COMPUTATION OF THE MAXIMUM $\mathbf{H}_\infty$-NORM OF PARAMETER-DEPENDENT LINEAR SYSTEMS BY A BRANCH AND BOUND ALGORITHM

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ABSTRACT

For linear systems that contain unspecified parameters that lie in given intervals, we present a branch and bound algorithm for computing the maximum $\mathbf{H}_\infty$-norm over the set of uncertain parameters.

Key Words: $\mathbf{H}_\infty$-norm, Branch-and-bound, Global optimization, Worst-case analysis, Robustness analysis

1. INTRODUCTION

We consider the family of linear time-invariant systems described by

$$\begin{array}{rclclcl} \dot{x} & = & Ax & + & B_u u & + & B_w w, \qquad\qquad x(0) = x_0, \\ y & = & C_y x & + & D_{yu} u & + & D_{yw} w, \\ z & = & C_z x & + & D_{zu} u & + & D_{zw} w, \\ u & = & \Delta\, y, & & & & \end{array} \tag{1}$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^{n_i}$, $z(t) \in \mathbf{R}^{n_o}$, $u(t), y(t) \in \mathbf{R}^p$, and A, B_u, B_w, C_y, C_z, D_{yu}, D_{yw}, D_{zu} and D_{zw} are real matrices of appropriate sizes. Δ is a diagonal *perturbation matrix.* In the sequel, we will assume that Δ is parametrized by a vector of parameters $q = [q_1, q_2, \ldots, q_m]$, and is given by

$$\Delta = \mathrm{diag}(q_1 I_1, q_2 I_2, \ldots, q_m I_m), \tag{2}$$

where I_i is an identity matrix of size p_i. Of course, $\sum_i^m p_i = p$. We will also assume that q lies in a rectangle $\mathcal{Q}_{\mathrm{init}} = [l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_m, u_m]$. A block diagram of the above family of linear systems is given in figure 1.

For future reference, we define

$$\begin{array}{rcl} P_{yu} & = & C_y(sI - A)^{-1}B_u + D_{yu} \\ P_{yw} & = & C_y(sI - A)^{-1}B_w + D_{yw} \\ P_{zu} & = & C_z(sI - A)^{-1}B_u + D_{zu} \\ P_{zw} & = & C_z(sI - A)^{-1}B_w + D_{zw}. \end{array}$$

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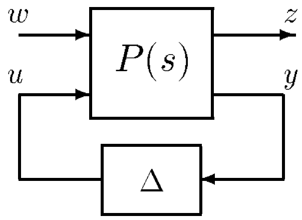

Fig. 1. System in standard form.

We may now write down an expression for the closed-loop transfer matrix from w to z:

$$P_{\mathrm{cl}}(q) = P_{zw} + P_{zu}\Delta(I - P_{yu}\Delta)^{-1}P_{yw}.$$

For uncertain systems described by the above framework, there are many inequivalent measures of stability: the stability margin [7], the minimum stability degree [1] *etc.* We will concern ourselves with computing the maximum $\mathbf{H}_\infty$-norm ($\mathcal{H}_{\mathrm{wc}}$) of the system, defined as

$$\mathcal{H}_{\mathrm{wc}}(\mathcal{Q}_{\mathrm{init}}) = \max_{q \in \mathcal{Q}_{\mathrm{init}}} \left\{ \max_{w(t)\neq 0} \frac{\|z\|_{RMS}}{\|w\|_{RMS}} \right\} = \max_{q \in \mathcal{Q}_{\mathrm{init}}} \|P_{\mathrm{cl}}(q)\|_\infty,$$

where $\|\cdot\|_\infty$ refers to the $\mathbf{H}_\infty$-norm:

$$\|G\|_\infty = \sup_{\mathrm{Re}\, s>0} \sigma_{\max}(G(s)).$$

($\sigma_{\max}(M)$ is the maximum singular value of M). $\mathcal{H}_{\mathrm{wc}}$ is just the *worst-case root mean square gain* (RMS-gain) of the system between the input $w(t)$ and the output $z(t)$.

There exist no methods that compute $\mathcal{H}_{\mathrm{wc}}$ exactly; however, there are several methods that provide good upper and lower bounds for $\mathcal{H}_{\mathrm{wc}}$. For example, lower bounds are provided by Monte Carlo methods where $\mathcal{H}_{\mathrm{wc}}$ is approximated by the largest value of $\|P_{\mathrm{cl}}(q)\|_\infty$ over many values of q drawn according to some distribution. Another class of methods that yield lower bounds are *local optimization methods*. Here a local search is made for the "worst" parameter, that is, one that finds a local maximum of $\|P_{\mathrm{cl}}(q)\|_\infty$. On the other hand, upper bounds are provided by conservative methods. These are usually based on some analytical result, such as a small gain theorem, or a Lyapunov theorem.

