COMPUTING BOUNDS FOR THE STRUCTURED SINGULAR VALUE VIA AN INTERIOR POINT ALGORITHM

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Abstract

We describe an interior point algorithm for computing the upper bound for the structured singular value described in [1]. We demonstrate the performance of the algorithm on a simple example.

1. Notation

$\mathbf{R}$ ($\mathbf{C}$) stands for the set of real (complex) numbers. $\mathbf{R}^{m\times n}$ ($\mathbf{C}^{m\times n}$) stands for the set of real (complex) $m\times n$ matrices. For $M\in\mathbf{C}^{m\times m}$, $\det(M)$ stands for the determinant, $\sigma_{\max}(M)$ the maximum singular value and M^* the complex conjugate of the transpose of M. I_n stands for the $n\times n$ identity matrix.

2. Introduction

An important quantity in robustness analysis in the presence of structured uncertainties is the structured singular value (SSV) of an $n\times n$ complex matrix M, defined as

$$\mu_{\mathcal{Q}}(M) \stackrel{\Delta}{=} \begin{cases} 0 & \text{if } \det(I_n+M\Delta)\neq 0 \text{ for all } \Delta\in\mathcal{Q} \\ \left(\min\limits_{\substack{\Delta\in\mathcal{Q} \\ \det(I_n+M\Delta)=0}} \sigma_{\max}(\Delta) \right)^{-1} & \text{else.} \end{cases}$$

Here $\mathcal{Q}$ is a subset of $\mathbf{C}^{n\times n}$ describing the uncertainty structure, where typically, every element Δ of $\mathcal{Q}$ is of the form

$$\begin{aligned} \Delta = \operatorname{diag}(&\delta_1^r I_{k_1},\ldots,\delta_p^r I_{k_p}, \\ &\delta_1^c I_{k_{p+1}},\ldots,\delta_q^c I_{k_{p+q}},\Delta_1^C,\ldots,\Delta_r^C) \end{aligned}$$

where $\delta_i^r\in\mathbf{R}$, $\delta_i^c\in\mathbf{C}$, and $\Delta_i^C\in\mathbf{C}^{c_i\times c_i}$.

Then the set $\mathcal{D}$ of matrices commuting with all elements of $\mathcal{Q}$ has elements of the form

$$\begin{aligned} D = \operatorname{diag}\big(&D_1^C,\ldots,D_p^C, \\ &D_{p+1}^C,\ldots,D_{p+q}^C, d_1^c I_{c_1},\ldots,d_r^c I_{c_r}\big) \end{aligned}$$

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where $D_i^C\in\mathbf{C}^{k_i\times k_i}$ and $d_i^C\in\mathbf{C}$.

$\mu_{\mathcal{Q}}(M)$ is hard to compute fast and reliably; in [1], Fan *et al.* describe an upper bound for $\mu_{\mathcal{Q}}(M)$ given by

$$\eta_{\mathcal{Q}}(M) \stackrel{\Delta}{=} \inf_{\substack{P\in\mathcal{P} \\ G\in\mathcal{G}}} \sqrt{\max\left(0, F(M,P,G)\right)} \tag{1}$$

where

$F(M,P,G) = \lambda_{\max}\left(M^*PM + j(GM-M^*G), P\right),$
$\mathcal{P} = \{P\in\mathcal{D} \mid P=P^*>0\}$, and
$\mathcal{G} = \{G\in\mathcal{D} \mid G\Delta \text{ is Hermitian for all } \Delta\in\mathcal{Q}\}$.

$\lambda_{\max}(X,Y)$ stands for the maximum generalized eigenvalue of the pair $X=X^T$, $Y=Y^T$ defined as

$$\lambda_{\max}(X,Y) \stackrel{\Delta}{=} \inf\{\ \lambda\in\mathbf{R} \mid \lambda Y - X > 0\ \}.$$

It can be shown that the computation of $\eta_{\mathcal{Q}}(M)$ is a nondifferentiable, quasi-convex optimization problem [2], so methods such as Kelley's cutting-plane algorithm or the ellipsoid algorithm of Shor, Nemirovksy, and Yudin are guaranteed to minimize it. In this paper we describe an interior point method for computing $\eta_{\mathcal{Q}}(M)$ more efficiently and describe its performance on a simple example.

