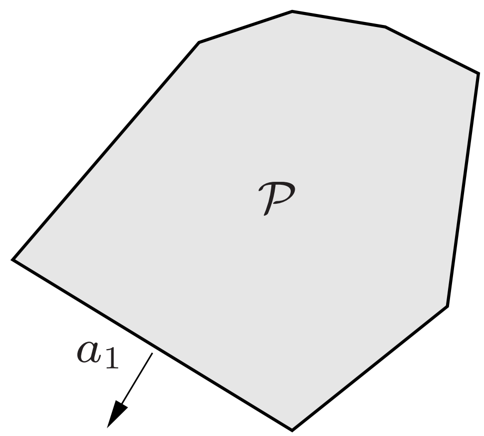
ISIT 02

# Two problems

polytope $\mathcal{P}$ described by linear inequalities, $a_i^T x \leq b_i$, $i = 1, \ldots, L$




**Problem 1a:** find minimum volume ellipsoid $\supseteq \mathcal{P}$

**Problem 1b:** find maximum volume ellipsoid $\subseteq \mathcal{P}$

are these (computationally) difficult? or easy?

problem 1a is **very difficult**

- in practice
- in theory (NP-hard)

problem 1b is **very easy**

- in practice (readily solved on small computer)
- in theory (polynomial complexity)

# Two more problems

find capacity of discrete memoryless channel, subject to constraints on input distribution

**Problem 2a:** find channel capacity, subject to:
no more than $30\%$ of the probability is concentrated on any $10\%$ of the input symbols

**Problem 2b:** find channel capacity, subject to:
at least $30\%$ of the probability is concentrated on $10\%$ of the input symbols

are problems 2a and 2b (computationally) difficult? or easy?

problem 2a is **very easy** in practice & theory

problem 2b is **very difficult**[1]

[1] I'm almost sure

# Moral

**very difficult** and **very easy** problems can look **quite similar**

. . . unless you're trained to recognize the difference

# Outline

- what's new in convex optimization
- some new standard problem classes
- generalized inequalities and semidefinite programming
- interior-point algorithms and complexity analysis

# Convex optimization problems

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0, \ldots, f_L(x) \leq 0, \quad Ax = b \end{array}$$

- $x \in \mathbf{R}^n$ is optimization variable

- $f_i : \mathbf{R}^n \to \mathbf{R}$ are **convex**, *i.e.*, for all $x$, $y$, $0 \leq \lambda \leq 1$,

$$f_i(\lambda x + (1 - \lambda)y) \leq \lambda f_i(x) + (1 - \lambda) f_i(y)$$

examples:

- linear & (convex) quadratic programs
- problem 1b & 2a (if formulated properly)

# Convex analysis & optimization

nice properties of convex optimization problems known since 1960s

- local solutions are global
- duality theory, optimality conditions
- simple solution methods like alternating projections

convex analysis well developed by 1970s (Rockafellar)

- separating & supporting hyperplanes
- subgradient calculus

# What's new (since 1990 or so)

- powerful primal-dual interior-point methods
  *extremely efficient, handle nonlinear large scale problems*
- polynomial-time complexity results for interior-point methods
  *based on self-concordance analysis of Newton's method*
- extension to generalized inequalities
  *semidefinite & maxdet programming*
- new standard problem classes
  *generalizations of LP, with theory, algorithms, software*
- lots of applications
  *control, combinatorial optimization, signal processing, circuit design, . . .*

## Recent history

- (1984–97) interior-point methods for LP
  - (1984) Karmarkar's interior-point LP method
  - theory (Ye, Renegar, Kojima, Todd, Monteiro, Roos, . . . )
  - practice (Wright, Mehrotra, Vanderbei, Shanno, Lustig, . . . )
- (1988) Nesterov & Nemirovsky's self-concordance analysis
- (1989–) semidefinite programming in control (Boyd, El Ghaoui, Balakrishnan, Feron, Scherer, . . . )
- (1990–) semidefinite programming in combinatorial optimization (Alizadeh, Goemans, Williamson, Lovasz & Schrijver, Parrilo, . . . )
- (1994) interior-point methods for nonlinear convex problems (Nesterov & Nemirovsky, Overton, Todd, Ye, Sturm, . . . )
- (1997–) robust optimization (Ben Tal, Nemirovsky, El Ghaoui, . . . )

# Some new standard (convex) problem classes

- second-order cone programming (SOCP)
- semidefinite programming (SDP), maxdet programming
- (convex form) geometric programming (GP)

for these new problem classes we have

- complete duality theory, similar to LP
- good algorithms, and robust, reliable software
- wide variety of new applications

# Second-order cone programming

**second-order cone program** (SOCP) has form

$$\begin{array}{ll} \text{minimize} & c_0^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

- variable is $x \in \mathbf{R}^n$
- includes LP as special case ($A_i = 0$, $b_i = 0$), QP ($c_i = 0$)
- nondifferentiable when $A_i x + b_i = 0$
- new IP methods can solve (almost) as fast as LPs

# Robust linear programming

**robust linear program:**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \;\; \text{for all} \;\; a_i \in \mathcal{E}_i \end{array}$$

- **ellipsoid** $\mathcal{E}_i = \{ \, \overline{a}_i + F_i p \mid \|p\|_2 \leq 1 \, \}$ describes **uncertainty** in constraint vectors $a_i$
- $x$ must satisfy constraints for all possible values of $a_i$
- can extend to uncertain $c$ & $b_i$, correlated uncertainties . . .