In this paper, we employ an approach where we first compute upper and lower bounds for $\mathcal{H}_{\mathrm{wc}}(\mathcal{Q}_{\mathrm{init}})$ using some of the methods described above; if these bounds are not satisfactory, that is, if they are not close enough, a branch and bound technique is used to systematically refine the bounds. At each stage of the algorithm, upper and lower bounds are maintained for $\mathcal{H}_{\mathrm{wc}}(\mathcal{Q}_{\mathrm{init}})$. The branch and bound technique used in this paper is described in detail in [1], where it is used to compute the minimum stability degree.

In the following section, we briefly describe the basic branch and bound algorithm; we then use it to compute $\mathcal{H}_{\mathrm{wc}}$ in subsequent sections.

2. THE BRANCH AND BOUND ALGORITHM

The branch and bound algorithm finds the maximum of a function $f : \mathbf{R}^m \to \mathbf{R}$ over an m-dimensional rectangle $\mathcal{Q}_{\text{init}}$ (the subscript "init" stands for *initial* rectangle).

For a rectangle $\mathcal{Q} \subseteq \mathcal{Q}_{\text{init}}$ we define

$$\Phi_{\max}(\mathcal{Q}) = \max_{q \in \mathcal{Q}} f(q).$$

Then, the algorithm computes $\Phi_{\max}(\mathcal{Q}_{\text{init}})$ to within an absolute accuracy of $\epsilon > 0$. The algorithm uses two functions $\Phi_{\text{lb}}(\mathcal{Q})$ and $\Phi_{\text{ub}}(\mathcal{Q})$ defined over $\{\mathcal{Q} : \mathcal{Q} \subseteq \mathcal{Q}_{\text{init}}\}$ which are easier to compute than $\Phi_{\max}(\mathcal{Q})$. These two functions must satisfy the two following conditions:

(R1) $$\Phi_{\text{lb}}(\mathcal{Q}) \leq \Phi_{\max}(\mathcal{Q}) \leq \Phi_{\text{ub}}(\mathcal{Q}) \quad .$$

(R2) As the maximum half-length of the sides of $\mathcal{Q}$, denoted by size($\mathcal{Q}$), goes to zero, the difference between upper and lower bounds *uniformly* converges to zero, *i.e.,*

$$\forall\, \epsilon > 0\ \exists\, \delta > 0\ \forall\, \mathcal{Q} \subseteq \mathcal{Q}_{\text{init}}\ \text{size}(\mathcal{Q}) \leq \delta \Longrightarrow \Phi_{\text{ub}}(\mathcal{Q}) - \Phi_{\text{lb}}(\mathcal{Q}) \leq \epsilon.$$

Roughly speaking, then, the bounds Φ_{lb} and Φ_{ub} become sharper as the rectangle shrinks to a point.

We describe the algorithm briefly (for a detailed description as well as for a discussion of convergence issues, see [1]). In what follows, k stands for the iteration index, $\mathcal{L}_k$ denotes the list of rectangles, L_k the lower bound and U_k the upper bound for $\Phi_{\max}(\mathcal{Q}_{\text{init}})$, at the end of k iterations.

The Algorithm

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$k = 0$;
$\mathcal{L}_0 = \{\mathcal{Q}_{\text{init}}\}$;
$L_0 = \Phi_{\text{lb}}(\mathcal{Q}_{\text{init}})$;
$U_0 = \Phi_{\text{ub}}(\mathcal{Q}_{\text{init}})$;
while $U_k - L_k > \epsilon$, {
        pick $\mathcal{Q} \in \mathcal{L}_k$ such that $\Phi_{\text{ub}}(\mathcal{Q}) = U_k$;
        split $\mathcal{Q}$ into $\mathcal{Q}_I$ and $\mathcal{Q}_{II}$ along the longest edge;
        $\mathcal{L}_{k+1} := (\mathcal{L}_k - \{\mathcal{Q}\}) \cup \{\mathcal{Q}_I, \mathcal{Q}_{II}\}$;
        $L_{k+1} := \max_{\mathcal{Q} \in \mathcal{L}_{k+1}} \Phi_{\text{lb}}(\mathcal{Q})$;
        $U_{k+1} := \max_{\mathcal{Q} \in \mathcal{L}_{k+1}} \Phi_{\text{ub}}(\mathcal{Q})$;
        $k = k + 1$;
}
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At the end of k iterations, U_k and L_k are upper and lower bounds respectively for $\Phi_{\max}(\mathcal{Q}_{\text{init}})$. Since $\Phi_{\text{lb}}(\mathcal{Q})$ and $\Phi_{\text{ub}}(\mathcal{Q})$ satisfy condition (R2), $U_k - L_k$ is guaranteed to converge to zero.

