Interactive Loop-Shaping Design of MIMO Controllers

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Abstract

We show that control system design via *classical loop shaping* and *singular value loop shaping* can be formulated as a *closed-loop convex* problem [3, 4, 11]. Consequently, loop shaping problems can be solved by efficient numerical methods. In particular, these numerical methods can always determine whether or not there exists a compensator that satisfies a given set of loop shaping specifications. Problems such as maximizing bandwidth subject to given margin and cutoff specifications can be directly solved. Moreover, any other closed-loop convex specifications, such as limits on step-response overshoot, tracking errors, and disturbance rejection, can be simultaneously considered.

These observations have two practical ramifications. First, closed-loop convex design methods can be used to synthesize compensators in a framework that is familiar to many control engineers. Second, closed-loop convex design methods can be used to aid the designer using classical loop shaping by computing absolute performance limits against which a classical design can be compared.

1 Introduction

We give a brief overview of classical and singular value loop shaping, which also serves to describe our notation.

1.1 Classical loop shaping

We first consider the standard classical one degree-of-freedom single-actuator, single-sensor (SASS) control system shown in figure 1. Here u is the actuator signal, y is the output signal, e is the (tracking) error signal, r is the reference or command signal, and d_{sensor} is a sensor noise. The plant and compensator are linear and time-invariant (LTI), with transfer functions given by P and C, respectively. The plant is given and the compensator is to be designed.

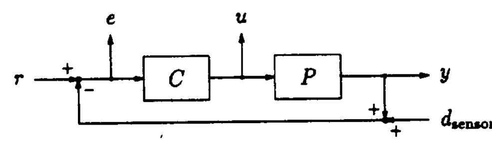

Figure 1 Classical 1-DOF control system.

In classical loop shaping, the designer focuses attention on the loop transfer function, given by

$$L \triangleq PC.$$

The *transmission* or input/output (I/O) transfer function

$$T \triangleq L/(1+L)$$

is the closed-loop transfer function from the reference input r to the output y. Its negative, $-T$, is the closed-loop transfer function from the sensor noise d_{sensor} to the output y.

The *sensitivity* transfer function is given by

$$S \triangleq 1/(1+L).$$

S is the transfer function from the reference input r to the tracking error e.

Classical loop shaping design is based on two important observations:

- the loop transfer function L has a very simple dependence on the compensator transfer function C, especially in a logarithmic (gain and phase) representation.
- many important requirements for the closed-loop system can be approximately reflected as requirements on the loop gain L.

Loop-shaping specifications constrain the magnitude and possibly the phase of the loop transfer function at each frequency. There are three basic types of loop-shaping specifications, which are imposed in different frequency bands:

- *In-band specifications.* At these frequencies we require $|L|$ to be large, so that S is small and $T \approx 1$. This ensures good command tracking, and low sensitivity to plant variations, two of the most important benefits of feedback.

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- *Cutoff specifications.* At these frequencies we require $|L|$ to be small, so that T is small. This ensures that the output y will be relatively insensitive to the sensor noise d_{sensor}, and that the system will remain closed-loop stable in the face of plant variations at these frequencies, for example, excess phase from small delays and unpredictable (or unmodeled) resonances.

- *Crossover (margin) specifications.* Crossover or transition band specifications are imposed between the control bands (where L is large) and cutoff bands (where L is small). At these frequencies the main concern is to keep L a safe distance away from the critical point -1 (closed-loop stability depends on the winding number of L with respect to -1). Classical specifications include gain margin and phase margin. More natural "modern" specifications exclude L from some circle about -1.

The Nyquist criterion (which constrains the winding number of L about -1) is also included as an implicit specification that ensures closed-loop stability.

A typical set of loop shaping specifications is:

$$
\begin{array}{ll}
|L(j\omega)| \geq l(\omega) & \text{for } 0 \leq \omega \leq \omega_B = 2, \\
|L(j\omega)| \leq u(\omega) & \text{for } \omega \geq \omega_C = 5, \\
-150^\circ \leq \angle L(j\omega) \leq 30^\circ & \text{for } \omega_B = 2 \leq \omega \leq \omega_C = 5
\end{array}
$$

where l and u are the frequency dependent constraint functions shown in figure 2. The in-band and cutoff constraints, which consist of frequency dependent restrictions on the magnitude of L, are conveniently shown on a Bode magnitude plot, while the margin constraint, which is often independent of frequency, is conveniently shown on a Nyquist plot (see figure 3).