# Robust LP as SOCP

robust LP is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \overline{a}_i^T x + \sup\{(F_i p)^T x \mid \|p\|_2 \le 1\} \le b_i \end{array}$$

which is the same as

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \overline{a}_i^T x + \|F_i^T x\|_2 \le b_i \end{array}$$

- an SOCP (hence, readily solved)
- term $\|F_i^T x\|_2$ is extra margin required to accommodate uncertainty in $a_i$

## Stochastic robust linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

where $a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$, $\eta \geq 1/2$ ($c$ and $b_i$ are fixed)
$i.e.$, each constraint must hold with probability at least $\eta$

equivalent to SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq 1, \quad i = 1, \ldots, m \end{array}$$

where $\Phi$ is CDF of $\mathcal{N}(0, 1)$ random variable

# Geometric programming

**log-sum-exp** function:

$$\mathbf{lse}(x) = \log\left(e^{x_1} + \cdots + e^{x_n}\right)$$

. . . a smooth **convex** approximation of the max function

**geometric program** (GP), with variable $x \in \mathbf{R}^n$:

$$\begin{array}{ll} \text{minimize} & \mathbf{lse}(A_0 x + b_0) \\ \text{subject to} & \mathbf{lse}(A_i x + b_i) \leq 0, \quad i = 1, \ldots, m \end{array}$$

where $A_i \in \mathbf{R}^{m_i \times n}$, $b_i \in \mathbf{R}^{m_i}$

new IP methods can solve large scale GPs (almost) as fast as LPs

# Dual geometric program

dual of geometric program is an **unnormalized entropy problem**

$$\begin{array}{ll} \text{maximize} & \sum_{i=0}^{m} \left( b_i^T \nu_i + \mathbf{entr}(\nu_i) \right) \\ \text{subject to} & \nu_i \succeq 0, \quad i = 0, \ldots, m, \quad \mathbf{1}^T \nu_0 = 1, \\ & \sum_{i=0}^{m} A_i^T \nu_i = 0 \end{array}$$

- dual variables are $\nu_i \in \mathbf{R}^{m_i}$, $i = 0, \ldots, m$
- (unnormalized) entropy is

$$\mathbf{entr}(\nu) = -\sum_{i=1}^{n} \nu_i \log \frac{\nu_i}{\mathbf{1}^T \nu}$$

- GP is closely related to problems involving entropy, KL divergence

# Example: DMC capacity problem

$x \in \mathbf{R}^n$ is distribution of input; $y \in \mathbf{R}^m$ is distribution of output
$P \in \mathbf{R}^{m \times n}$ gives conditional probabilities: $y = Px$

**primal channel capacity problem**:

$$\begin{array}{ll} \text{maximize} & -c^T x + \mathbf{entr}(y) \\ \text{subject to} & x \succeq 0, \quad \mathbf{1}^T x = 1, \quad y = Px \end{array}$$

where $c_j = -\sum_{i=1}^{m} p_{ij} \log p_{ij}$

**dual channel capacity problem** is a simple GP:

$$\begin{array}{ll} \text{minimize} & \mathbf{lse}(u) \\ \text{subject to} & c + P^T u \succeq 0 \end{array}$$

# Generalized inequalities

with proper convex cone $K \subseteq \mathbf{R}^k$ we associate **generalized inequality**

$$x \preceq_K y \iff y - x \in K$$

**convex optimization problem with generalized inequalities:**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \preceq_{K_1} 0, \ldots, f_L(x) \preceq_{K_L} 0, \quad Ax = b \end{array}$$

$f_i : \mathbf{R}^n \to \mathbf{R}^{k_i}$ are $K_i$-convex: for all $x$, $y$, $0 \leq \lambda \leq 1$,

$$f_i(\lambda x + (1-\lambda)y) \preceq_{K_i} \lambda f_i(x) + (1-\lambda) f_i(y)$$

# Semidefinite program

**semidefinite program** (SDP):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_0 + x_1 A_1 + \cdots + x_n A_n \preceq 0, \qquad Cx = d \end{array}$$

- $A_i = A_i^T \in \mathbf{R}^{m \times m}$
- inequality is matrix inequality, *i.e.*, $K$ is positive semidefinite cone
- single constraint, which is affine (hence, matrix convex)

# Maxdet problem

extension of SDP: **maxdet problem**

$$\begin{array}{ll} \text{minimize} & c^T x - \log \det_+(G_0 + x_1 G_1 + \cdots + x_m G_m) \\ \text{subject to} & A_0 + x_1 A_1 + \cdots + x_n A_n \preceq 0, \qquad Cx = d \end{array}$$