Now we apply the branch and bound algorithm to the problem of computing the $\mathcal{H}_{\text{wc}}$ of systems described by equations (1).

3. COMPUTATION OF $\mathcal{H}_{\mathrm{wc}}$

Recall that our objective is to compute

$$\mathcal{H}_{\mathrm{wc}}(\mathcal{Q}_{\mathrm{init}}) = \max_{q \in \mathcal{Q}_{\mathrm{init}}} \|P_{\mathrm{cl}}(q)\|_\infty .$$

Then, following the notation used to describe the branch and bound algorithm, we have $f(q) = \|P_{\mathrm{cl}}(q)\|_\infty$ and $\Phi_{\mathrm{max}}(\mathcal{Q}) = \mathcal{H}_{\mathrm{wc}}(\mathcal{Q})$. The task that remains before the branch and bound algorithm can be applied to this problem is the computation of a lower bound $\Phi_{\mathrm{ub}}(\mathcal{Q})$ and an upper bound $\Phi_{\mathrm{lb}}(\mathcal{Q})$ for $\mathcal{H}_{\mathrm{wc}}$.

Given any parameter rectangle, we may first apply a loop transformation so that we have $\mathcal{Q} = [-1,1]^m$ (see [4] or [1] for details). Therefore, we will consider only the case $\mathcal{Q} = [-1,1]^m$. Note that from equation (2), $\|\Delta\|_\infty = 1$.

A simple lower bound for $\mathcal{H}_{\mathrm{wc}}(\mathcal{Q})$ is just the $\mathbf{H}_\infty$-norm of the closed-loop system with the parameter vector set to the midpoint of the parameter region $\mathcal{Q}$:

$$\Phi_{\mathrm{lb}}(\mathcal{Q}) = \|P_{\mathrm{cl}}(0)\|_\infty = \|P_{zw}\|_\infty . \tag{3}$$

We now describe a simple scheme for computing an upper bound that is based on a small gain based robust stability condition due to Doyle [5] and Safonov [9] (see [3, p239-241]).

We define

$$P_\beta = \begin{bmatrix} \dfrac{P_{zw}}{\beta} & \dfrac{P_{zu}}{\sqrt{\beta}} \\ \dfrac{P_{yw}}{\sqrt{\beta}} & P_{yu} \end{bmatrix}, \tag{4}$$

where $\beta > 0$. Then

$$\|P_\beta\|_\infty < 1 \Longrightarrow \sup_{\|\Delta\|_\infty \le 1} \left\| \left[P_{zw} + P_{zu}\Delta(I - P_{yu}\Delta)^{-1} P_{yw} \right] \right\|_\infty < \beta .$$

Our upper bound is:

$$\Phi_{\mathrm{ub}}(\mathcal{Q}) = \inf\left\{\beta : \|P_\beta\|_\infty < 1\right\}, \tag{5}$$

with the convention that the infimum of a function over the empty set is infinity. The condition in (5) is readily checked by forming an appropriate Hamiltonian matrix and checking its eigenvalues (see [2]); a simple bisection can be used to compute Φ_{ub}.

Of course, more sophisticated bounds can be used. A local optimization procedure can be used to search for a (locally) worst parameter value, which would give a good lower bound. The upper bound can be vastly improved by scaling (see Doyle [5], Safonov [8]) or other techniques for approximating the structured singular value (see Fan and Tits [6]).

4. An Example

We consider a mechanical plant consisting of two masses connected by a spring with the lefthand mass driven by a force, as shown in figure 2 below.

The parameters are the masses and spring constant, each of which varies in a range between 2/3 and 3/2:

$$2/3 \le m_1 \le 3/2, \qquad 2/3 \le m_2 \le 3/2, \qquad 2/3 \le k \le 3/2.$$

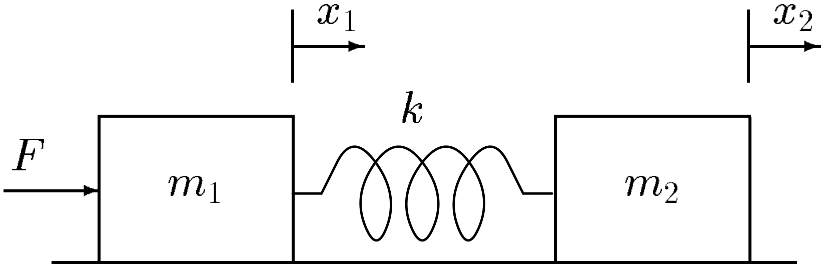

Fig. 2. The plant consisting of two masses connected by a spring.