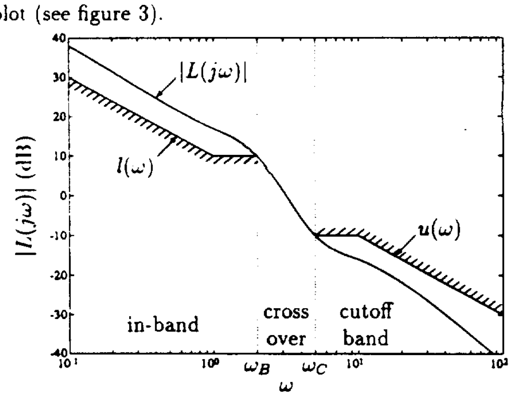


Figure 2 A typical set of in-band and cutoff specifications. In the in-band region, $\omega \leq \omega_B$, the loop gain magnitude $|L|$ is required to exceed the frequency dependent lower bound $l(\omega)$. In the cutoff region, $\omega \geq \omega_C$, the loop gain magnitude $|L|$ is required to be below the upper bound $u(\omega)$. In the crossover region, $\omega_B < \omega < \omega_C$, the loop gain crosses $|L| = 0\mathrm{dB}$.

In this example, the in-band region is $\omega \leq \omega_B$. Over this region, the large loop gain will ensure good command tracking ($T \approx 1$), and low sensitivity ($|S| \ll 1$). In the cutoff region, $\omega \geq \omega_C$, the small loop gain ensures that sensor noise will not affect the output, and small time-delays and variations in P will not destabilize the closed-loop system. In the in-band region, L cannot be close to the critical point -1 since $|L|$ exceeds $+10\mathrm{dB}$ there; similarly, in the cutoff region, $|L|$ is less than $-10\mathrm{dB}$ and so cannot be close to -1. The margin specification ensures that L cannot be too close to -1 in the transition region $\omega_B \leq \omega \leq \omega_C$ by constraining $\angle L$. Of course, the phase bounds in the margin constraint can be frequency dependent.

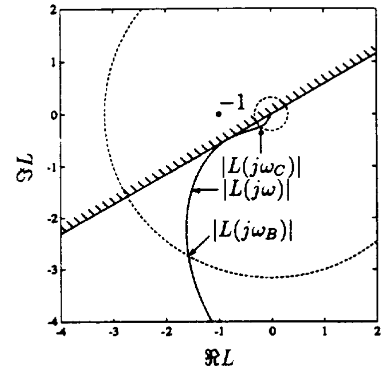


Figure 3 A typical margin specification requires the phase of the loop transfer function to lie between -150° and $+30^\circ$ over the frequency band $\omega_B \leq \omega \leq \omega_C$. The dotted circles shown correspond to the magnitude constraints $|L(j\omega)| > +10\mathrm{dB}$ and $|L(j\omega)| < -10\mathrm{dB}$ which must be satisfied for $\omega \leq \omega_B$ and $\omega \geq \omega_C$, respectively.

While many important closed-loop properties can be specified via L, some cannot. For example, loop shaping does not explicitly include specifications on $C/(1 + PC)$ (actuator effort) and $P/(1 + PC)$ (effect of input-referred process noise on y). A design will clearly be unsatisfactory if either of these transfer functions is too large. The specification that these transfer functions should not be too large is usually included as implicit "side information" in a classical loop shape design. Specifications that limit the size of these transfer functions are closed-loop convex, however, and so are readily incorporated in a closed-loop convex formulation.

Given a desired set of loop shaping specifications, the compensator C is typically synthesized by adding dynamics until the various requirements on the loop transfer function L are satisfied (or until the designer suspects that the loop shaping specifications cannot be met).

Classical loop shaping is described in many texts; see, for example, [1, 8, 10, 7]. The discussions found in these references emphasize techniques that help the engineer "do" loop-shaping design. With the exception of Bode's work on optimal cutoff characteristics and integral constraints, these references do not consider questions such as:

- Is there a compensator that meets a given set of loop-shaping specifications?

- For a given set of in-band and margin specifications and shape of the cutoff specification, what is the smallest cutoff frequency that can be achieved?

- For a given set of cutoff and margin specifications, how large can the loop gain be made in the in-band region?

The main point of this paper is that such questions are readily answered.