- $x \in \mathbf{R}^n$ is variable
- $A_i = A_i^T \in \mathbf{R}^{m \times m}$, $G_i = G_i^T \in \mathbf{R}^{p \times p}$
- $\det_+(Z) = \begin{cases} \det Z & \text{if } Z \succ 0 \\ 0 & \text{otherwise} \end{cases}$

# Semidefinite & maxdet programming

- nearly complete duality theory, similar to LP
- interior-point algorithms that are efficient in theory & practice
- applications in many areas:
  - control theory
  - combinatorial optimization & graph theory
  - structural optimization
  - statistics
  - signal processing
  - circuit design
  - geometrical problems
  - algebraic geometry

# Chebyshev bounds

**generalized Chebyshev inequalities**: lower bounds on

$$\mathbf{Prob}(X \in C)$$

- $X \in \mathbf{R}^n$ is a random variable with $\mathbf{E}\,X = a$, $\mathbf{E}\,XX^T = S$
- $C$ is an open polyhedron $C = \{x \mid a_i^T x < b_i,\ i = 1, \ldots, m\}$

cf. classical Chebyshev inequality on $\mathbf{R}$

$$\mathbf{Prob}(X < 1) \geq \frac{1}{1 + \sigma^2}$$

if $\mathbf{E}\,X = 0$, $\mathbf{E}\,X^2 = \sigma^2$

# Chebyshev bounds via SDP

$$
\begin{array}{ll}
\text{minimize} & 1 - \sum_{i=1}^m \lambda_i \\
\text{subject to} & a_i^T z_i \geq b_i \lambda_i, \quad i = 1, \ldots, m \\
& \sum_{i=1}^m \left[ \begin{array}{cc} Z_i & z_i \\ z_i^T & \lambda_i \end{array} \right] \preceq \left[ \begin{array}{cc} S & a \\ a^T & 1 \end{array} \right] \\
& \left[ \begin{array}{cc} Z_i & z_i \\ z_i^T & \lambda_i \end{array} \right] \succeq 0, \quad i = 1, \ldots, m
\end{array}
$$

- an SDP with variables $Z_i = Z_i^T \in \mathbf{R}^{n \times n}$, $z_i \in \mathbf{R}^n$, and $\lambda_i \in \mathbf{R}$
- optimal value is a (sharp) lower bound on $\mathbf{Prob}(X \in C)$
- can construct a distribution with $\mathbf{E}\, X = a$, $\mathbf{E}\, XX^T = S$ that attains the lower bound

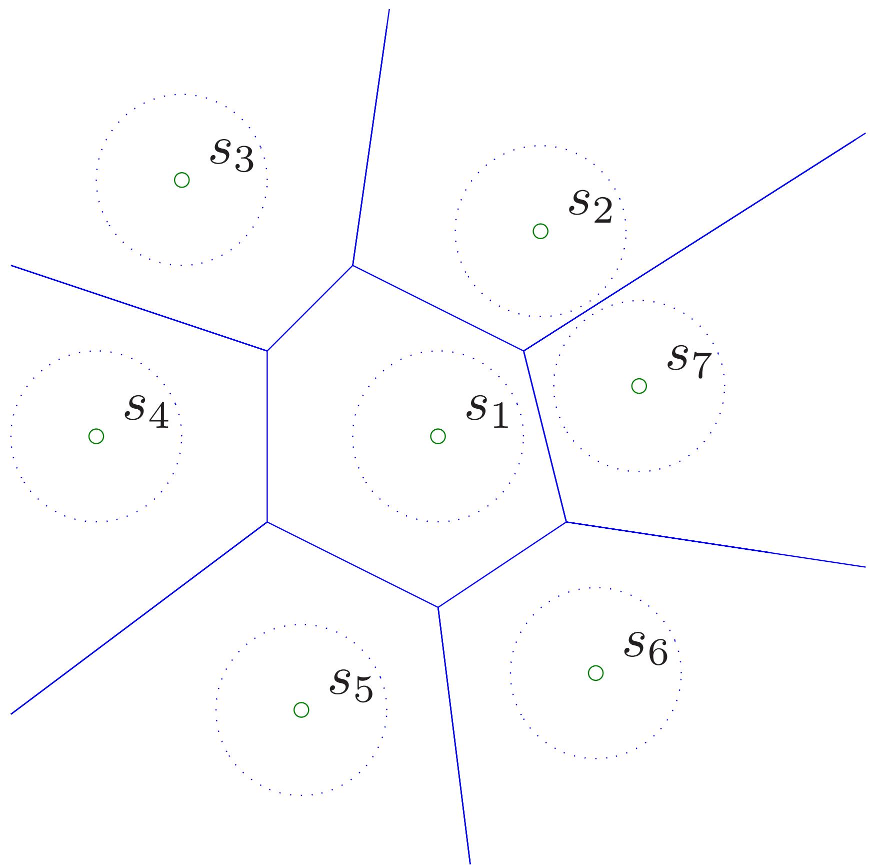
# Detection example

$$x = s + v$$

- $x \in \mathbf{R}^n$: received signal
- $s$: transmitted signal $s \in \{s_1, s_2, \ldots, s_N\}$ (one of $N$ possible symbols)
- $v$: noise with $\mathbf{E}\, v = 0$, $\mathbf{E}\, vv^T = I$ (but otherwise unknown distribution)