Thus, these physical parameters can vary over a range exceeding 2 : 1.

With $[x_1\ \dot{x}_1\ x_2\ \dot{x}_2]^T$ as the state, we employ a state-feedback law $F = -k_{\rm LQR}$ which is LQR optimal for the nominal parameter values $m_1 = m_2 = k = 1$ (with weights $Q = I$, $\rho = 1$), and consider the sensitivity transfer function (from w to z):

$$H_{\rm cl} = \frac{1}{1 + k_{\rm LQR} H}$$

where H denotes the transfer matrix from the input F to the state x, as shown in figure 3. Thus, $\mathcal{H}_{\rm wc}$ is the worst case peak of the sensitivity transfer function induced by the parameter variations. From LQR theory we know that with the nominal parameters, $\|H_{\rm cl}\|_\infty = 1$.

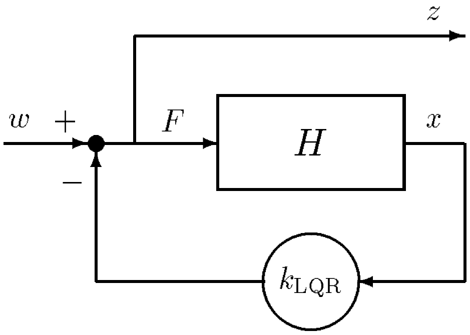

Fig. 3. The closed-loop system with LQR-optimal state feedback.

Figure 4 shows the convergence of the upper and lower bounds as a function of iterations. We observe that by about 50 iterations, the upper bound on $\mathcal{H}_{\rm wc}$ is finite, indicating that the system is robustly stable. At the end of 500 iterations, the algorithm guarantees that $2.21 \leq \mathcal{H}_{\rm wc} \leq 2.34$. The algorithm takes about 1100 iterations to return $\mathcal{H}_{\rm wc} = 2.25$ to within an absolute accuracy of 0.01; thus relatively large parameter variations, even in the worst case, only degrade system performance a little bit. Of course, control theory folklore holds that LQR state feedback is quite "robust". But the folklore does not help us with this specific problem—for example, if the LQR optimal regulator for weights $Q = I$, $\rho = 10$ is used instead, not much can be concluded from conventional LQR wisdom, while our algorithm rapidly determines that $\mathcal{H}_{\rm wc} = \infty$, *i.e.* , the system is not robustly stable.

The algorithm returns the worst-case parameters $m_1 = 3/2$, $m_2 = 2/3$ and $k = 3/2$, which happen to lie on a vertex of the parameter box. Needless to say, this

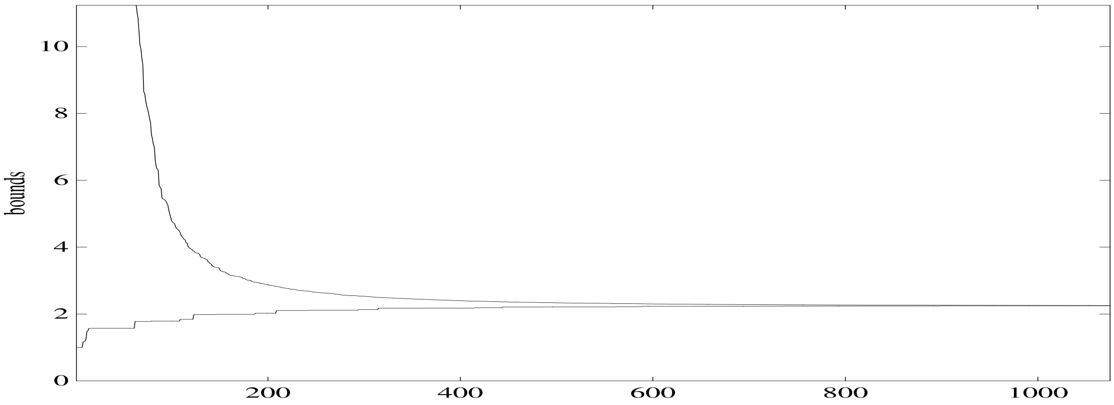

Fig. 4. Bounds for the maximum $\mathbf{H}_\infty$-norm.

is not the case in general. It is likely that local optimization methods would find this set of parameters fairly quickly. However, unlike our algorithm, local methods have no way of guaranteeing that the maximum they find is the global maximum.

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