1.2 Singular value loop shaping

We now consider the case in which there are multiple actuators and multiple sensors (MAMS) in the control system shown in figure 1. The plant P and compensator C are given by transfer matrices: P is n_{sens} by n_{act} and C is n_{act} by n_{sens}, where n_{sens} is the number of sensors, and n_{act} is the number of actuators.

Unlike the SASS case, there is no longer a unique choice for the "loop transfer function." A common choice is the loop transfer matrix cut at the sensors:

$$L \triangleq PC.$$

The transmission or input/output (I/O) transfer matrix is

$$T \triangleq (I+L)^{-1}L,$$

and the sensitivity transfer matrix is given by

$$S \triangleq (I+L)^{-1}.$$

These transfer matrices have interpretations that are are similar to those in SASS case. For example, if the plant transfer matrix P changes to $(I+\Delta)P$, then the I/O transfer matrix T, to first order, changes to $(I+S\Delta)T$. (Note that Δ can be interpreted as the output-referred the output-referred fractional change in the I/O transfer matrix T is then given b $S\Delta$ [5, 3].)

In contrast, the loop transfer matrix cut at the actuators is denoted $\tilde{L}$, the complementary loop transfer matrix:

$$\tilde{L} \triangleq CP.$$

Note that the loop transfer matrix and the complementary loop transfer matrix may have different dimensions: L is n_{sens} by n_{sens}, while $\tilde{L}$ is n_{act} by n_{act}. Moreover, loop specifications on L and $\tilde{L}$ are in general different and inequivalent. For example, it is possible for L to be "large" (in the sense to be described below), while $\tilde{L}$ is not "large."

At in-band frequencies, singular value loop shaping specifications have the form

$$\sigma_{\min}(L(j\omega)) \geq l(\omega) > 1,$$

where l is some frequency dependent bound. For cutoff frequencies, singular value loop shaping specifications have the form

$$\sigma_{\max}(L(j\omega)) \leq u(\omega) < 1,$$

where u is some frequency dependent bound.

These specifications are often depicted on a singular value Bode plot, as in figure 4.

(This discussion assumes that there are at least as many actuators as actuators, *i.e.*, $n_{act} \geq n_{sens}$. If not, the in-band

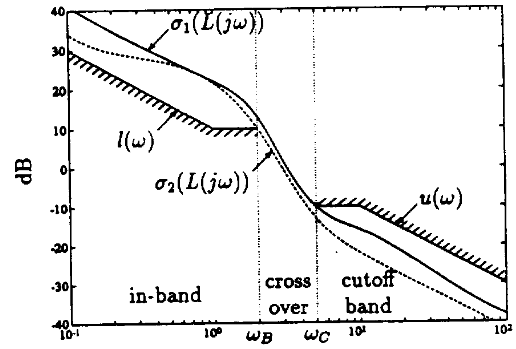


Figure 4 Examples of in-band and cutoff specifications on the loop gain L in a system with two sensors. In the in-band region, $\omega \leq \omega_B$, both singular values of L are required to exceed the lower bound. In the cutoff region, $\omega \geq \omega_C$, both singular values of L are required to be below the upper bound. In the crossover region each singular value of L crosses 0dB.

specifications above are guaranteed to be infeasible since at all frequencies at least one singular value of L is zero. In this case, similar specifications can be imposed on $\tilde{L}$.)

It is difficult to formulate margin specifications that are directly analogous to the gain or phase margin constraints used in the SASS case. The general idea is to ensure that $L+I$ stays "sufficiently invertible" in the crossover band. One effective method simply limits the minimum singular value of this matrix:

$$\sigma_{\min}(L+I) \geq r > 0,$$

or equivalently,

$$\sigma_{\max}(S) \leq 1/r.$$

2 A Closed-Loop Convex Formulation

A design specification is *closed-loop convex* if it is equivalent to some closed-loop transfer function or matrix (*e.g.*, the sensitivity S) belonging to a convex set.

As a specific example, consider the specification

$$|L(j\omega)| \geq 3 \quad \text{for } 0 \leq \omega \leq 1. \tag{1}$$

We will see that this is equivalent to

$$|S(j\omega) - 1/8| \leq 3/8 \quad \text{for } 0 \leq \omega \leq 1. \tag{2}$$

Now, the set of transfer functions S that satisfy (2) is convex, since if $S^{(a)}$ and $S^{(b)}$ both satisfy (2), then so does $(S^{(a)} + S^{(b)})/2$. Therefore the specification (1) is closed-loop convex. See [3, 4] for extensive discussions.