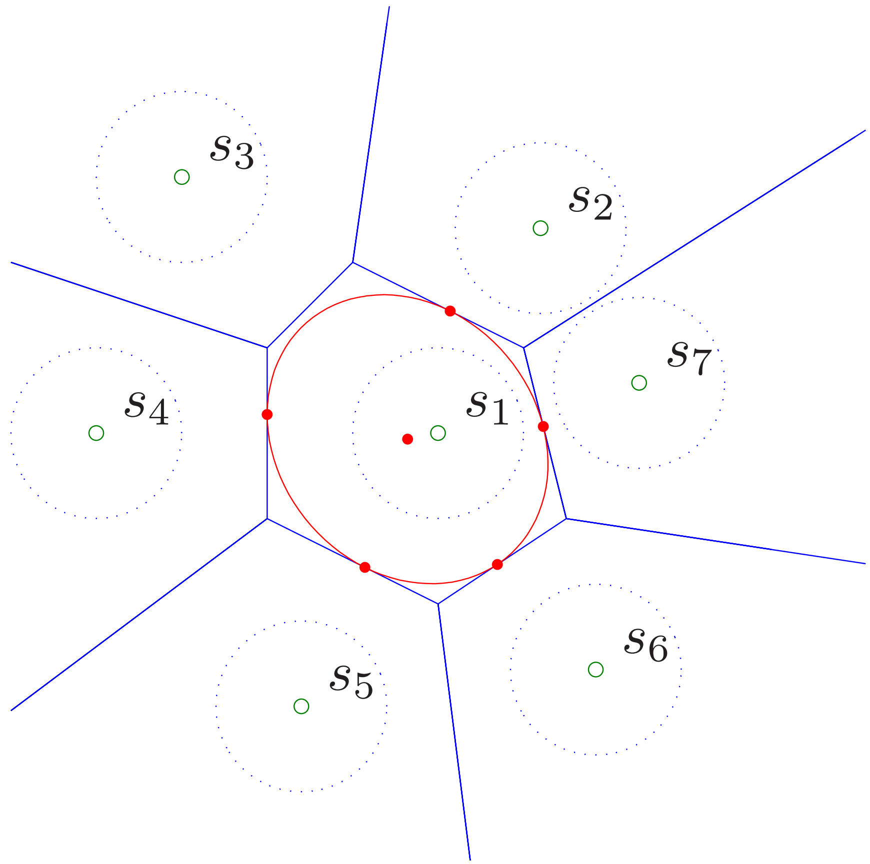
**detection problem**: given observed value of $x$, estimate $s$

**example** $(n = 2,\ N = 7)$




- detector selects symbol $s_k$ closest to received signal $x$
- correct detection if $s_k + v$ lies in the Voronoi region around $s_k$

**example**: bound on probability of correct detection of $s_1$ is $0.205$




solid circles: distribution with probability of correct detection $0.205$

# Boolean least-squares

$x \in \{-1, 1\}^n$ is transmitted; we receive $y = Ax + v$, $v \sim \mathcal{N}(0, I)$

ML estimate of $x$ found by solving **boolean least-squares problem**

$$\begin{array}{ll} \text{minimize} & \|Ax - y\|^2 \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

- could check all $2^n$ possible values of $x$ . . .
- an NP-hard problem
- many heuristics for approximate solution

# Boolean least-squares as matrix problem

$$\begin{aligned}\|Ax-y\|^2 &= x^TA^TAx - 2y^TA^Tx + y^Ty \\ &= \mathbf{Tr}\,A^TAX - 2y^TA^Tx + y^Ty\end{aligned}$$

where $X = xx^T$

hence can express BLS as

$$\begin{array}{ll}\text{minimize} & \mathbf{Tr}\,A^TAX - 2y^TA^Tx + y^Ty \\ \text{subject to} & X_{ii} = 1, \quad X \succeq xx^T, \quad \mathrm{rank}(X) = 1\end{array}$$

. . . still a very hard problem

# SDP relaxation for BLS

ignore rank one constraint, and use

$$X \succeq xx^T \iff \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0$$

to obtain **SDP relaxation** (with variables $X$, $x$)

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, A^T A X - 2y^T A^T x + y^T y \\ \text{subject to} & X_{ii} = 1, \quad \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 \end{array}$$

- optimal value of SDP gives **lower bound** for BLS
- if optimal matrix is rank one, we're done

# Stochastic interpretation and heuristic

- suppose $X$, $x$ are optimal for SDP relaxation
- generate $z$ from normal distribution $\mathcal{N}(x, X - xx^T)$
- take $x = \mathbf{sgn}(z)$ as approximate solution of BLS
  (can repeat many times and take best one)