3. Interior Point Algorithm

It is readily shown that (1) can be recast as an optimization problem of the form

$$\min \quad \lambda_{\max}(A(x),B(x)) \tag{2}$$

where the vector x (of length N, say) contains optimization variables (consisting of the independent entries of P and G), and A, B are affine functions from $\mathbf{R}^N$ into the spaces of real symmetric matrices of size $n\times n$:

$$\begin{aligned} A(x) &\stackrel{\Delta}{=} A_0 + \textstyle\sum_{i=1}^N x_i A_i, \\ B(x) &\stackrel{\Delta}{=} B_0 + \textstyle\sum_{i=1}^N x_i B_i, \end{aligned} \tag{3}$$

where $A_i = A_i^T$, $B_i = B_i^T \in \mathbf{R}^{n\times n}$.

The algorithm is based on the notion of the *analytic center* of an affine matrix inequality, say $D(x) = D_0 + \sum_{i=1}^N x_i D_i > 0$. With $\mathbf{X}$ denoting the 'feasible' set (which we assume is bounded)

$$\mathbf{X} \triangleq \{\ x \in \mathbf{R}^N \ \mid D(x) > 0\ \},$$

the analytic center x^* of the inequality $D(x) > 0$ is defined as

$$x^* = \mathrm{argmin}_{x\in\mathbf{X}} \log\det D(x)^{-1}.$$

The function $\log\det D(x)^{-1}$ is finite if and only if $x \in \mathbf{X}$, and becomes infinite as x approaches the boundary of $\mathbf{X}$, *i.e.*, it is a *barrier function* for $\mathbf{X}$. There are many other barrier functions for $\mathbf{X}$, but this one enjoys many special properties. For more details about this barrier function, see [3] and [4], where Nesterov an Nemirovski give sharp bounds on the number of computations needed to find x^*.

Starting with any feasible $x^{(0)}$, and a $\lambda^{(0)} = \lambda_{\max}(A(x^{(0)}), B(x^{(0)}))$, the algorithm proceeds as follows:

$$\begin{aligned} \lambda^{(i+1)} &:= (1-\theta)\lambda_{\max}(A(x^{(i)}), B(x^{(i)})) + \theta\lambda^{(i)} \\ x^{(i+1)} &:= \text{analytic center of } \lambda^{(i+1)}B(x) - A(x) > 0. \end{aligned}$$

Here, $\theta \in (0,1)$ is a parameter which is typically small. It enables one to take $x^{(i)}$ as an initial guess for the Newton-type method that finds the analytic center of the inequality $\lambda^{(i+1)}B(x) - A(x) > 0$. Indeed, for $\theta = 0$, $x^{(i)}$ is not a valid initial guess, as $\lambda^{(i+1)}B(x^{(i)}) - A(x^{(i)})$ is singular.

A proof of convergence for this algorithm is given in [5]. We have found that it performs considerably better than other competing methods, such as ellipsoid or cutting plane algorithms.

A number of assumptions on A, B must be made in order to ensure convergence, such as compactness of the 'level sets' $\{x \mid \lambda B(x) - A(x) > 0\}$. These assumptions are satisfied by optimizing P over the set $\overline{\mathcal{P}} = \{P \in \mathcal{P} \mid \mathbf{Tr}P \leq n\}$ instead of $\mathcal{P}$ in (1).

4. An example

We consider a simple example with

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix},$$

with $\mathcal{Q}$ consisting of 4×4 matrices of the form $\mathrm{diag}(\delta_1^r I_2, \delta_2^r I_2)$, where the δ_i^r's are arbitrary real parameters. Our algorithm returns $\eta_{\mathcal{Q}}(M) = 1$ to within an absolute accuracy of 0.001, with the corresponding optimal matrices

$$\begin{aligned} P_{\mathrm{opt}} &= \begin{bmatrix} 2.2527 & 0 & 0 & 0 \\ 0 & 1.2403 & 0 & 0 \\ 0 & 0 & 0.5070 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ G_{\mathrm{opt}} &= j\begin{bmatrix} 0 & -0.50620 & 0 & \\ 0.5062 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.7465 \\ 0 & 0 & 1.7465 & 0 \end{bmatrix}. \end{aligned}$$

Note that the optimal matrix P_{opt} is not positive definite.

5. Conclusion

In this paper, we have presented an interior point algorithm to reliably compute the upper bound $\mu_{\mathcal{Q}}(M)$ described in [1]. Similar algorithms can be applied to many other important problems in control (see for example [6, 7]).

References

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