The main result of this paper is that many classical and singular value loop-shaping specifications are closed-loop convex.

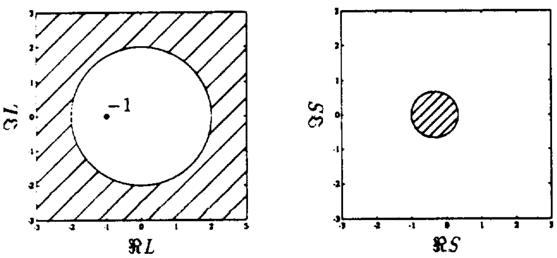
2.1 SASS case

2.1.1 In-band specifications

We first consider the in-band specification $|L(j\omega)| \geq \alpha$, where $\alpha > 1$. It is closed-loop convex since it is equivalent to the following convex specification on the sensitivity S, (a closed-loop transfer function):

$$|L(j\omega)| \geq \alpha > 1 \iff \left|S(j\omega) - \frac{1}{\alpha^2-1}\right| \leq \frac{\alpha}{\alpha^2-1}. \quad (3)$$

Figure 5 illustrates this correspondence for $\alpha = 2$.



(a) (b)

Figure 5 The region $|L| \geq 2$ in the L-plane is shown in (a). The loop shaping specification $|L| \geq 2$ requires the Nyquist plot of L to lie in the shaded region in (a). The corresponding region in the S-plane is shown in (b), which is a disk that includes but is not centered at 0. This region is convex, and hence the loop gain specification $|L| \geq 2$ is closed-loop convex.

2.1.2 Cutoff specifications

We now consider the cutoff specification $|L(j\omega)| \leq \alpha$, where $\alpha < 1$. It is also closed-loop convex since it is equivalent to the following convex specification on the sensitivity:

$$|L(j\omega)| \leq \alpha < 1 \iff \left|S(j\omega) - \frac{1}{1-\alpha^2}\right| \leq \frac{\alpha}{1-\alpha^2}. \quad (4)$$

Figure 6 illustrates this correspondence for $\alpha = 0.5$.

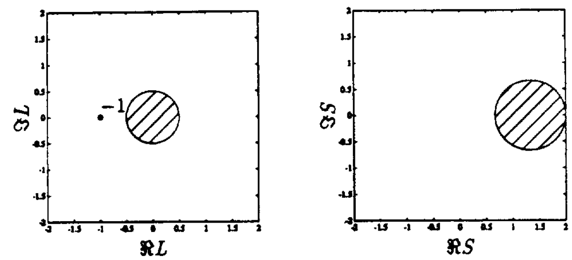
The results (3) and (4) are easily established. Since we give a careful proof for the more general MAMS case, we give a simple discussion here. The in-band loop specification (3) requires L to lie outside a circle of radius α in the complex plane. Since $\alpha > 1$, the critical point -1 lies in the interior of this circle. Since S and L are related by the bilinear transformation $S = 1/(1+L)$, this circle maps to another circle in the S-plane. To find this circle, we note that the points $L = \pm\alpha$ map to $S = 1/(1 \pm \alpha)$, and the circle must be symmetric with respect to the real axis. Moreover since the critical point -1 is mapped to $S = \infty$, the exterior of the $|L| = \alpha$ circle maps to the interior of the circle in the S-plane.

We note that the specifications requiring L to be "not too big,"

$$|L(j\omega)| \leq \alpha \quad \text{where } \alpha > 1,$$

and requiring L to be "not too small,"

$$|L(j\omega)| \geq \alpha \quad \text{where } \alpha < 1,$$



(a) (b)

Figure 6 The region $|L| \leq 0.5$ in the L-plane is shown in (a). The loop shaping specification $|L| \leq 0.5$ requires the Nyquist plot of L to lie in the shaded region in (a). The corresponding region in the S-plane is shown in (b), which is a disk that includes but is not centered at 1. This region is convex, and hence the loop gain specification $|L| \leq 0.5$ is closed-loop convex.

are not closed-loop convex, since these specifications are equivalent to $S(j\omega)$ lying outside of the shaded disks in figures 5(b) and 6(b). These specifications, however, are not likely to be used in a practical design. It is interesting that the sensible specifications on $|L|$, given in (3) and (4), turn out to coincide exactly with the specifications on $|L|$ that are closed-loop convex.