# Interior-point methods

- handle linear and **nonlinear** convex problems (Nesterov & Nemirovsky)
- based on Newton's method applied to 'barrier' functions that trap $x$ in **interior** of feasible region (hence the name IP)
- worst-case complexity theory: # Newton steps $\sim \sqrt{\text{problem size}}$
- in practice: # Newton steps between 5 & 50 (!)
- 1000s variables, 10000s constraints feasible on PC; far larger if structure is exploited

## Log barrier

for convex problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

we define **logarithmic barrier** as

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x))$$

- $\phi$ is convex, smooth on interior of feasible set
- $\phi \to \infty$ as $x$ approaches boundary of feasible set

# Central path

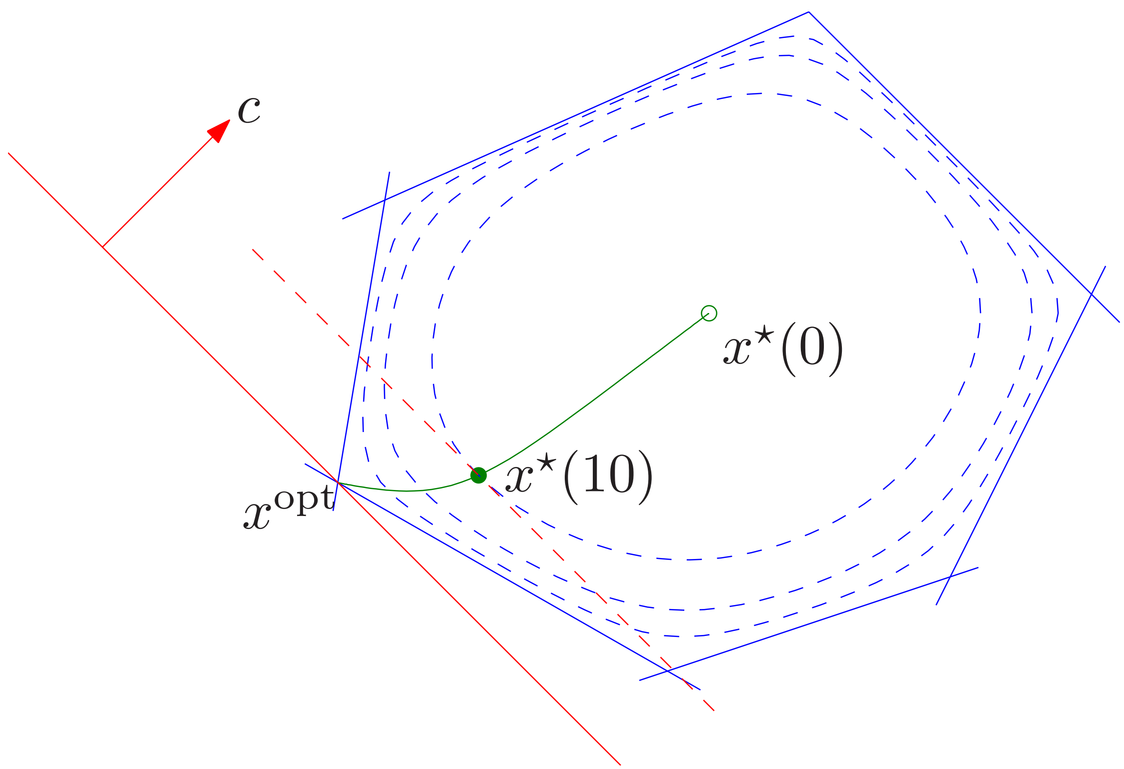
**central path** is curve

$$x^\star(t) = \underset{x}{\operatorname{argmin}} \left( t f_0(x) + \phi(x) \right), \qquad t \geq 0$$

- $x^\star(t)$ is strictly feasible, $i.e.$, $f_i(x) < 0$
- $x^\star(t)$ can be computed by, $e.g.$, Newton's method
- intuition suggests $x^\star(t)$ converges to optimal as $t \to \infty$
- using duality can prove $x^\star(t)$ is $m/t$-suboptimal

# Example: central path for LP

$$x^\star(t) = \mathrm{argmin}_x \left( tc^T x - \sum_{i=1}^6 \log(b_i - a_i^T x) \right)$$



# Barrier method

a.k.a. **path-following method**

**given** strictly feasible $x$, $t > 0$, $\mu > 1$

**repeat**

1. compute $x := x^\star(t)$

   (using Newton's method, starting from $x$)

2. **exit if** $m/t <$ tol

3. $t := \mu t$

duality gap reduced by $\mu$ each outer iteration

# Trade-off in choice of $\mu$

large $\mu$ means

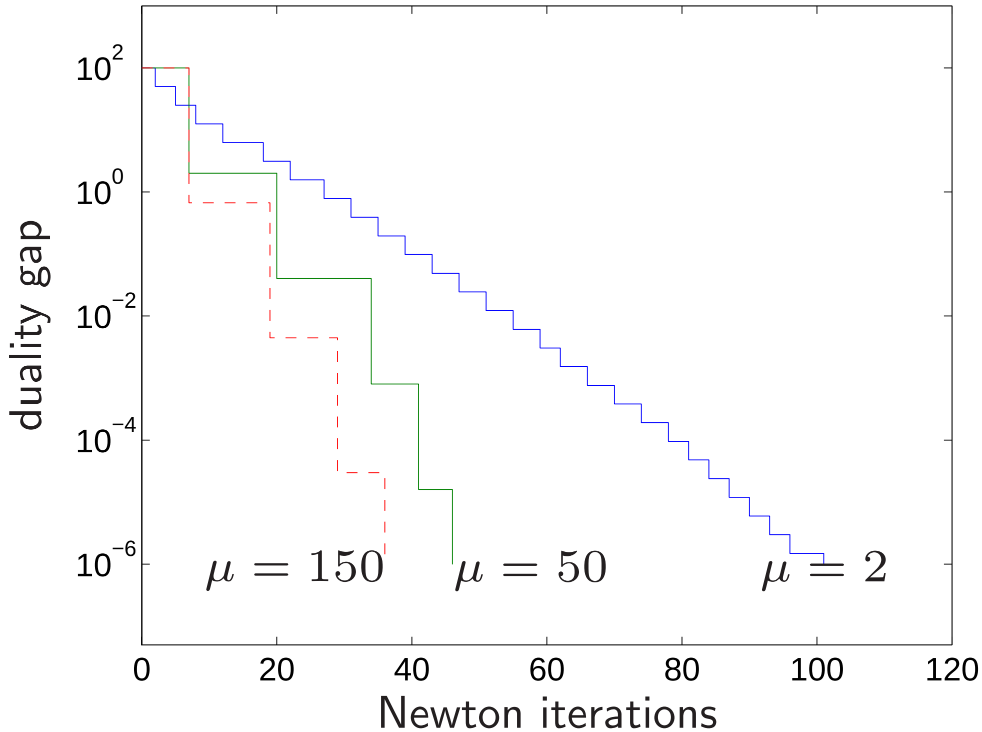
- fast duality gap reduction (fewer outer iterations), but
- many Newton steps to compute $x^\star(t^+)$
  (more Newton steps per outer iteration)

total effort measured by total number of Newton steps

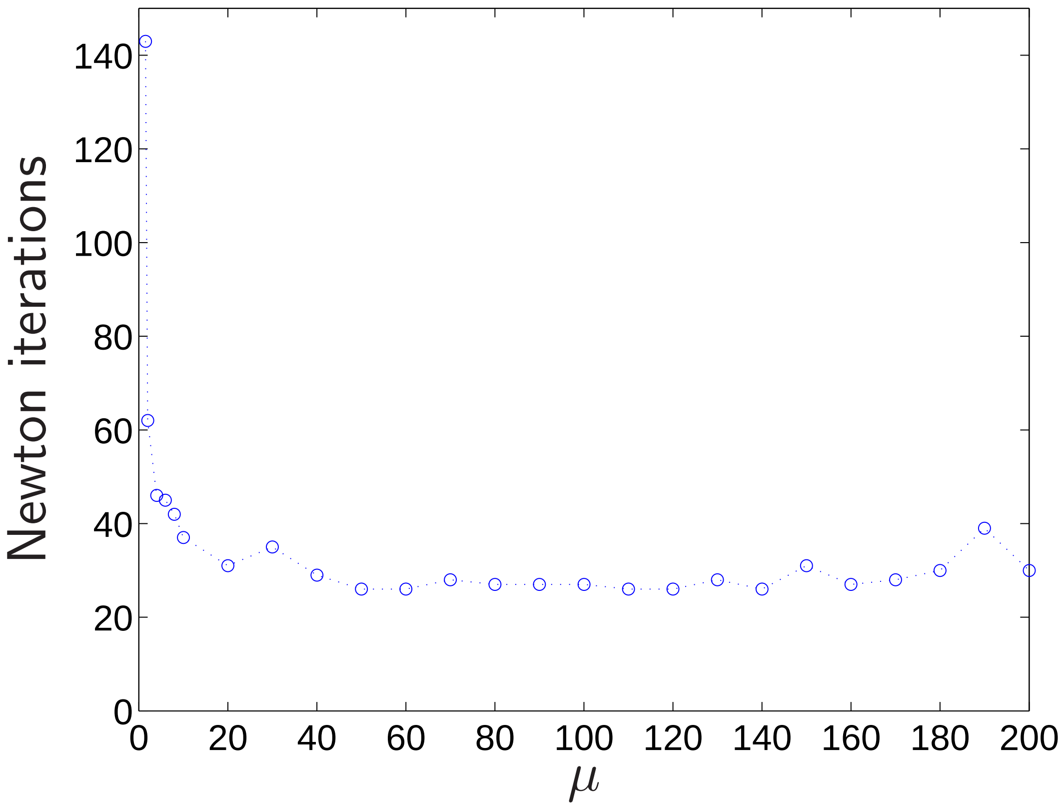
# Typical example

GP with $n = 50$ variables, $m = 100$ constraints, $m_i = 5$

- wide range of $\mu$ works well
- very typical behavior (even for large $m$, $n$)