2.1.3 Phase margin specifications

A common form for a margin specification limits the phase of the loop transfer function in the crossover band:

$$\theta_{\min} \leq \angle L(j\omega) \leq \theta_{\max},$$

where $-180° < \theta_{\min} < 0°$ and $0° < \theta_{\max} < 180°$. It turns out that such a specification is closed-loop convex if and only if $\theta_{\max} - \theta_{\min} \leq 180°$, in which case S must lie in the intersection of two disks:

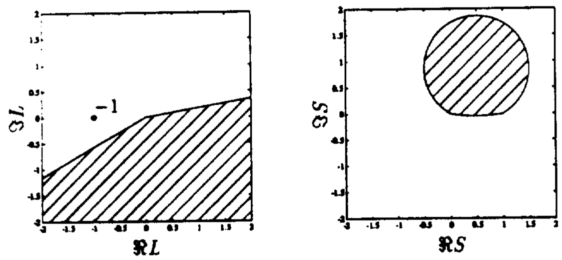
$$|2S(j\omega) - (1 + j/\tan\theta_{\max})| \leq 1/\sin\theta_{\max} \text{ and} \quad (5)$$

$$|2S(j\omega) - (1 + j/\tan\theta_{\min})| \leq 1/\sin -\theta_{\min}. \quad (6)$$

This is shown in figure 7 for the case $\theta_{\min} = -150°$, $\theta_{\max} = 10°$. The phase margin specification $-150° \leq \angle L \leq 10°$ is equivalent to requiring the sensitivity S to lie in the convex set shown in figure 7(b).

2.1.4 General circle specifications

All of the specifications above—in-band, cutoff, and phase margin, are special cases of *general circle specifications.* Consider any generalized circle in the complex plane (*i.e.*, a circle or a line, which we consider a "circle" centered at ∞) that does not pass through the critical point -1. Such a circle divides the complex plane into two regions, one of which includes the critical point -1. The specification that the loop transfer function must lie in the region that does not contain -1 is what we call a *generalized circle constraint*, and is readily shown (by a mapping argument) to be closed-loop convex since it is equivalent to S lying inside a circle or half-plane.



(a) (b)

Figure 7 The region $-150° \le \angle L \le 10°$ in the L-plane is shown in (a). The phase margin specification $-150° \le \angle L \le 10°$ requires the Nyquist plot of L to lie in the shaded region in (a). The corresponding region in the S-plane is the intersection of two disks, shown in (b), and hence the phase margin specification $-150° \le \angle L \le 10°$ is closed-loop convex.

We note that generalized circle constraints have appeared in many contexts. Examples include the circle criterion, used in stability and robustness analysis of nonlinear systems (see [16, 9, 14, 13]), the Popov criterion (with a fixed Popov parameter) [12], and many of the specifications in [6, 15].

2.2 MAMS Case

2.2.1 In-band and Cutoff Specifications

The analogous results for the MAMS case are:

$$\alpha < 1: \quad \sigma_{\max}(L) \le \alpha \iff \sigma_{\max}((1-\alpha^2)S - I) \le \alpha \quad (7)$$
$$\alpha > 1: \quad \sigma_{\min}(L) \ge \alpha \iff \sigma_{\max}((1-\alpha^2)S - I) \le \alpha \quad (8)$$

(we have suppressed the frequency arguments for simplicity). Note that the right-hand sides of (7) and (8) are the same. Thus, the inequality on the right-hand side expresses in one formula all reasonable in-band and cutoff loop shaping specifications:

$$\sigma_{\max}((1-\alpha^2)S - I) \le \alpha \qquad \text{both in-band } (\alpha > 1) \text{ and cutoff } (\alpha < 1). \quad (9)$$

(The same correspondences hold with $\tilde{L}$ and $\tilde{S}$.)

We now establish (7). Since $\alpha < 1$, S is nonsingular, and we have

$$\begin{aligned} \sigma_{\max}(L) \le \alpha &\iff \sigma_{\max}(S^{-1} - I) \le \alpha \\ &\iff (S^{-1} - I)^*(S^{-1} - I) \le \alpha^2 I. \end{aligned}$$

Multiplying the last inequality by S^* on the left and S on the right, and multiplying by $1 - \alpha^2 > 0$ (since $\alpha < 1$), gives

$$\begin{aligned} \sigma_{\max}(L) \le \alpha &\iff (1-\alpha^2)^2 S^*S - (1-\alpha^2)S^* \\ &\qquad -(1-\alpha^2)S + I \le \alpha^2 I \\ &\iff \sigma_{\max}((1-\alpha^2)S - I) \le \alpha, \end{aligned}$$

which is (7).