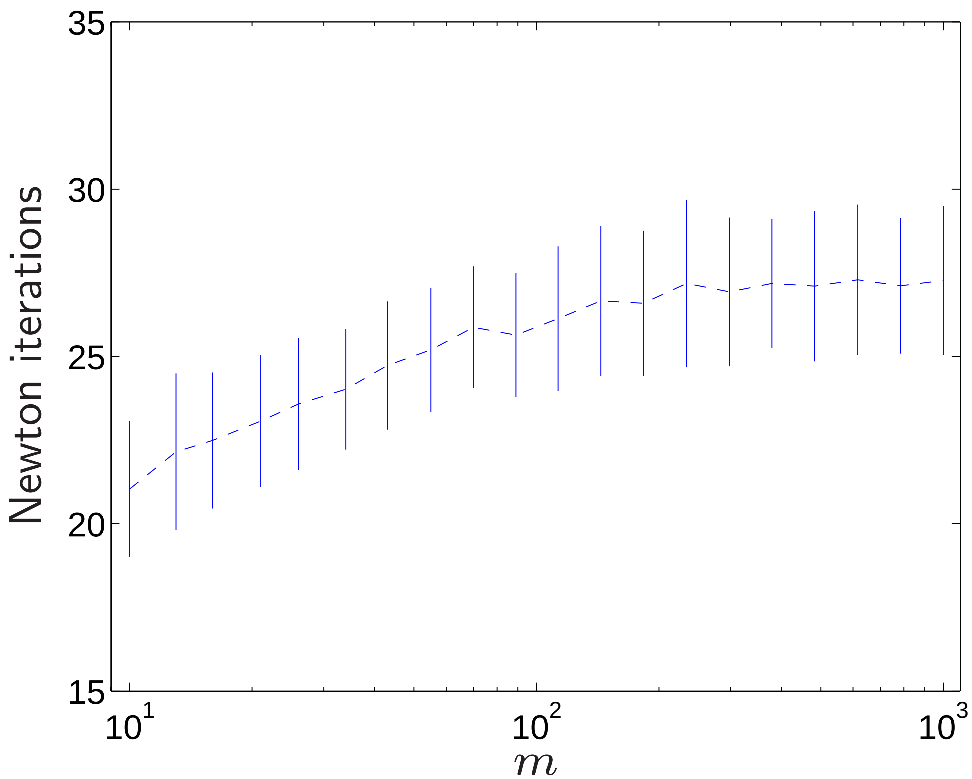
# Effect of $\mu$




barrier method works well for $\mu$ in large range

# Typical effort versus problem dimensions

- LPs with $n = 2m$ vbles, $m$ constraints
- $100$ instances for each of $20$ problem sizes
- avg & std dev shown



# Other interior-point methods

more sophisticated IP algorithms

- primal-dual, incomplete centering, infeasible start
- use same ideas, *e.g.*, central path, log barrier
- readily available (commercial and noncommercial packages)

typical performance: $10 - 50$ Newton steps (!)
— over wide range of problem dimensions, problem type, and problem data

# Complexity analysis of Newton's method

- classical result: if $|f'''|$ small, Newton's method converges fast
- classical analysis is local, and coordinate dependent
- need analysis that is global, and, like Newton's method, coordinate invariant

# Self-concordance

**self-concordant** function $f$ (Nesterov & Nemirovsky, 1988): when restricted to any line,

$$|f'''(t)| \leq 2f''(t)^{3/2}$$

- $f$ SC $\Longleftrightarrow$ $\tilde{f}(z) = f(Tz)$ SC, for $T$ nonsingular (*i.e.*, SC is coordinate invariant)

- a large number of common convex functions are SC

$$x \log x - \log x, \quad \log\det X^{-1}, \quad -\log(y^2 - x^T x), \quad \ldots$$

## Complexity analysis of Newton's method for self-concordant functions

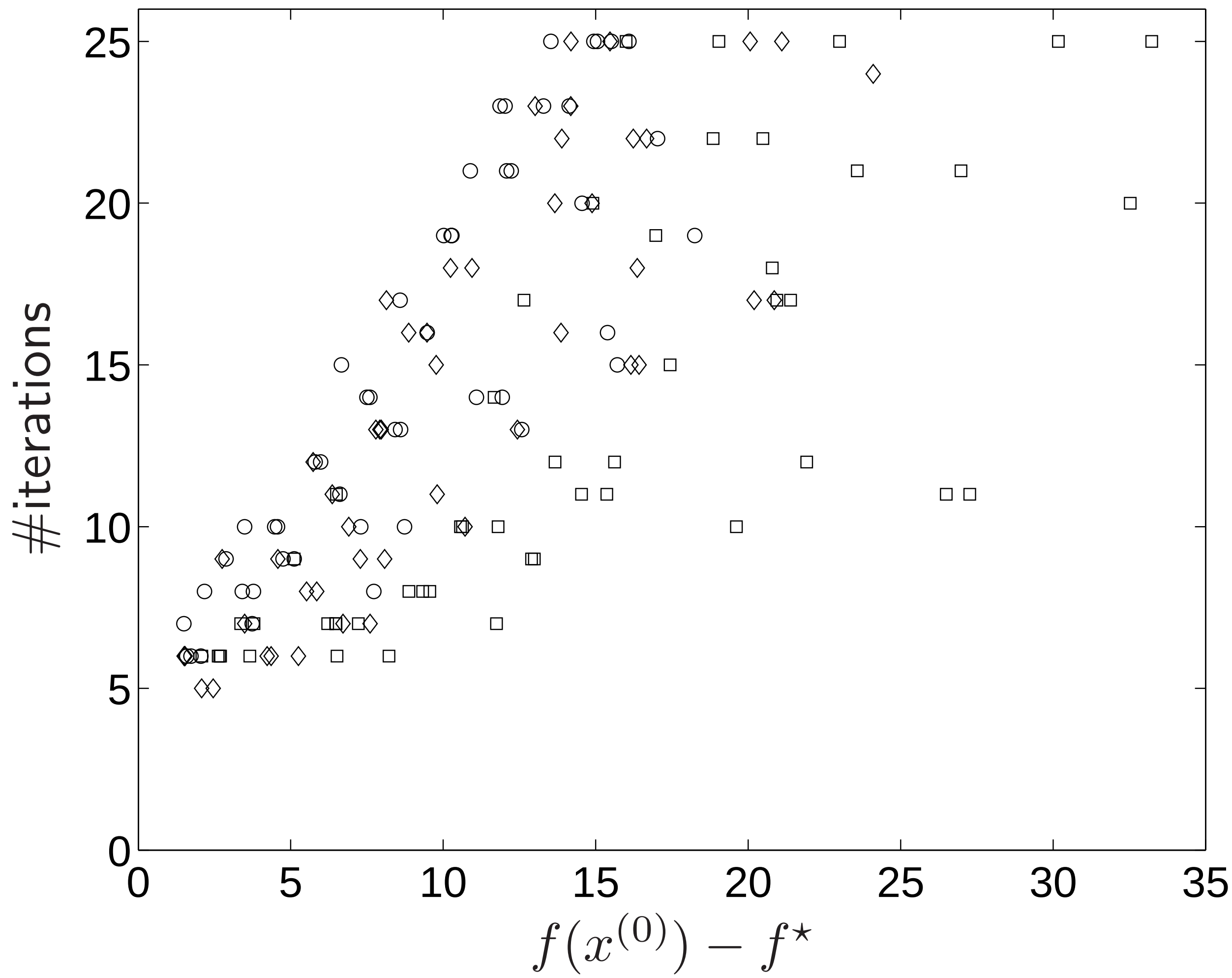
for self-concordant function $f$, with minimum value $f^\star$,

- **theorem:** #Newton steps to minimize $f$, starting from $x$:

$$\#\text{steps} \leq 11(f(x) - f^\star) + 5$$

- **empirically:** $\#\text{steps} \approx 0.6(f(x) - f^\star) + 5$

note absence of unknown constants, problem dimension, etc.



25
20
15
10
5
0
0
5
10
15
20
25
30
35
#iterations
$f(x^{(0)}) - f^\star$

## Complexity of path-following algorithm

- to compute $x^\star(\mu t)$ starting from $x^\star(t)$,

$$\#\text{steps} \leq 11m(\mu - 1 - \log\mu) + 5$$

  using N&N's self-concordance theory, duality to bound $f^\star$

- number of outer steps to reduce duality gap by factor $\alpha$: $\lceil \log\alpha / \log\mu \rceil$

- **total number of Newton steps** bounded by product,

$$\left\lceil \frac{\log\alpha}{\log\mu} \right\rceil (11m(\mu - 1 - \log\mu) + 5)$$

  . . . captures trade-off in choice of $\mu$

# Complexity analysis conclusions

- for any choice of $\mu$, #steps is $O(m \log 1/\epsilon)$, where $\epsilon$ is final accuracy
- to optimize complexity bound, can take $\mu = 1 + 1/\sqrt{m}$, which yields #steps $O(\sqrt{m} \log 1/\epsilon)$
- in any case, IP methods work extremely well in practice

# Conclusions

since 1985, lots of advances in theory & practice of convex optimization

- complexity analysis
- semidefinite programming, other new problem classes
- efficient interior-point methods & software
- **lots of applications**

# Some references

- *Semidefinite Programming*, SIAM Review 1996
- *Determinant Maximization with Linear Matrix Inequality Constraints*, SIMAX 1998
- *Applications of Second-order Cone Programming*, LAA 1999
- *Linear Matrix Inequalities in System and Control Theory*, SIAM 1994
- *Interior-point Polynomial Algorithms in Convex Programming*, SIAM 1994, Nesterov & Nemirovsky
- *Lectures on Modern Convex Optimization*, SIAM 2001, Ben Tal & Nemirovsky

# Shameless promotion

*Convex Optimization*, Boyd & Vandenberghe

- to be published 2003
- pretty good draft available at Stanford EE364 (UCLA EE236B) class web site as course reader