The in-band result (8) is established in a similar manner. Since $\alpha > 1$, S is nonsingular, and so

$$\begin{aligned} \sigma_{\min}(L) \ge \alpha &\iff \sigma_{\min}(S^{-1} - I) \ge \alpha \\ &\iff (S^{-1} - I)^*(S^{-1} - I) \ge \alpha^2 I. \end{aligned}$$

We proceed as before, except that $1 - \alpha^2 < 0$ (since $\alpha > 1$), so the inequality is reversed:

$$\begin{aligned} \sigma_{\min}(L) \ge \alpha &\iff (1-\alpha^2)^2 S^*S - (1-\alpha^2)S^* \\ &\qquad -(1-\alpha^2)S + I \le \alpha^2 I \\ &\iff \sigma_{\max}((1-\alpha^2)S - I) \le \alpha, \end{aligned}$$

which is (8).

2.2.2 General sector specifications

The in-band and cutoff specifications (7) and (8) are special forms of *general sector specifications*, which we now describe. Given complex matrices C and R such that $(I + C)^*(I + C) > R^*R$, the specification

$$(L - C)^*(L - C) \le R^*R \quad (10)$$

is closed-loop convex. This specification can be interpreted as requiring L to be in a neighborhood of "radius" R about the "center" C that excludes $-I$. The specification (10) reduces to (7) when $C = 0$ and $R = \alpha I$.

Similarly, given complex matrices C and R such that $(I + C)^*(I + C) < R^*R$, then

$$(L - C)^*(L - C) \ge R^*R \quad (11)$$

is closed-loop convex. This specification can be interpreted as requiring L to be outside a neighborhood of "radius" R about C that includes $-I$. The specification (11) reduces to (8) when $C = 0$ and $R = \alpha I$.

The specifications (10) and (11) are closely connected to the conic sector conditions developed by Zames [17] and Safonov [13]. For example, if C, R, R^{-1}, and L are stable transfer matrices and (10) is imposed at all frequencies, then, using the terminology of Safonov, (10) implies that Graph(L) is inside Cone(C, R). These sector conditions form the basis of various MAMS generalizations of classical frequency domain stability and robustness criteria.

With $C = -I$ and $R = \alpha I$, (11) excludes L from a neighborhood about the critical point $-I$:

$$\sigma_{\min}(L + I) \ge \alpha.$$

This is equivalent to

$$\sigma_{\max}(S) \le 1/\alpha, \quad (12)$$

which limits the size of the closed-loop sensitivity transfer matrix. Specifications such as (12) that limit the size of a closed-loop transfer matrix, when imposed at all frequencies, can be interpreted as circle criterion constraints that guarantee robustness in the face of various types and locations of nonlinearities.

3 Conclusions

We have shown that many classical and singular value loop shaping problems are closed-loop convex. Consequently, loop shaping problems can be solved by efficient numerical methods. In particular, it can be determined whether or not a compensator exists that satisfies a given set of loop shaping specifications. Loop shaping design problems that are formulated as classical optimization problems, *e.g.*, maximizing bandwidth subject to given margin and cutoff specifications, can be solved by direct numerical methods for quasiconvex optimization.

A consequence of these observations is that closed-loop convex design methods can be used to do compensator design in a classical loop shaping framework which is familiar to many control engineers. In contrast with classical compensator design methods, in which the designer must decide how to vary *parameters* (such as poles, zeros, and gain) in such a way that the loop transfer function meets the specifications, the designer can directly manipulate the loop shaping *specifications*, since the step of finding a suitable compensator, or determining that none exists, can be automated.

We comment, however, that classical loop shaping is an indirect design technique originally developed before the advent of computers, and is not a particularly good method for compensator design, and especially, computer-aided compensator design. In our opinion, every specification that can be expressed in terms of the loop transfer function or matrix can be more directly expressed in terms of some closed-loop transfer function or transfer matrix, so that compensator design directly from closed-loop (convex) specifications is more direct and natural (see [2, 3]).

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